Golden mean renormalization for a generalized Harper equation: The strong coupling fixed point

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Golden mean renormalisation for the Harper equation: the strong coupling fixed point

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Abstract

We construct a renormalisation fixed point corresponding to the strong coupling limit of the golden mean Harper equation. We give an analytic expression for this fixed point, establish its existence and uniqueness, and verify properties previously seen only in numerical calculations. The spectrum of the linearisation of the renormalisation operator at this fixed point is also explicitly determined. This strong coupling fixed point also helps describe the onset of a strange nonchaotic attractor in quasiperiodically forced systems.

1 Introduction

The finite difference eigenvalue equation known as the Harper equation (also known as the almost Mathieu equation):

$$\psi_{i+1} + \psi_{i-1} + 2\lambda \cos(2\pi(i\omega + \phi))\psi_i = E\psi_i$$  \hspace{1cm} (1.1)

is important in the study the localization transition in incommensurate systems. (For a review see [12].) Here $E$ is the eigenvalue corresponding to eigenfunction $\psi_i$ which is defined on the one-dimensional integer lattice with sites labelled by $i \in \mathbb{Z}$.

In this paper we study the self-similar fluctuations of the exponentially localized eigenstates of this equation occurring when the interaction parameter $\lambda > 1$. The exponential decay of the wave function $\psi_i$ leads us to write

$$\psi_i = e^{-\gamma|i|}\eta_i,$$  \hspace{1cm} (1.2)

where $\gamma = \log \lambda$ (the Lyapunov exponent) [1]. The fluctuations may thus be analysed by considering the evolution of $\eta_i$. For $i > 0$ the Harper equation now reduces to

$$\lambda^{-1}\eta_{i+1} + \lambda\eta_{i-1} + 2\lambda \cos(2\pi(i\omega + \phi))\eta_i = E\eta_i.$$  \hspace{1cm} (1.3)

We shall be exclusively concerned here with the golden mean case $\omega = (\sqrt{5} - 1)/2$ which satisfies $\omega^2 + \omega = 1$. According to the work of Ketoja and Satija [6], if the phase $\phi$ is chosen so that the wave function has its peak at $i = 0$, so that the fluctuations are bounded, the entire localized phase may be characterised by a
single universal \textit{strong} coupling fixed point of the renormalisation operator associated with the recursion

\begin{equation}
    t_{n+1}(x) = t_n(-\omega x) t_{n-1}(\omega^2 x + \omega).
\end{equation}

In this paper we determine a fixed point of this recursion and thereby rigorously establish the results numerically obtained in [6].

The outline of this paper is as follows. In section 2 we review the work of Ketoja and Satija focussing on a derivation of (1.4). After establishing some notation in section 3, in section 4 we formulate our main results, which we begin to prove in section 6, having established some preliminary results in section 5. We shall prove that the recursion (1.4) has a real analytic, entire fixed point, unique up to a choice of ‘universality class’. We shall give an expression for this fixed point and determine some of its analytic properties, thereby confirming the numerical observations reported by Ketoja and Satija in [6] and [7]. In section 7 we solve an associated eigenvalue problem, which in turn we use in sections 8 and 9 to completely determine the spectrum of the derivative of the renormalisation operator at the fixed point. In section 10 we point out a connection between the functional equations studied here and the classical Schröder and Abel functional equations.

There is subtle dependence of the universal characteristic functions on the phase \(\phi\) in (1.1). We shall say little about how this affects the renormalisation in this paper, confining our attention to a few specifically chosen simple cases. We hope to be able to elaborate on this matter in a forthcoming publication. We also remark that it is not yet completely clear how the results generalise to other incommensuracies.

Our work here will also shed some light on the problem of the onset of a strange nonchaotic attractor, in which an identical renormalisation scheme determines critical behaviour [9]. In this setting, the dependence on (the equivalent of) the phase \(\phi\) leads to a multifractal nature of the strange nonchaotic attractor.

## 2 Golden-mean renormalisation theory for the Harper equation

In this section we briefly review the renormalisation/decimation approach of Ketoja and Satija [6], who show how a fixed point of the recursion (1.4) helps to explain the universality of the supercritical localized regime of the golden mean Harper equation. Note that in [6] the iteration occurs in the form

\begin{equation}
    \tilde{t}_{n+1}(x) = \tilde{t}_n(-\omega x) \tilde{t}_{n-1}(\omega^2 x + \omega),
\end{equation}

but the substitution \(\tilde{t}_n = -t_n\) renders it equivalent to (1.4).

Remarkably, Kuznetsov \textit{et al} [9] derive this same equation, (1.4), in their analysis of the birth of a strange nonchaotic attractor. In [7] these two seemingly distinct scenarios are linked, and indeed an analogy with the critical dissipative standard map is also drawn.

In the decimation scheme of Ketoja and Satija (first introduced in [4]) we consider only sites with index a Fibonacci number \(F_n\), where we define \(F_0 = 0\), \(F_1 = 1\), and \(F_n = F_{n-1} + F_{n-2}\) for \(n > 1\). This results in a \textit{decimation} of the tight-binding model (1.3) of the form

\begin{equation}
    s_{n+1}(i) = s_n(i) t_{n+1}(i + F_n) + t_n(i) \eta_i,
\end{equation}

where, simply by using the defining property of the Fibonacci numbers, the functions \(t_n\) and \(s_n\) may be seen to obey the explicit recursions

\begin{equation}
    t_{n+1}(i) = -t_n(i) \left( t_{n-1}(i + F_n) + s_{n-1}(i + F_n) \right) / \left( 1 + s_n(i) \left( t_{n-1}(i + F_n) + s_{n-1}(i + F_n) \right) \right)
\end{equation}

\begin{equation}
    s_{n+1}(i) = s_{n-1}(i + F_n) s_n(i + F_n) / \left( 1 + s_n(i) \left( t_{n-1}(i + F_n) + s_{n-1}(i + F_n) \right) \right),
\end{equation}

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for $n > 1$. The initial conditions for these recursions are

\begin{align}
    t_1(i) &= 0, \\
    s_1(i) &= 1; \\
    t_2(i) &= -\left(\frac{E}{\lambda} - 2 \cos(2\pi(i + 1)\omega + \phi)\right)^{-1}, \\
    s_2(i) &= \lambda^{-2} \left(\frac{E}{\lambda} - 2 \cos(2\pi(i + 1)\omega + \phi)\right)^{-1};
\end{align}

given by equation (2.2) with $n = 1$ and $n = 2$ respectively, referring to the original version of the tight-binding model (1.3). (In their earlier work [5] Ketoja and Satija develop a general decimation scheme for arbitrary irrational $\omega$ based on Farey paths.)

