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On the non-existence of surface waves trapped by submerged obstructions having exterior cusp points

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Trapped modes, i.e. localized unforced oscillations of fluid in presence of floating structures and bottom topography, have been a topic of considerable interest over many years and substantial effort has been put, for a variety of different geometries, into finding the solutions and conditions of their existence or non-existence. However, the class of geometries having cusp points has been avoided in existing proofs of non-existence of trapped modes, though problems of scattering of water waves by such structures have been considered and examples of trapped modes are known. In the work we consider the case when the depth profile or submerged obstacles have exterior cusp points with horizontal tangents. Under some geometrical restrictions a proof of non-existence of trapped modes is given with the help of the so-called Maz’ya’s integral identity. The identity suggested in [1, 2, 3] states that a quadratic form in the potential of trapped mode is equal to zero, which yields non-existence of trapped modes if the form, depending on the geometry, is non-negative. For the obstruction under consideration a generalization of the identity is derived including coefficients of local asymptotics of the trapped mode potential near cusp points of the contour.

1 Introduction

In the paper we consider a three-dimensional ocean with cylindrical boundaries; examples of such geometry are a fluid layer with long canyons or ridges, long circular cylinders submerged parallel to the free surface and a plane beach. We are interested in modes of fluid oscillation which are trapped by the structure, i.e. in unforced time harmonic motions which do not radiate energy to large distances and which are spatially periodic along the generator of the cylindrical geometry.

In the context of the standard linearised theory it was found that some cylindrical structures can support trapped modes; the origin of the theory goes back to Stokes edge waves of 1846 and the results of Ursell [4, 5] and Jones [6] in 1951–1953. Since that time a

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number of papers have appeared concerned with constructing trapped modes and proving their existence or establishing their non-existence in a variety of problems of the surface wave theory. We shall not present the results here for reasons of space and because an extensive review of the subject can be found in [7]. In the paper we consider the case of submerged obstruction having exterior cusp points, which was avoided in the known proofs of non-existence of trapped modes, despite the fact that scattering problems for this class of obstructions have been under consideration and examples of trapped modes for this class of geometry have been constructed (see [8, 9] and references therein).

In order to prove non-existence of trapped modes we shall apply a generalization of the integral identity, which was suggested by Maz’ya in [2, 3] (in a weaker form in [1]). The identity involves an arbitrary vector field and an arbitrary function and states that a quadratic form in the potential of trapped mode is equal to zero. Thus, non-existence of trapped modes is established if the function and the vector field in the domain containing fluid can be chosen such that the quadratic form is non-negative, or, in other words, the vector field nowhere enters the obstruction. Interesting analysis of the scheme can be found in [10, 11, 12, 7], where interpretations and generalizations of the Maz’ya identity are presented. For the geometries considered in the present paper the identity also includes coefficients of local asymptotics of the potential near the cusp points of the contour. The local asymptotics derived in the present work can also be used to generalize other known proofs based on integral identities, e.g. the scheme applied by Simon and Ursell in [13].

Now, the contents of the paper will be briefly summarized. In § 2 we introduce notations and present the mathematical problem for trapped modes along with a remark on correlation of the trapped modes existence with uniqueness of scattering problems. Maz’ya’s identity is presented in § 3, and it is generalized for the case of contours with cusp points in § 4, where, thus, the class of contours, for which the identity guarantees non-existence of trapped modes, is described. The local asymptotics of trapped mode potential near a corner point of the contour is derived in Appendix A.

2 Statement of the problem

We suppose that the fluid occupies a cylindrical domain of the form $W \times \mathbb{R}$, where $W$ denotes an unbounded open set in $\mathbb{R}^2$. We decompose the boundary of $W$ as $F \cup S$, where $F$ is the mean free surface and $S$ is the wetted part of submerged obstructions including the bottom and some submerged obstacles. The surface $S$ is assumed to be piecewise smooth and is allowed to have corner and cusp points, so that any finite part of the contour contains only finite number of corner points.

Let $t$ be a time variable and a Cartesian coordinate system $(x, y, z)$ be attached to the free surface of the fluid so that $y$ is the vertical coordinate decreasing with depth
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and equal to zero in the free surface and $x, z$ are horizontal coordinates, such that the geometry of obstruction is constant in $z$. The notation is illustrated in fig. 1, where the case of constant-depth ocean with a deepening is shown.