Under repeated iteration it is found that $s_n \to 0$, and so asymptotically (in $n$) we may ignore $s_n$ and are left with the single recursion

\begin{equation}
    t_{n+1}(i) = -t_n(i)t_{n-1}(i + E_n).
\end{equation}

We need to be careful however with regard to the initial conditions for this recursion. Looking at equation (2.3) and assuming that we have $s_n = 0$ for $n > 1$ and $t_1 = 0$, leads us to write

\begin{equation}
    t_3(i) = -t_2(i)t_2(i + E_n),
\end{equation}

where $t_2$ is given by (2.7). Given $t_2$, we thus begin by defining $t_3$ in this manner and then use equation (2.9) for $n > 2$.

Henceforth, we shall consider these limiting equations at the “upper band edge”, at which $\lim E/\lambda = +2$, and we shall also take the phase $\phi = 0$. (The lower band edge where $\lim E/\lambda = -2$ can be treated similarly. In that case we set $\phi = 1/2$. A period-three orbit results rather than a fixed point.) For other choices of the phase $\phi$ the analysis is far from complete. We hope to be able to investigate the resulting periodic orbits that arise for certain other choices of $\phi$ in a forthcoming publication. (Some preliminary findings concerning this subtle dependence on the phase were also discussed in the context of strange nonchaotic attractors in [9].)

To analyse the scaling we now replace the discrete lattice index $i$ by the continuous variable $x = (-\omega)^{-n}\{n\omega\}$ where $\{\cdot\}$ denotes fractional part. Care must be taken when doing this as the definition of $x$ depends on the index $n$ of the function. The result is that our recursion becomes

\begin{equation}
    t_{n+1}(x) = -t_n(-\omega x)t_{n-1}(\omega^2 x + \omega),
\end{equation}

for $n > 2$, with initial conditions

\begin{align}
    t_2(x) &= -\left(2\left(1 - \cos(2\pi(\omega^2 x + \omega))\right)\right)^{-1}, \\
    t_3(x) &= -t_2(-\omega x)t_2(\omega^2 x + \omega).
\end{align}

We have thus derived the recursion (2.1), or equivalently (1.4). However, we shall not be concerned directly with the iteration itself starting with initial conditions arising from (1.3). Rather we seek the fixed point to which the starting conditions converge.

Note that a fixed point of (1.4) is a solution to the equation

\begin{equation}
    t(x) = t(-\omega x)t(\omega^2 x + \omega),
\end{equation}

and we remark that if $t(x)$ is a fixed point then so too is $t(x)^\alpha$ for any real number $\alpha$, so it is not immediately clear how we should normalise this equation. Normalisation corresponds to selecting the ‘correct’ universality class, and for this reason it is important to pay attention to the initial conditions. We observe that $t_2$ has a pole of order two at $x = 1$, and that this singularity in preserved (and indeed propagated) by the recursion. Thus we shall look for a solution to (2.14) with a pole of order two at $x = 1$. In fact we shall show that there exists a fixed point with a pole (or zero) of any order, i.e., there are many universality classes parametrised by the nature of the singularity at $x = 1$. 

\begin{equation}
    t(x) = t(-\omega x)t(\omega^2 x + \omega),
\end{equation}
It what follows it will at times be convenient to express the second order recursion (1.4) as a first order recursion of pairs of functions. This is achieved (as in [6]) by writing the original iteration (1.4) as an iteration for a pair of functions \((t_{n-1}(-\omega x), t_n(x)) = (u(x), t(x))\), say, thereby defining the renormalisation operator

\[ R : (u(x), t(x)) \mapsto (t(-\omega x), t(-\omega x)u(-\omega x - 1)). \] (2.15)

In our analysis we will also have recourse to an additive version of this operator

\[ (U(x), T(x)) \mapsto (T(-\omega x), T(-\omega x) + U(-\omega x - 1)). \] (2.16)

### 3 Iterated function system and the fundamental interval

Let

\[ \phi_1(z) = -\omega z, \quad \phi_2(z) = \omega^2 z + \omega, \] (3.1)

where \(\omega = (\sqrt{5} - 1)/2\) is the golden mean which satisfies \(\omega^2 + \omega = 1\). Then we may write equation (2.14) in the form

\[ t(z) = t(\phi_1(z))t(\phi_2(z)). \] (3.2)

Associated with this equation is an iterated function system (IFS) on \(C\) given by the two contractions \(\phi_1\), \(\phi_2\) satisfying the following properties.

1. \(\phi_1\) and \(\phi_2\) are linear contractions on \(C\) with fixed points 0 and 1 respectively, and with \(\phi'_1(z) = -\omega\) and \(\phi'_2(z) = \omega^2\).

2. The interval \(I = [-\omega, 1]\) is the fixed point set for the IFS. Indeed

\[ \phi_1([-\omega, 1]) = [-\omega, \omega^2], \quad \phi_2([-\omega, 1]) = [\omega^2, 1], \] (3.3)

so that

\[ \phi_1(I) \cup \phi_2(I) = I. \] (3.4)

We shall refer to \(I\) as the fundamental interval.

3. The fundamental interval \(I\) is the attractor for the IFS. Indeed given any compact subset \(K \subseteq C\) and any open neighbourhood \(U\) of \(I\) in \(C\), there exists \(N \in \mathbb{N}\) such that for any \(k \geq N\) and any choice \(i_1, \ldots, i_k \in \{1, 2\}\) we have

\[ \phi_{i_1} \circ \cdots \circ \phi_{i_k}(z) \in U \] (3.5)

for any \(z \in K\).

4. For any open neighbourhood \(U\) of \(I\) in \(C\) there is a neighbourhood \(V \subseteq U\) of \(I\) in \(C\) with \(\phi_1(V) \subseteq V\), \(\phi_2(V) \subseteq V\).

For notational convenience we shall assume in what follows that all sums and products over a finite sequence \(i_1, \ldots, i_k\) of indices are assumed to be over an unrestricted choice from the set \(\{1, 2\}\) unless otherwise indicated. We shall also often just write \(\Sigma i_j\) for the sum of the indices, the range of \(j\) being clear from context.

### 4 Statement of main results

In this section we state our main results which put the numerical results of Ketola and Satija in [6] and [7] on a firm basis. Proofs are given in later sections.

Firstly we have the following existence and uniqueness result.
Theorem 1. Let $n \in \mathbb{N}$ be given. Then there exists a unique, real analytic, entire function $t : \mathbb{C} \to \mathbb{C}$ satisfying the fixed point equation

$$t(z) = t(\phi_1(z))t(\phi_2(z)),$$

with

1. $t(1) = 0$
2. $t^{(j)}(1) = 0$ for $j = 1, \ldots, n-1$, $t^{(n)}(1) \neq 0$, so that $t$ has a zero of order $n$ at $z = 1$; and
3. $t(z) > 0$ for $z \in (-\omega^{-1}, 1)$.

Moreover

$$t(z) = t_*(z)^n,$$

where $t_*$ is the entire function given by

$$t_*(z) = \frac{1 - z}{1 - \omega} \prod_{k=1}^{\infty} \prod_{i_1=1}^{i_k} \frac{1 - \phi_{i_k} \circ \cdots \circ \phi_{i_1}(z)}{1 - \phi_{i_k} \circ \cdots \circ \phi_{i_1}(\omega)}.$$ 

Figure 1: The function $t_*$. 

The function $t_*$ in theorem 1 is plotted in figure 1. Some of its properties are given in the following theorem.

Theorem 2. The function $t_*$ in theorem 1 satisfies:

1. the zeros of $t_*$ are the points $1$ and $\phi_{i_k}^{-1} \circ \cdots \circ \phi_{i_1}^{-1}(1)$, where $k \geq 1$, $i_1 = 1$ and $i_2, \ldots, i_k \in \{1, 2\}$ (We note that $-\omega^{-1}$ is of this form);
2. $t_*(\omega) = 1, t_*(-\omega) = \omega^{-2}, t_*(\omega^2) = \omega^{-1};$
3. $t_*$ has a unique maximum at $z_c$ on $(-\omega^{-1}, 1)$ with $z_c \in (-\omega, 0)$.

In section 8 we define a Banach space $F$ on which the derivative of the operator $R$ (2.15) acts, and prove the following theorem.

Theorem 3. Let $n \in \mathbb{N}$ and let $t$ be the solution of (4.1) given by theorem 1. Let $u(z) = t(-\omega z)$. Then

1. The derivative of $R$ at $(u, t)$, $L = dR(u, t)$, is a compact operator on $F$. 

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2. The spectrum of $L$ consists of 0 together with eigenvalues

$$\lambda = \pm \omega^{-n}, \pm \omega^{-(n-1)}, \ldots, \pm \omega^{-1}, \pm \omega, \pm \omega^2, \ldots.$$  \hspace{1cm} (4.4)

Each of these eigenvalues is simple except for $\omega^{-1}$ which is a double eigenvalue with a one-dimensional eigenspace and further one-dimensional generalised eigenspace.

Remark. The universal scaling ratio $|t(0)|$, numerically calculated in [6] to be 0.172586410945 (with error bound one in the last digit), is just $t_*(0)^{-2}$. Indeed the universal function $t(x)$ (figure 1 in [7]) is just $-t_*(x)^{-2}$.

The spectrum given by theorem 3 when $n = 2$ accords with that studied by Ketoja and Satija in [6]. A similar spectrum is also noted by Ostlund and Pandit in their analysis of the critical golden mean Harper equation [11].

To prove theorem 1 we shall consider an additive version of the fixed point equation:

$$T(z) = T(\phi_1(z)) + T(\phi_2(z)).$$  \hspace{1cm} (4.5)

Our proof involves firstly obtaining a fixed point of the linear operator $R$ given by

$$R(f)(z) = f(\phi_1(z)) + f(\phi_2(z)) - f(\phi_1(\omega)) - f(\phi_2(\omega)).$$  \hspace{1cm} (4.6)

We observe that a fixed point $T_*$ of $R$ that is analytic on $(-\omega^{-1}, 1)$ is a solution of (4.5). For if

$$T_*(z) = T_*(\phi_1(z)) + T_*(\phi_2(z)) - T_*(\phi_1(\omega)) - T_*(\phi_2(\omega)),$$  \hspace{1cm} (4.7)

then, evaluating at $z = \omega$, we see that $T_*(\omega) = 0$. We now evaluate at $z = 0$ to obtain

$$T_*(0) = T_*(0) + T_*(\omega) - T_*(\phi_1(\omega)) - T_*(\phi_2(\omega)),$$  \hspace{1cm} (4.8)

which immediately gives $T_*(\phi_1(\omega)) + T_*(\phi_2(\omega)) = 0$, and so $T_*(z) = T_*(\phi_1(z)) + T_*(\phi_2(z))$.

5 Preliminary Results

In this section we establish some preliminary results to help us prove theorem 1.

Specifically, let $r > 1$ and consider a function $f$ analytic on the disc $\Delta(0, r) = \{z \in \mathbb{C} : |z| < r\}$. Let

$$f(z) = \sum_{i=0}^{\infty} f_i \left(\frac{z}{r}\right)^i.$$  \hspace{1cm} (5.1)

We define the $\ell_1$-norm, $\|f\|$, with respect to the basis functions $1, \frac{z}{r}, \frac{z^2}{r^2}, \ldots$, by

$$\|f\| = \sum_{i=0}^{\infty} |f_i|.$$  \hspace{1cm} (5.2)

We denote by $B_r$ the complex Banach space of analytic functions on $\Delta(0, r)$ with $\|f\| < \infty$.

Now if $r > 1$ we have

$$\phi_1(\Delta(0, r)) \subseteq \Delta(0, r), \quad \phi_2(\Delta(0, r)) \subseteq \Delta(0, r).$$  \hspace{1cm} (5.3)

For if $|z| < r$ then $|\phi_1(z)| = |\omega| |z| < \omega r < r$, and $|\phi_2(z)| = |\omega^2 z + \omega| \leq \omega^2 |z| + \omega < \omega^2 r + \omega < (\omega^2 + \omega)r = r$, where in the penultimate step we have used $r > 1$.

Our basic tool for the construction of a solution to (4.5), $T_*$, is the observation that for $r$ chosen sufficiently large, the operator $R$ is a contraction on $B_r$ to the zero function. Indeed we have the following lemma

**Lemma 1.** There exist $r > 1$ and $0 < c < 1$ such that $R$ is a bounded linear operator on $B_r$ with operator norm $\|R\| \leq c < 1$. Indeed $r = 2$ suffices.
Proof. It is clear that $R$ is linear and maps a function analytic on $\Delta(0, r)$ to another such function. Since the norm on $B_r$ is the $\ell_1$-norm we have that
\[
\|R\| = \sup_{i \geq 0} \|R(e_i)\|, \tag{5.4}
\]
where $e_i$ is the basis function $e_i(z) = (z/r)^i$. If we show that $\|R\| \leq c$ for some $c < 1$ then we shall have shown that $R(B_r) \subseteq B_r$ and $\|R\| \leq c$.