The fluid is assumed to be ideal incompressible. Under these assumptions its irrotational motion can be described in frame of the linearised surface wave theory (see e.g. [14, ch. 3]) by a velocity potential $U(x, y, z, t)$ satisfying the equations

$$
\begin{align*}
(\partial_x^2 + \partial_y^2 + \partial_z^2) U &= 0 \quad \text{in} \quad W \times \mathbb{R}, \\
\partial_y U + g \partial_y U &= 0 \quad \text{on} \quad F \times \mathbb{R}, \\
\partial_n U &= 0 \quad \text{on} \quad S \times \mathbb{R},
\end{align*}
$$

where we use the notation $\partial_a$ for the partial derivative in variable $a$, $g$ is the acceleration due to gravity and $n$ is the unit normal vector directed into the domain $W$.

In the present note we are interested in trapped modes, i.e. solutions of the form

$$U(x, y, z, t) = \text{Re}\{u(x, y)e^{i(\omega t + \kappa z)}\},$$

where $k$ and $\omega$ are real numbers and $u(x, y)$ is a real-valued function submitted to the ‘localization property’ i.e. the condition of finiteness of energy over the cross-section of the fluid domain:

$$\int_W \left(|\nabla u|^2 + \nu^2 u^2\right) \, dx \, dy < \infty.$$  

A solution to (2.1)–(2.5) corresponds to a harmonic wave propagating in the $z$-direction without distortion and having finite transverse energy.

It follows by substituting (2.4) into (2.1)–(2.3), that the potential $u$, which is defined on $W$, satisfies the following boundary value problem

$$
\begin{align*}
\nabla^2 u - k^2 u &= 0 \quad \text{in} \quad W, \\
\partial_y u - \nu u &= 0 \quad \text{on} \quad F, \\
\partial_n u &= 0 \quad \text{on} \quad S,
\end{align*}
$$

Figure 1: A sketch of geometric notation
where \( \nabla = (\partial_x, \partial_y) \) and \( \nu = \omega^2/g \). We mean a solution of (2.5), (2.6)–(2.8) in the classical sense, so that \( u \in C^2(W) \cap C(\overline{W}) \) and the potential \( u \) has regular normal derivative at all regular points of contours where the normal is well-defined.

The problem (2.6)–(2.8) also arises when considering surface waves in an infinitely long channel with vertical walls \( z = \pm b \) spanned by a horizontal cylindrical obstruction. In this case the velocity potential can be fixed in the form
\[
U(x, y, z, t) = \Re \{ u(x, y) e^{i\omega t} \} \cos kz,
\]
where \( kb = n\pi \), \( (n = 1, 2, \ldots) \), so as to satisfy (2.8) on the walls. A similar solution with a sine dependence on \( z \) can be taken provided \( kb = (2n - 1)\pi/2 \), \( (n = 1, 2, \ldots) \).

Non-existence of the trapped modes is closely related to the uniqueness in radiation and diffraction problems. Let, for example, the domain \( W \) be a layer of fluid (see fig. 1), having a constant depth \( h \) for \( x > a \). Consider the problem, which describes scattering by the topography of oblique water waves having wavenumber \( \nu \) and whose crests make an angle \( \theta \) with the plane \( Oxz \), so that \( k = \nu \sin \theta \) in (2.4). The problem for the potential \( u \) consists of (2.6), (2.7), the non-homogeneous Neumann condition \( \partial_n u = f \) on \( S \) and the condition at infinity \( \partial_x u \mp i\ell_\pm u \to 0 \) as \( x \to \pm\infty \), where \( \ell_\pm = (\lambda_\pm^2 - k^2)^{1/2} \) and \( \lambda_\pm \) is the unique, real, positive root of the equation \( \nu = \lambda_\pm \tanh \lambda_\pm h_\pm \). Since the contours are assumed to have corner points we also demand \( u \in H^1_{\text{loc}}(W) \). Then, following the scheme of [1, § 2] or [10, § 2] and taking into account the asymptotics at the cusp points in the way we use in § 4 of this work, it can be shown that the difference of two solutions to the problem in question satisfies the condition (2.5) and, thus, represents a trapped mode.