Let us therefore consider the action of $R$ on the basis function $e_n$.
\[
R(e_n)(z) = (-\omega z/r)^n + ((\omega^2 z + \omega)/r)^n - (\omega^2/r)^n - ((\omega^3 + \omega)/r)^n
\]
\[
= ((-\omega)^n + \omega^{2n}) (z/r)^n + \sum_{j=1}^{n-1} \binom{n}{j} \omega^{2(n-j)} (z/r)^{n-j} (\omega/r)^{j} + \omega^2 (\omega^3 + \omega)^n (\omega/r)^n.
\tag{5.5}
\]
Thus
\[
\|R(e_n)\| = \|(-\omega)^n + \omega^{2n} + \sum_{j=1}^{n-1} \binom{n}{j} \omega^{2(n-j)} (\omega/r)^{j} + (\omega/r)^n + \omega^2 (\omega^3 + \omega)^n (\omega/r)^n
\]
\[
= \omega^n + (\omega^2 + \omega/r)^n + \omega^2 (\omega^3 + \omega)^n (\omega/r)^n. \tag{5.7}
\]
We note that this expression decreases as $n$ increases and as $r$ increases. Moreover for fixed $r > 1$ it has limit $0$ as $n \to \infty$. Indeed it can be easily checked that for $n \geq 3, r \geq 2$ we have
\[
\|R(e_n)\| \leq \omega^3 + (\omega^2 + \omega/2)^3 + \omega^4/4 + (\omega^3 + \omega)^3/8 = (35\omega - 19)/4 < 1. \tag{5.8}
\]
For $n = 0$ we have $R(e_0)(z) = 0$ for all $z$. For $n = 1$
\[
\|R(e_1)\| = |(-\omega) + \omega^2| + |\omega/r + \omega^2/r - (\omega^3 + \omega)/r|
\]
\[
= \omega - \omega^2 + \omega^2/r - \omega^2/r - \omega^3/r - \omega/r
\]
\[
= \omega - \omega^2 + \omega^2/r - \omega^3/r
\]
\[
= (\omega - \omega^2)(1 + \omega/r)
\]
\[
= \omega^3(1 + \omega/r) < \omega^2 < 1, \tag{5.9}
\]
for $r > 1$.

Finally for $n = 2$ we have (taking $r \geq 2$)
\[
\|R(e_2)\| = |\omega^2 + \omega^4| + |2\omega^3/r + \omega^2/r^2 - (\omega^3 + \omega)^2/r^2|
\]
\[
\leq \omega^2 + \omega^4 + \omega^3 + (\omega^3 + \omega)^2/4 + \omega^4/4 - \omega^2/4
\]
\[
= (19 - 25\omega)/4 < 1, \tag{5.10}
\]
as can readily be checked.

Taking the maximum of these numbers for $n = 0, 1, 2 \ldots$ it is clear that
\[
\|R\| = \sup_{i \geq 0} \|R(e_i)\| \leq c, \tag{5.11}
\]
for some $c < 1$ provided $r \geq 2$. Thus the lemma is proved. \qed
One immediate consequence of this lemma is the following

**Corollary 1.** The only entire solution of (4.5) is the zero solution: $T(z) = 0$ for all $z$.

**Proof.** A solution of (4.5) is a fixed point of $R$ on $B_2$ and as we have seen in the lemma

$$0 \leq \| R(T) \| \leq \| R \| \| T \| \leq c \| T \| < \| T \|,$$  

(5.12)

so if $R(T) = T$ we must have $\| T \| = 0$, i.e., $T(z) = 0$ for all $z \in \Delta(0, 2)$. Hence $T(z) = 0$ for all $z \in \mathbb{C}$. □

This corollary will be used to establish the uniqueness of the solution.

In the proof of theorem 1 we shall also have use of the following lemma that allows us to extend solutions of equation (4.1) to the whole of the complex plane.

**Lemma 2 (The extension lemma).** Let the function $t$ be analytic and satisfy equation (4.1) on an open neighbourhood $U \supseteq [-\omega, 1]$ in $\mathbb{C}$. Then there is an entire extension of $t$ to the whole of $\mathbb{C}$ satisfying equation (4.1) for all $z \in \mathbb{C}$.

**Proof.** From section 3 property 4, we may, by restricting to a smaller set if necessary, assume that $\phi_1(U)$, $\phi_2(U) \subseteq U$. Let $z \in \mathbb{C} \cap U$. Then, since the iterated function system consisting of $\phi_1$ and $\phi_2$ has fixed point set $[-\omega, 1]$, there exists $k \geq 1$ such that for all $i_1, \ldots, i_k \in \{1, 2\}$ we have

$$\phi_{i_1} \circ \cdots \circ \phi_{i_k}(z) \in U.$$  

(5.13)

We define

$$t(z) = \prod_{i_1, \ldots, i_k} t(\phi_{i_1} \circ \cdots \circ \phi_{i_k}(z)).$$  

(5.14)

We firstly observe that this definition is independent of the choice of $k$. For if $1 \leq k_1 < k_2$ are such that for all $i_1, \ldots, i_{k_1}$, we have $\phi_{i_1} \circ \cdots \circ \phi_{i_{k_1}}(z) \in U$ and for all $i_1, \ldots, i_{k_2}$, we have $\phi_{i_1} \circ \cdots \circ \phi_{i_{k_2}}(z) \in U$, then

$$\prod_{i_1, \ldots, i_{k_1+1}} t(\phi_{i_1} \circ \cdots \circ \phi_{i_{k_1+1}}) = \prod_{i_1, \ldots, i_{k_1}} t(\phi_{i_1} \circ \cdots \circ \phi_{i_{k_1}}(\phi_1(z)) t(\phi_{i_1} \circ \cdots \circ \phi_{i_k}(\phi_2(z))) = \prod_{i_1, \ldots, i_{k_1}} t(\phi_{i_1} \circ \cdots \circ \phi_{i_k}(z)).$$  

(5.15)

Here we have used the fact that $t$ satisfies equation (4.1) on $U$. Continuing in this way we see that

$$\prod_{i_1, \ldots, i_{k_2}} t(\phi_{i_1} \circ \cdots \circ \phi_{i_{k_2}}(z)) = \prod_{i_1, \ldots, i_{k_1}} t(\phi_{i_1} \circ \cdots \circ \phi_{i_{k_1}}(z)).$$  

(5.16)

Thus the definition is independent of the choice of $k$.

We now observe that $t$ satisfies equation (4.1). For if $z \in U$, and $k \geq 2$ are such that $\phi_{i_1} \circ \cdots \circ \phi_{i_k}(z) \in U$ for all choices $i_1, \ldots, i_k \in \{1, 2\}$, we have

$$t(z) = \prod_{i_1, \ldots, i_k} t(\phi_{i_1} \circ \cdots \circ \phi_{i_k}(z))$$

$$= \prod_{i_1, \ldots, i_{k-1}} t(\phi_{i_1} \circ \cdots \circ \phi_{i_{k-1}}(\phi_1(z))) \prod_{i_1, \ldots, i_{k-1}} t(\phi_{i_1} \circ \cdots \circ \phi_{i_{k-1}}(\phi_2(z)))$$

$$= t(\phi_1(z)) t(\phi_2(z)).$$  

(5.17)

Finally, let $z \in \mathbb{C}$. Then there exists $k \geq 1$ and an open disc $V$ in $\mathbb{C}$ containing $z$ such that

$$\phi_{i_1} \circ \cdots \circ \phi_{i_k}(V) \subseteq U$$  

(5.18)

for all $i_1, \ldots, i_k \in \{1, 2\}$. Then for all $w \in V$

$$t(w) = \prod_{i_1, \ldots, i_k} t(\phi_{i_1} \circ \cdots \circ \phi_{i_k}(w))$$  

(5.19)

is a finite product of analytic maps and is therefore analytic. Thus we have defined an analytic extension of $t$ to $\mathbb{C}$. □
6 Proofs of theorems 1 and 2

6.1 Proof of theorem 1

6.1.1 Existence

We construct the solution of (4.5) as follows. Let

\[ f_0(z) = \log(1 - z), \]  

(6.1)

where we take the principal branch of the logarithm function, analytic for \(|z| > 0, \arg(z) \in (-\pi, \pi)\). Let

\[ f_1(z) = f_0(\phi_1(z)) - f_0(\phi_1(\omega)) = \log \left( \frac{1 - \phi_1(z)}{1 - \phi_1(\omega)} \right), \]  

(6.2)

and, for \(n \geq 2\), let

\[ f_n(z) = R^{n-1}(f_1)(z) = R(f_{n-1})(z), \]  

(6.3)

with the linear operator \(R\) defined in (4.6). We observe that

\[ f_k(z) = \sum_{i_1, \ldots, i_k} \log \left( \frac{1 - \phi_{i_1} \circ \cdots \circ \phi_{i_k}(z)}{1 - \phi_{i_1} \circ \cdots \circ \phi_{i_k}(\omega)} \right). \]  

(6.4)

We set

\[ T_*(z) = \sum_{n=0}^{\infty} f_n(z). \]  

(6.5)

We observe that \(f_0\) is analytic on \(\Delta(0, 1)\), and \(f_1\) is analytic on \(\phi_1^{-1}(\Delta(0, 1)) = \Delta(0, \omega^{-1})\). Furthermore we observe that if \(z \in \Delta(0, 2)\) then

\[ |\phi_1^2(z)| = |\omega^2 z| \leq \omega^2 r < 1, \]  

(6.6)

and

\[ |\phi_1 \circ \phi_2(z)| = |1 - \omega^2 z + \omega^2| \leq \omega^2 |z| + \omega^2 \leq \omega^3 r + \omega^2 < 1. \]  

(6.7)

Thus

\[ f_2(z) = R(f_1)(z) = \log \left( \frac{1 - \phi_1 \circ \phi_1(z)}{1 - \phi_1 \circ \phi_1(\omega)} \right) + \log \left( \frac{1 - \phi_1 \circ \phi_2(z)}{1 - \phi_1 \circ \phi_2(\omega)} \right) \]  

(6.8)

is analytic on a domain \(\Delta(0, 2 + \varepsilon)\) for some \(\varepsilon > 0\), and thus \(f_2 \in B_2\). It follows that for \(n \geq 2\)

\[ \|f_n\| = \|R^{n-2}(f_2)\| \leq \|R\|^{n-2}\|f_2\| \leq c^{n-2}\|f_2\|, \]  

(6.9)

where \(c < 1\) is given by lemma 1. Thus the function

\[ S_*(z) = T_*(z) - f_0(z) - f_1(z) = \sum_{n=2}^{\infty} f_n(z) \]  

(6.10)

is absolutely convergent on \(B_2\), and thus \(T_*\) is well defined and analytic on \(\Delta(0, 1)\). Moreover

\[ R(S_*)(z) = R(T_*)(z) - R(f_0)(z) - R(f_1)(z) = R \left( \sum_{n=2}^{\infty} f_n(z) \right) \]

\[ = \sum_{n=2}^{\infty} R(f_n)(z) = \sum_{n=2}^{\infty} f_n(z) \]

\[ = S_*(z) - f_2(z). \]  

(6.11)

But

\[ R(f_0)(z) = \log \left( \frac{1 - \phi_1(z)}{1 - \phi_1(\omega)} \right) + \log \left( \frac{1 - \phi_2(z)}{1 - \phi_2(\omega)} \right) \]

\[ = f_1(z) + \log \left( \frac{1 - (\omega^2 z + \omega)}{1 - (\omega^3 + \omega)} \right) \]

\[ = f_1(z) + \log \left( 1 - \omega^2 - \omega \right) \]

\[ = f_1(z) + f_0(z). \]  

(6.12)
Hence

\[ R(T_\ast)(z) = R(S_\ast + f_0 + f_1)(z) \]
\[ = R(S_\ast)(z) + R(f_0)(z) + R(f_1)(z) \]
\[ = S_\ast(z) - f_2(z) + f_0(z) + f_1(z) + f_2(z) \]
\[ = S_\ast(z) + f_0(z) + f_1(z) \]
\[ = T_\ast(z). \]  \hfill (6.13)

Thus \( T_\ast \) is a fixed point of \( R \). We now define

\[ t_\ast(z) = \exp(T_\ast(z)) = \exp(f_0(z)) \exp(f_1(z)) \exp(S_\ast(z)) \]
\[ = \frac{(1 - z)}{(1 - \phi_1(z))} \prod_{k \geq 2} \prod_{i_1, i_2, \ldots, i_k} \frac{1 - \phi_{i_1} \circ \cdots \circ \phi_{i_k}(z)}{1 - \phi_{i_1} \circ \cdots \circ \phi_{i_k}(\omega)}. \]
\hfill (6.14)

We note that the expression on the right hand side is analytic on \( \Delta(0, 2) \) (the exponential has removed the singularities of \( f_0 \) and \( f_1 \)). Moreover, let \( \tilde{R} \) denote \( \exp(R) \), i.e.,

\[ \tilde{R}(f)(z) = \frac{f(\phi_1(z))}{f(\phi_1(\omega))} \frac{f(\phi_2(z))}{f(\phi_2(\omega))}. \]
\hfill (6.15)