3 Maz'ya's integral identity

In this section we present a version of the integral identity known as Maz'ya's one. In order to obtain the variant of the identity we follow the derivation of [7, § 4.2], where the identity was applied for the case \( k \neq 0 \) in the condition (2.6), and the scheme of [1] to make the identity applicable for obstacles having outlets to infinity. We consider a real vector field \( V = (V_1, V_2) \) and a real function \( H \) defined on \( W \). We also assume that \( V \in C^1(\overline{W}), H \in C^2(\overline{W}) \) and start with the identity
\[
2(V \cdot \nabla u + Hu) \nabla^2 u = 2 \nabla \cdot \{ (V \cdot \nabla u + Hu) \nabla u \} + (Q \nabla u) \cdot \nabla u - \nabla \cdot \{ |\nabla u|^2 V + u^2 \nabla H \} + u^2 \nabla^2 H,
\]
(3.1)
where the elements of the matrix \( Q \) are defined as follows:
\[
Q_{ij} = (\nabla \cdot V - 2H) \delta_{ij} - (\partial_x V_i + \partial_y V_j), \quad i, j = 1, 2, \quad x_1 = x, \quad x_2 = y,
\]
and \( \delta_{ij} \) is the Kronecker delta.

Further we integrate the identity (3.1) over \( W_R \), where \( W_R \) is a domain bounded internally by the obstruction \( D \) with the wetted surface \( S = \partial D \), externally by a part of
semicircle \( C_R = \{|x + iy| = R\} \setminus D \) and by a part of the free surface \( F_R \) from above (see fig. 2). We also use the notation \( S_R = S \cap \{(x, y) : |x + iy| \leq R, y \leq 0\} \). Then, taking into account the condition (2.6) we have

\[
0 = \int_{W_R} \left\{ (Q \nabla u) \cdot \nabla u - 2k^2 u (V \cdot \nabla u + Hu) + u^2 \Delta^2 H \right\} \, dx \, dy \\
+ \int_{C_R \cup S_R} \left\{ |\nabla u|^2 V \cdot n + u^2 \partial_n H - 2\partial_n u (V \cdot \nabla u + Hu) \right\} \, ds \\
+ \int_{F_R} \left\{ 2\partial_y u (V \cdot \nabla u + Hu) - |\nabla u|^2 V_2 - u^2 \partial_y H \right\} \, dx.
\]

Figure 2: Auxiliary geometrical notation

It is to note at this point that though the potential \( u \) is continuous over the surface of obstruction, its derivatives can be unbounded at corners. From results of Appendix A it follows that the potential is \( \mathcal{O}(\rho^{\pi/\alpha}) \) and its derivatives are \( \mathcal{O}(\rho^{\pi/\alpha - 1}) \), where \( \rho \) denotes the radial distance from the corner of angle \( \alpha \), the angle is measured through the fluid domain. Under the assumption that \( \alpha \in (0, 2\pi) \) for any corner point of \( S \) the latter estimates are sufficient to ensure the existence of integrals in the last formula and those appearing subsequently in this section.

Using the condition (2.7) we write

\[
\int_{F_R} \partial_y u \left[ V_1 \partial_x u + V_2 \partial_y u + Hu \right] \, dx = \nu \int_{F_R} \left[ u^2 (\nu V_2 + H) + u V_1 \partial_x u \right] \, dx, 
\]
where

\[
2 \int_{F_R} u V_1 \partial_x u \, dx = \int_{F_R} V_1 \partial_x (u^2) \, dx = \left[ u(x, 0) \right] V_1(x, 0) \bigg|_{x=-R}^{x=R} - \int_{F_R} u^2 \partial_x V_1 \, dx.
\]

Also, we have

\[
2 \int_{W_R} u \nabla \cdot u \, dx \, dy = \int_{W_R} \nabla \cdot (u^2) \, dx \, dy \\
= - \int_{W_R} u^2 \nabla \cdot V \, dx \, dy + \int_{F_R} u^2 V_2 \, dx - \int_{C_R \cup S_R} u^2 V \cdot n \, ds
\]

(3.5)
Using the formulas (3.3)–(3.5) to transform (3.2) we find
\[
\int_{F_R} \left\{ u^2 \left[ 2\nu^2 V_2 + 2\nu H - \nu \partial_z V_1 - k^2 V_2 - \partial_y H \right] - |\nabla u|^2 V_2 \right\} \, dx \\
+ \int_{S_R} \left\{ \mathbf{V} \cdot \mathbf{n} \left( |\nabla u|^2 + k^2 u^2 \right) + u^2 \partial_n H - 2 \partial_n u \left( \mathbf{V} \cdot \nabla u + Hu \right) \right\} \, ds \] (3.6)
\[+ \int_{W_R} \left\{ (\mathbf{Q} \nabla u) \cdot \nabla u + u^2 \left[k^2 (\nabla \cdot \mathbf{V} - 2H) + \nabla^2 H \right]\right\} \, dx \, dy = \alpha(R; u, \mathbf{V}, H),
\]
where
\[
\alpha(R; u, \mathbf{V}, H) = \int_{C_R} \left\{ 2 \partial_n u \left( \mathbf{V} \cdot \nabla u + Hu \right) - \mathbf{V} \cdot \mathbf{n} \left( |\nabla u|^2 + k^2 u^2 \right) - u^2 \partial_n H \right\} \, ds \\
+ \nu \left[u(-R, 0)\right]^2 V_1(-R, 0) - \nu \left[u(R, 0)\right]^2 V_1(R, 0). \] (3.7)

Asymptotics of the term \( \alpha \) for large \( R \) is established in the following assertion.

**Proposition 3.1.** Let \( u \) be a solution to the problem (2.5), (2.6)–(2.8) and let the functions \( V_i, H \) and \( \partial H \) have estimate \( O(R) \) as \( R \to \infty \) uniform in \( \theta \), where \( Re^{i\theta} = x + iy \). Then,
\[
\liminf_{R \to \infty} \alpha(R; u, \mathbf{V}, H) = 0. \] (3.8)

**Proof.** By the assumption on finiteness of energy we have
\[
\int_{W_R} \left( |\nabla u|^2 + \nu^2 u^2 \right) \, dx \, dy = \int_{R} \, d\rho \int_{C_R} \left( |\nabla u|^2 + \nu^2 u^2 \right) \, ds \leq \frac{1}{2}.
\]
From the convergence of the integral over \((R, \infty)\) in the last formula it obviously follows that there exists a sequence \( \{R_n\} \), such that \( R_n \to \infty \) as \( n \to \infty \) and,
\[
R_n \int_{C_{R_n}} \left( |\nabla u|^2 + \nu^2 u^2 \right) \, ds \to 0 \quad \text{as} \quad n \to \infty. \] (3.9)

Further we consider the integral identity
\[
\int_{W_{R_n}} u \nabla^2 u \, dx \, dy = - \int_{W_{R_n}} |\nabla u|^2 \, dx \, dy + \int_{F_{R_n}} u \partial_y u \, dx - \int_{C_{R_n}} u \partial_n u \, ds,
\]
which can be rewritten with the help of (2.6) and (2.7) as follows
\[
\int_{W_{R_n}} \left(|\nabla u|^2 + k^2 u^2\right) \, dx \, dy + \int_{C_{R_n}} u \partial_n u \, ds = \nu \int_{F_{R_n}} u^2 \, dx. \] (3.10)

Using the Cauchy-Buniakowsky inequality and the formula (3.9) it can be shown that the second integral in (3.10) tends to zero as \( n \to \infty \). Thus, from the condition (2.5) it
follows that $\int_{F_{R_n}} u^2 \, dx$ has a finite limit as $n \to \infty$. Since $\int_{F_R} u^2 \, dx$ is monotonic in $R$, the function $\int_{F_R} u^2 \, dx$ has the same limit as $R \to \infty$, so that $\int_{F} u^2 \, dx < \infty$ and we write

$$\int_{W \setminus W_R} \left( |\nabla u|^2 + \nu^2 u^2 \right) \, dx \, dy + \nu \int_{F \setminus F_R} u^2 \, dx = \int_{R} d\rho \left\{ \nu [u(\rho, 0)]^2 + \nu [u(-\rho, 0)]^2 + \int_{C_\rho} (|\nabla u|^2 + \nu^2 u^2) \, ds \right\} < \infty,$$

and convergence of the integral implies that

$$\liminf_{\rho \to \infty} \rho \left\{ \nu [u(\rho, 0)]^2 + \nu [u(-\rho, 0)]^2 + \int_{C_\rho} (|\nabla u|^2 + \nu^2 u^2) \, ds \right\} = 0.$$