Then for \( z \in \Delta(0, 2) \)

\[ \tilde{R}(t_\ast)(z) = \frac{(1 - \phi_1(z))}{(1 - \phi_1(\omega))} \frac{(1 - \phi_2(z))}{(1 - \phi_2(\omega))} \frac{(1 - \phi_3(z))}{(1 - \phi_3(\omega))} \frac{(1 - \phi_4(z))}{(1 - \phi_4(\omega))} \exp(R(S_\ast)(z)) \]
\[ = \frac{(1 - z)}{(1 - \phi_1(z))} \frac{(1 - \phi_2(z))}{(1 - \phi_2(\omega))} \frac{(1 - \phi_3(z))}{(1 - \phi_3(\omega))} \frac{(1 - \phi_4(z))}{(1 - \phi_4(\omega))} \exp(f_2(z) + S_\ast(z) - f_2(z)) \]
\[ = \frac{(1 - z)}{(1 - \phi_1(\omega))} \frac{(1 - \phi_2(z))}{(1 - \phi_2(\omega))} \exp(S_\ast(z)) \]
\[ = t_\ast(z). \]  \hfill (6.16)

Thus \( \tilde{R}(t_\ast)(z) = t_\ast(z) \) for \( z \in \Delta(0, 2) \), and hence

\[ t_\ast(z) = \frac{t_\ast(\phi_1(z))}{t_\ast(\phi_1(\omega))} \frac{t_\ast(\phi_2(z))}{t_\ast(\phi_2(\omega))}. \]
\hfill (6.17)

Evaluating at \( z = \omega \) and then at \( z = 0 \) gives \( t_\ast(\phi_1(\omega))t_\ast(\phi_2(\omega)) = 1 \) and so

\[ t_\ast(z) = t_\ast(\phi_1(z))t_\ast(\phi_2(z)). \]
\hfill (6.18)

for \( z \in \Delta(0, 2) \). We also note that \( t_\ast \) is real analytic. Finally we use the extension lemma to extend to the whole of \( \mathbb{C} \).

We also observe that \( t_\ast(1) = 0 \) and \( t'_\ast(1) \neq 0 \), and, in view of (6.14), \( t_\ast(z) \neq 0 \) for \( z \in \Delta(0, 2) \setminus \{ -\omega^{-1}, 1 \} \) and indeed \( t_\ast(z) > 0 \) for \( z \in (-\omega^{-1}, 1) \), so that \( t_\ast \) is the desired solution of equation (4.1) in the case \( n = 1 \). The desired solution for \( n \geq 1 \) is simply \( t(z) = t_\ast(z)^n \).

### 6.1.2 Uniqueness

We shall now prove that the solution constructed is unique. Let \( t_1, t_2 \) be two entire solutions of equation (4.1) satisfying properties 1–3 of theorem 1. We shall show that \( t_1(z) = t_2(z) \) for all \( z \in \mathbb{C} \). Consider the function

\[ v(z) = \frac{t_1(z)}{t_2(z)}. \]
\hfill (6.19)

Then, in view of properties 1–3 of theorem 1, we have that \( v \) has a removable singularity at \( z = 1 \) and so we may define \( v(1) = \lim_{z \to 1} v(z) \neq 0 \) so that \( v \) is analytic at \( z = 1 \); \( v(z) \neq 0 \) for \( z \in [-\omega, 1] \); and on a neighbourhood \( U \) of \([-\omega, 1] \) in \( \mathbb{C} \), \( t_1(z), t_2(z) \neq 0 \) except at \( z = 1 \) and hence \( v \) is well defined, analytic and nonzero on \( U \).
We may now apply the extension lemma (lemma 2) to extend \( v \) to the whole complex plane giving an entire function. Moreover we see that \( v(z) \neq 0 \) for all \( z \in \mathbb{C} \). For if \( z \in \mathbb{C} \), then we may choose \( k \geq 1 \) such that 
\[
\phi_{i_1} \circ \cdots \circ \phi_{i_k}(z) \in U \quad \text{for all } i_1, \ldots, i_k \in \{1, 2\}. \tag{6.21}
\]
Then 
\[
v(z) = \prod_{i_1, \ldots, i_k} v(\phi_{i_1} \circ \cdots \circ \phi_{i_k}(z)) \neq 0,
\]
since \( v(z) \neq 0 \) on \( U \).

Now \( v \) is entire and nonzero, so we may write \( v(z) = \exp(V(z)) \), where \( V \) is an entire function, defined up to an integer multiple of \( 2\pi i \). Now 
\[
\exp(V(z)) = v(z) = v(\phi_1(z))v(\phi_2(z)) = \exp(V(\phi_1(z)) + V(\phi_2(z))),
\]
for \( z \in \mathbb{C} \), so that 
\[
\exp(V(z) - V(\phi_1(z)) - V(\phi_2(z))) = 1,
\]
for all \( z \in \mathbb{C} \). It follows that 
\[
V(z) = V(\phi_1(z)) + V(\phi_2(z)).
\]

From corollary 1 it follows that \( V(z) = 0 \), and hence \( v(z) = 1 \), so that \( t_1(z) = t_2(z) \) for all \( z \in \mathbb{C} \).

This completes the proof of theorem 1.

### 6.2 Proof of theorem 2

1. Since \( t_*(z) > 0 \) on \((-\omega^{-1}, 1)\) it follows that there is an open neighbourhood \( U \) of \([-\omega, 1)\) such that \( z = 1 \) is the only zero of \( t_*(z) \) in \( U \).

Suppose \( z \in \mathbb{C} \) with \( t_*(z) = 0 \). Then let \( k \geq 1 \) be such that 
\[
\phi_{i_1} \circ \cdots \circ \phi_{i_k}(z) \in U \quad \text{for all choices } i_1, \ldots, i_k \in \{1, 2\}. \tag{6.26}
\]
Then since 
\[
0 = t_*(z) = \prod_{i_1, \ldots, i_k} t_*(\phi_{i_1} \circ \cdots \circ \phi_{i_k}(z))
\]
and we must have 
\[
\phi_{i_1} \circ \cdots \circ \phi_{i_k}(z) = 1
\]
for some choice of \( i_1, \ldots, i_k \in \{1, 2\} \). Then clearly \( z = \phi_{i_k}^{-1} \circ \cdots \circ \phi_{i_1}^{-1}(1) \).