Under the assumption on behaviour of $V_i$, $H$ and $|\nabla H|$ at infinity, the latter formula obviously substantiates the formula (3.8).

4 Non-existence of modes trapped by obstructions with non-overlapping horizontal exterior cusps

We note that under the particular choice $V = (-x, 0)$, $H = -1/2$, many of the terms in the formula (3.6) disappear and we arrive at the identity (cf. (18.1) in [1]):

$$2 \int_{W_R} |\partial_x u|^2 \, dx \, dy - \int_{S_R} x n_x (|\nabla u|^2 + k^2 u^2) \, ds + \int_{S_R} \partial_u (2 x \partial_x u + u) \, ds = \alpha(R; u), \quad (4.1)$$

where $n = (n_x, n_y)$ and the function $\alpha$ is given in (3.7).

Consider the case when the contours do not contain cusp points and let $x n_x \leq 0$ on $S$. Then the first two integrals in (4.1) are monotonic in $R$, the third integral vanishes in view of (2.8), and, thus, from (3.8) it follows that $\lim_{R \to \infty} \alpha(R; u) = 0$ and we arrive at

$$\int_{S} x n_x (|\nabla u|^2 + k^2 u^2) \, ds = 2 \int_{W} |\partial_x u|^2 \, dx \, dy. \quad (4.2)$$

Since $x n_x \leq 0$ on $S$, from (4.2) it obviously follows that $u_x \equiv 0$ in $W$ and, hence, in view of (2.7) $u \equiv 0$ in $W$. The condition $x n_x \leq 0$ can be satisfied e.g. for deepening (not raising) of the depth profile and for a system of non-overlapping bodies of semi-infinite extent (docks).

In the case of contours with cusp points we cannot apply the identity (4.1) directly. We consider the identity for the domain $W$ with small vicinities of cusp points excluded and then shrink their size to zero. Further we shall suppose that $x n_x \leq 0$ at all regular points of $S$ and that the number of cusp points is finite. We shall also consider only the case of exterior cusps with horizontal one-side tangents.
We introduce discs of radius $\varepsilon$ with centres at the cusp points

$$B^\pm_i(\varepsilon) = \{ (x, y) : |x - x^\pm_i + i(y - y^\pm_i)| \leq \varepsilon \}, \quad i = 1, \ldots, N_\pm,$$

and denote by $W(\varepsilon)$ the fluid domain with $\varepsilon$-vicinities of the cusp points excluded and by $S(\varepsilon)$ the wetted surface of the bodies with $\varepsilon$-vicinities of the cusp points added

$$W(\varepsilon) = W \setminus \bigcup_{\pm} \bigcup_{i=1}^{N_\pm} B^\pm_i(\varepsilon), \quad S(\varepsilon) = \partial W(\varepsilon) \setminus F.$$

Following the arguments we used to obtain (4.2) from (4.1), we arrive at the identity

$$
2 \int_{W(\varepsilon)} |\partial_x u|^2 \, dx \, dy - \int_{S(\varepsilon)} x \, n_x (|\nabla u|^2 + k^2 u^2) \, ds \\
+ \sum_{\pm} \sum_{i=1}^{N_\pm} \int_{S^\pm_i(\varepsilon)} \partial_n u (2 x \partial_x u + u) \, ds = 0, \tag{4.3}
$$

where $S^\pm_i(\varepsilon) = \partial B^\pm_i(\varepsilon) \cap W$ (see fig. 3). Further we shall make use of the asymptotic representation (A.5) to compute asymptotics as $\varepsilon \to 0$ of the integrals over contours $S^\pm_i(\varepsilon)$ in the latter formula. Omitting the index $i$ for brevity we define the polar system of coordinates with origin at the point $P^\pm_i$

$$x = x^\pm \pm \rho^\pm \cos \theta^\pm, \quad y = y^\pm - \rho^\pm \sin \theta^\pm,$$

so that the value $\theta^\pm = 0 (2\pi)$ corresponds to the lower (upper) side of the cusp as $\rho^\pm \to 0$. 

---

**Figure 3: Auxiliary geometrical notation**

We denote by $P^+_i = (x^+_i, y^+_i)$, $i = 1, \ldots, N_+$, and $P^-_j = (x^-_j, y^-_j)$, $j = 1, \ldots, N_-$ the cusp points turning to the right and to the left resp. The numeration of the points is e.g. as follows

$$x^+_1 \leq \ldots \leq x^+_N, \quad x^-_1 \leq \ldots \leq x^-_N.$$
We use the asymptotics (A.5) of the potential $u$ near the cusp points and write
\begin{align}
\partial_x u(\rho^\pm, \theta^\pm) &= \mp 2^{-1} c^\pm (\rho^\pm)^{-1/2} \cos(\theta^\pm/2) + O((\rho^\pm)^{1/2}), \\
\partial_y u(\rho^\pm, \theta^\pm) &= -2^{-1} c^\pm (\rho^\pm)^{-1/2} \sin(\theta^\pm/2) + O((\rho^\pm)^{1/2}).
\end{align}
(4.4)

By (A.5) and (4.4) we have
\begin{align}
\int_{S^\pm(\varepsilon)} \partial_n u (2x \partial_x u + u) \, ds \xrightarrow{\varepsilon \to 0} 
\mp x^\pm (c^\pm)^2 \int_0^{2\pi} \cos^2 \frac{\theta^\pm}{2} \, d\theta^\pm = \mp \pi x^\pm (c^\pm)^2, \\
\int_{S^\pm(\varepsilon)} x n_x (|\nabla u|^2 + k^2 u^2) \, ds \xrightarrow{\varepsilon \to 0} 
\frac{x^\pm (c^\pm)^2}{2} \int_0^{2\pi} \cos \theta^\pm \, d\theta^\pm = 0.
\end{align}
(4.5)

Finally, combining (4.3), (4.5) and taking the limit $\varepsilon \to 0$ we arrive at the following generalization of Maz’ya’s identity involving coefficients of singularities of the velocity field:
\begin{align}
\pi \sum_{\pm} \sum_{i=1}^{N_x} \pm x_i^\pm (c_i^\pm)^2 = \int_W |\partial_x u|^2 \, dx \, dy - \int_S x n_x (|\nabla u|^2 + k^2 u^2) \, ds. \tag{4.6}
\end{align}

We apply the equality (4.6) to prove the following assertion.

**Theorem 4.1.** Let for any horizontal line $\gamma = \{ y = -H \}$, $H > 0$, the set $\gamma \cap \overline{W}$ be empty or consist of only one segment and the segment contains the point $(0, -H)$. Then the problem (2.5), (2.6)–(2.8) for the geometry $S$ has only the trivial solution.

**Proof.** From the assumptions imposed it follows that $x n_x \leq 0$ at all regular points of the contour $S$, the latter was preposed in derivation of the identity (4.6). Besides, under the assumptions of the assertion, we have $\pm x_i^\pm \leq 0$. Hence, from (4.6) we find
\begin{align}
2 \int_W |\partial_x u|^2 \, dx \, dy - \int_S x n_x (|\nabla u|^2 + k^2 u^2) \, ds \leq 0. \tag{4.7}
\end{align}

From the last inequality it follows that $\int_W |\partial_x u|^2 \, dx \, dy = 0$ and, thus, $u \equiv 0$ in $W$. \qed

Two important cases of the geometry, for which Theorem 4.1 yields non-existence of trapped modes are the case of deepening of bottom shown in fig. 1, when Theorem 4.1 generalizes results of Vainberg and Maz’ya [1], and the case when the submerged obstruction is a system of non-overlapping horizontal semi-infinite barriers. Consideration of the problem for semi-infinite barriers can be found in [8] (see also references therein).