Conversely if \( z = \phi_{i_k}^{-1} \circ \cdots \circ \phi_{i_1}^{-1}(1) \) for some \( k \) and some choice of \( i_1, \ldots, i_k \in \{1, 2\} \) then 
\[
t_*(z) = \prod_{i_1, \ldots, i_k} t_*(\phi_{i_1} \circ \cdots \circ \phi_{i_k}(z)) = 0. \tag{6.27}
\]

2. Substituting \( z = 0 \) into equation (4.1) we immediately obtain 
\( t_*(\omega) = 1 \) since \( t_*(0) \neq 0 \).

Differentiating equation (4.1) and substituting \( z = 1 \) gives 
\[
t'_*(1) = -\omega t'_*(\omega)t_*(1) + \omega^2 t_*(\omega)t'_*(1) \tag{6.28}
\]
Since \( t_*(1) = 0, t'_*(1) \neq 0 \) we obtain 
\( t_*(\omega) = \omega^{-2} \).

Substitution of \( z = -\omega \) into equation (4.1) gives 
\( t_*(-\omega) = t_*(\omega^2)^2 \), which, since \( t_*(\omega^2) > 0 \) gives 
\( t_*(\omega^2) = \omega^{-1} \).

3. We observe that \( T_*(\omega) \) (given by equation (6.5)) is a sum of terms each of which have negative second derivative on \( R \). Indeed for \( x \in (-\omega^{-1}, 1) \) we have 
\[
\frac{d^2}{dx^2} \log \frac{1 - \phi_{i_1} \circ \cdots \circ \phi_{i_k}(x)}{1 - \phi_{i_1} \circ \cdots \circ \phi_{i_k}(\omega)} = -\frac{((\phi_{i_1} \circ \cdots \circ \phi_{i_k})'(x))^2}{(1 - \phi_{i_1} \circ \cdots \circ \phi_{i_k}(x))^2} < 0. \tag{6.29}
\]
It follows that $T''_\ast(x) < 0$.

Since $\lim_{x \to -1 -} T_\ast(x) = -\infty$ and $\lim_{x \to -\omega^{-1} -} T_\ast(x) = -\infty$, we have that $T_\ast$ (and hence $t_\ast = \exp(T_\ast)$) has a unique maximum in the interval $(-\omega^{-1}, 1)$.

Furthermore, differentiating equation (4.5), we obtain

$$T'_\ast(x) = -\omega T'_\ast(\phi_1(x)) + \omega^2 T'_\ast(\phi_2(x)),$$

which, evaluated at $x = 0$, gives

$$(1 + \omega)T'_\ast(0) = \omega^2 T'_\ast(\omega).$$

Now, from part 2 above, we see $T'_\ast(-\omega) = -2\log \omega > -\log \omega = T'_\ast(\omega^2) > T'_\ast(\omega) = \log 1 = 0$, hence $T'_\ast(\omega) < 0$. It follows that $T'_\ast(0) < 0$, and so the maximum is to the left of zero.

Evaluating at $x = -\omega$, we see that

$$T'_\ast(-\omega) = (\omega^2 - \omega)T'_\ast(\omega^2).$$

But $\omega^2 > 0$ and hence $T'_\ast(\omega^2) < 0$, and thus $T'_\ast(-\omega) > 0$, so the maximum occurs to the right of $-\omega$.

This completes the proof of theorem 2.

# 7 An associated eigenvalue problem

Associated with the additive version of the operator $\mathcal{R}$ defined in equation (2.16) we may consider the eigenproblem

$$\lambda(U(z), T(z)) = (T(-\omega z), T(-\omega z) + U(-\omega z - 1)),$$

with $\lambda \neq 0$, which on eliminating $U$ becomes

$$T(z) = \lambda^{-1} T(\phi_1(z)) + \lambda^{-2} T(\phi_2(z)).$$

As well as being an interesting problem in its own right, this problem will also arise in our analysis in the next sections of the spectrum of the derivative of the operator $\mathcal{R}$ at its fixed point. In that case we will be interested in eigenfunctions (and a generalised eigenfunction) of (7.2) which are either polynomial, or which possess a pole at $z = 1$.

## 7.1 Polynomial Solutions

Firstly we look for polynomial solutions of (7.2). If $T$ is a polynomial of degree $k$:

$$T(z) = a_0 + a_1 z + \cdots + a_k z^k,$$

with $a_k \neq 0$, then consideration of the leading degree gives the necessary condition

$$\lambda^2 = (-\omega)^k \lambda + \omega^{2k},$$

i.e., $\lambda = \omega^k((-1)^k \pm \sqrt{5})/2$, which are just

$$\lambda = \begin{cases} 
\omega^{k-1}, & \text{if } k \text{ is even}; \\
\omega^{k+1}, & \text{if } k \text{ is odd}.
\end{cases}$$

We see that, given $\lambda$ satisfying (7.4) and a nonzero but otherwise arbitrary $a_k$, there is a unique polynomial solution of degree $k$ with coefficients given by the recurrence relations

$$\lambda a_i(\lambda - (-\omega)^i) = \sum_{\ell=i}^{k} \binom{\ell}{i} \omega^{\ell+i} a_\ell, \quad i = k - 1, \ldots, 0,$$

i.e.

$$a_i(\lambda^2 - (-\omega)^i \lambda - \omega^{2i}) = \sum_{\ell=i+1}^{k} \binom{\ell}{i} \omega^{\ell+i} a_\ell, \quad i = k - 1, \ldots, 0.$$
The full list of possible eigenvalues corresponding to polynomial eigenfunctions is thus
\[ \omega^{-1}, -1, \pm \omega, \pm \omega^2, \pm \omega^3, \ldots \] (7.8)
and these are nondegenerate within the space of polynomials.

Remark. Note that \( \lambda = 1 \) is not possible, so that there do not exist nonzero polynomial fixed points of our original additive problem (4.5).

### 7.2 Nonpolynomial solutions

We now construct some nonpolynomial solutions of equation (7.2) with real \( \lambda \) with \( |\lambda| > 1 \). The solutions are similar in structure to the solution (4.3) of theorem 1, although our method of proof will be more direct.

We shall see in the next sections that the solutions we construct will correspond to eigenfunctions of the derivative of the operator \( R \) at its fixed point with eigenvalues
\[ \lambda = \pm \omega^{-1}, \pm \omega^{-3}, \ldots \] (7.9)

In what follows let \( (1 - z)^\sigma \) be given by \( \exp(\sigma \log(1 - z)) \), where \( \log \) is the principal branch of the logarithm function.

#### 7.2.1 Solutions with \( |\lambda| > \omega^{-1} \)

We first prove the following theorem.