Let the potential $u$ satisfy the following general form of the Helmholtz equation
\begin{align}
\nabla^2 u + \lambda u = 0 \quad \text{in} \quad W. \tag{4.8}
\end{align}
For the problem with the equation (4.8) we can repeat literally the arguments of the Sections 3 and 4 leading to the inequality (4.7) where the term $k^2$ should be replaced by $-\lambda$. Since the second integral in (4.7) vanishes if horizontal barriers are considered, Theorem 4.1 for this geometry is valid for the problem consisting of the conditions (2.5), (2.7), (2.8) and (4.8).

**Conclusion**

A boundary-value problem describing localized unforced harmonic motions of an ideal unbounded fluid in presence of submerged obstacles and a bottom topography is considered in the case when the geometry has exterior cusps. A generalization of the Maz’ya integral identity has been obtained including coefficients of singularities of the velocity field. The identity has been used to prove non-existence of trapped modes for some classes of geometries having exterior cusps with horizontal one-side tangents.

Further work will endeavour to prove an analogue of Theorem 4.1 for three-dimensional water wave problem in the case when the submerged obstructions have angular points with horizontal tangents. It is supposed that non-existence of trapped modes can be established when any cross-section of the fluid domain by the plane $y = -H$, $H > 0$, is starlike with regard to the $y$-axis.

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**Appendix A. Asymptotics of potential near corners of obstacles**

In this appendix we shall derive asymptotics near a corner of a submerged obstacle for a function $u$ satisfying the Helmholtz equation (4.8) in the fluid domain $W$, the homogeneous Neumann condition (2.8) on the boundary of the obstacle $S$ and the condition of finiteness of energy $u \in H^1_{\text{loc}}(W)$. We introduce a system of polar coordinates $(\rho, \theta)$ with origin at the corner point. Let $\alpha$ be the angle between the one-side tangents in the corner point, where $\alpha \in (0, 2\pi]$, measured through the fluid. Let the contour $S$ be smooth at distance $\ell$ from the corner and let $\chi_{a,b}(s) \in C^\infty(\mathbb{R})$ be a cut-off function, which is equal to one for $s \in [0, a]$ and to zero for $s \geq b$. We define the potential

$$U(\rho, \theta) = \chi_{a,b}(\rho) u(\rho, \theta), \quad a < b < \ell.$$  

(A.1)
Since \( U \equiv 0 \) for \( \rho > b \), further we shall consider the potential in the subset \( \Omega \) of the fluid domain, where \( \Omega = W \cap \{(\rho, \theta) : \rho < c\}, b < c < \ell \).

Taking into account (4.8) we find

\[
\nabla^2 U = \rho^2 \partial_\rho^2 U + \rho \partial_\rho U + \partial_\theta^2 U = \rho^2 \left( u \partial_\rho^2 \chi + 2 \partial_\rho u \partial_\rho \chi + \chi \partial_\rho^2 u \right) \\
+ \rho \left( u \partial_\rho + \chi \partial_\rho \right) u + \chi \partial_\rho^2 u = \chi \left( \nabla^2 u + \lambda u \right) + F = F,
\]

where

\[
F(\rho, \theta; u) = \rho^2 \left[ u(\rho, \theta) \partial_\rho^2 \chi(\rho) + 2 \partial_\rho u(\rho, \theta) \partial_\rho \chi(\rho) \right] \\
+ \rho u(\rho, \theta) \partial_\rho \chi(\rho) - \lambda \chi(\rho) u(\rho, \theta).
\]

(A.2)

Consider also the normal derivative of the potential \( U \) on the boundary of the domain \( \Omega \). By (A.1) and (2.8) we find

\[
\partial_n U = u \partial_n \chi \equiv G.
\]

(A.3)

Let us now suppose that the potential \( u \) is fixed so that \( F(x, y; u) = F(x, y) \) and \( G(x, y; u) = G(x, y) \), then the potential \( U \) can be considered as a solution to the Neumann problem for the Poisson equation in the domain \( \Omega \):

\[
\nabla^2 U = F \quad \text{in} \quad \Omega, \\
\partial_n U = G \quad \text{on} \quad \partial \Omega.
\]

(A.4)

We shall make use of the results by [15, ch. 2] and have to introduce here the functional spaces used in this book. Let the space \( V^l_\gamma(\Omega) \) \((l = 0, 1, \ldots ; \gamma \in \mathbb{R})\) consisting of functions on \( \Omega \) be defined as closure of \( C^0_0(\Omega \setminus \partial \Omega) \) in the norm

\[
\|u; V^l_\gamma(\Omega)\| = \left( \int_{\Omega} \sum_{i=0}^{l-1} \sum_{j=0}^{l-i} \rho^{2(\gamma-l+i+j)} |\partial_x^i \partial_y^j u(x, y)|^2 \, dx \, dy \right)^{\frac{1}{2}}
\]

and the spaces \( V^{l-1/2}_\gamma(\partial \Omega) \) \((l = 1, 2, \ldots)\) consists of traces on \( \partial \Omega \) of functions from \( V^l_\gamma(\Omega) \) with the norm defined by

\[
\|u; V^{l-1/2}_\gamma(\partial \Omega)\| = \inf \{ \|v; V^{l}_\gamma(\Omega)\| : v = u \text{ on } \partial \Omega \setminus \partial \Omega \}.
\]

From the condition (2.5) and the definition (A.2) it obviously follows that \( F \in V^l_\gamma(\Omega) \) for \( l = 1 \) and \( \gamma > 1 \). In its turn the function \( G \) is equal to zero in \( \{(\rho, \theta) \in \Omega : \rho < a\} \). Under the assumption on smoothness of the boundary \( u \in C^\infty(\Omega \setminus 0) \) and, in view of (A.3), \( G \) belongs to the space \( V^{m+1/2}_\gamma(\partial \Omega) \) for any \( m > 0 \), in particular, for \( m = l = 1 \), which is preposed in Theorem 4.2 in [15, ch. 2]. In order to satisfy further conditions of the theorem, where it is assumed that \( \gamma - l - 1 \in (0, \pi \alpha^{-1}) \), we choose \( \gamma = 2 + \varepsilon \), where \( \varepsilon \)}
is a small positive value. Then, the theorem guarantees existence of the unique solution to (A.4), \( \mathcal{U} \in V_{l+2}^1(\Omega) \). Appealing to the definitions of the functions \( \mathcal{U} \) and \( F \) we find that \( F \in V_{2+e}^3(\Omega) \) and, obviously, \( F \in V_{4+e}^3(\Omega) \), so that we can apply Theorem 4.2 in [15, ch. 2] again. Repeating the procedure we find that \( \mathcal{U} \in V_{2n+1}^{2n+1}(\Omega) \) for all integers \( n \geq 0 \).

In order to satisfy the assumptions of Theorem 4.4 in [15, ch. 2] we shall consider \( \mathcal{U} \) as an element of a wider class, so that \( \mathcal{U}, F \in V_l^1(\Omega) \), where \( l = 2n + 1, \gamma = 2n + e, e > 0 \), and the asymptotics of \( \mathcal{U} \) as \( \rho \to 0 \) is given by Theorem 4.4 in [15, ch. 2] as follows:

\[
\mathcal{U}(x,y) = X(\rho) \left\{ c_0 + c_{01} \log \rho + \sum_{j=1}^{m} c_j \rho^{j\pi/\alpha} \cos \left( j\pi\theta\alpha^{-1} \right) \right\} + w(x,y), \tag{A.5}
\]

where \( c_i \) are constants, \( w \in V_{l+2}^1(\Omega) = V_{2n+3}^{2n+3}(\Omega) \), \( n \geq 0 \) and the integer \( m \) is defined by the condition of Theorem 4.4 in [15, ch. 2] that the value \( \alpha^{-1}(l + 1 - \gamma) \) should belong to the interval \((m, m + 1)\), so that \( m = 0, 1, 2, 3 \) for \( m/2 < \alpha^{-1} \leq (m + 1)/2 \). From the fact that \( w \in V_{2n+3}^{2n+3}(\Omega) \) it follows, in particular, that \( \partial_x \partial_y w = O(\rho^{2-i-j}) \) as \( \rho \to 0 \). We note that \( c_{01} = 0 \) in view of the condition (2.5).

References