**Theorem 4.** Let \( \lambda \in \mathbb{R} \) with \( |\lambda| > \omega^{-1} \), let
\[ \sigma = \frac{\log |\lambda|}{\log \omega}, \] (7.10)
and let \( 1 < r < \omega^{-1} \). Then there exists a solution \( T \) of (7.2) given by
\[ T(z) = (1 - z)^\sigma + S(z), \] (7.11)
where \( S \) is analytic on \( \Delta(0, r) \) and is given by
\[ S(z) = \sum_{k=1}^{\infty} \sum_{t_1, \ldots, t_k} \frac{1}{\lambda^{2t_j}} (1 - \phi_{i_1} \circ \cdots \circ \phi_{i_k}(z))^\sigma. \] (7.12)

Of interest in the following sections will be eigenfunctions possessing a pole at \( z = 1 \). Condition (7.10) then shows that
\[ \lambda = \pm \omega^{-2}, \pm \omega^{-3}, \ldots \] (7.13)
are eigenvalues corresponding eigenfunctions with a pole at \( z = 1 \).

**Proof.** Since \( r > 1 \) we have that (5.3) holds. Furthermore we have for \( z \in \Delta(0, r) \) that
\[ |1 - \phi_1(z)| = |1 + \omega z| \geq 1 - |\omega z| \geq 1 - \omega r > 0, \] (7.14)
since \( r < \omega^{-1} \). We thus have a lower bound for \( |1 - \phi_1(z)| \) on \( \Delta(0, r) \).

We wish to demonstrate the uniform convergence of the function \( S \) on \( \Delta(0, r) \). We define
\[ S_n(z) = \sum_{k=1}^{n} \sum_{t_1, \ldots, t_k} \frac{1}{\lambda^{2t_j}} (1 - \phi_{i_1} \circ \cdots \circ \phi_{i_k}(z))^\sigma, \] (7.15)
and
\[ T_n(z) = (1 - z)^\sigma + S_n(z). \] (7.16)

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Now
\[
\left| \sum_{i_1, \ldots, i_k} \frac{1}{\lambda^{\Sigma i_j}} (1 - \phi_{i_1} \circ \cdots \circ \phi_{i_k}(z))^\sigma \right| \leq \sum_{i_1, \ldots, i_k} \frac{1}{|\lambda|^{\Sigma i_j}} |1 - \phi_{i_1} \circ \cdots \circ \phi_{i_k}(z)|^\sigma \\
\leq \frac{1}{|\lambda|} \sum_{i_2, \ldots, i_k} \frac{1}{|\lambda|^z} |1 - \phi_{i_1}(z)|^\sigma,
\]
(7.17)
where
\[
z_1 = \phi_{i_2} \circ \cdots \circ \phi_{i_k}(z) \in \Delta(0, r)
\]
in view of (5.3). Now, since $|\lambda| > 1$, we have $\sigma < 0$ and so
\[
|1 - \phi_{i_1}(z)|^\sigma \leq (1 - \omega r)^\sigma,
\]
(7.19)
and hence
\[
\left| \sum_{i_1, \ldots, i_k} \frac{1}{\lambda^{\Sigma i_j}} (1 - \phi_{i_1} \circ \cdots \circ \phi_{i_k}(z))^\sigma \right| \leq \frac{(1 - \omega r)^\sigma}{|\lambda|} \sum_{i_2, \ldots, i_k} \frac{1}{|\lambda|^z} |1 - \phi_{i_1}(z)|^\sigma,
\]
(7.20)
where $\ell$ and $m$ are the number of occurrences of the digits 1 and 2 in the list $i_2, \ldots, i_k$.

Since we are summing over all possible choices of $i_2, \ldots, i_k \in \{1, 2\}$ we have
\[
\sum_{i_2, \ldots, i_k} \frac{1}{|\lambda|^z} = \sum_{i=0}^{k-1} \binom{k-1}{i} \left\lfloor \frac{1}{|\lambda|^z} \right\rfloor = \left( \frac{1}{|\lambda|} + \frac{1}{|\lambda|^2} \right)^{k-1},
\]
(7.21)
and thus for $k \geq 2$
\[
\left| \sum_{i_1, \ldots, i_k} \frac{1}{\lambda^{\Sigma i_j}} (1 - \phi_{i_1} \circ \cdots \circ \phi_{i_k}(z))^\sigma \right| \leq \frac{(1 - \omega r)^\sigma}{|\lambda|} \left( \frac{1}{|\lambda|} + \frac{1}{|\lambda|^2} \right)^{k-1}.
\]
(7.22)

Now, since $|\lambda| > \omega^{-1}$, we have $1/|\lambda| + 1/|\lambda|^2 < 1$, and thus the bound in (7.22) is a convergent geometric progression, and hence, by the Weierstrass M-test, we have that the series $S(z) = \lim S_n(z)$ converges uniformly on $\Delta(0, r)$ to an analytic function.

We now show that $T = \lim T_n$ satisfies (7.2). We note that
\[
(1 - \phi_2(z))^\sigma = \omega^{2\sigma}(1 - z)^\sigma = \lambda^2(1 - z)^\sigma.
\]
(7.23)

Now
\[
\lambda^{-1} T_n(\phi_1(z)) + \lambda^{-2} T_n(\phi_2(z)) = \lambda^{-1} (1 - \phi_1(z))^\sigma
\]
\[
+ \lambda^{-1} \sum_{k=1}^n \sum_{i_1, \ldots, i_k} \frac{1}{\lambda^{\Sigma i_j}} (1 - \phi_{i_1} \circ \cdots \circ \phi_{i_k} \circ \phi_1(z))^\sigma
\]
\[
+ \lambda^{-2} (1 - \phi_2(z))^\sigma
\]
\[
+ \lambda^{-2} \sum_{k=1}^n \sum_{i_1, \ldots, i_k} \frac{1}{\lambda^{\Sigma i_j}} (1 - \phi_{i_1} \circ \cdots \circ \phi_{i_k} \circ \phi_2(z))^\sigma
\]
\[
= (1 - z)^\sigma + \lambda^{-1} (1 - \phi_1(z))^\sigma
\]
\[
+ \sum_{k=2}^n \sum_{i_1, \ldots, i_k} \frac{1}{\lambda^{\Sigma i_j}} (1 - \phi_{i_1} \circ \cdots \circ \phi_{i_k}(z))^\sigma
\]
\[
= T_{n+1}(z).
\]
(7.24)
Taking the limit $n \to \infty$ we have that
\[
T(z) = \lim_{n \to \infty} T_{n+1}(z) = \lim_{n \to \infty} \lambda^{-1}T_n(\phi_1(z)) + \lim_{n \to \infty} \lambda^{-2}T_n(\phi_2(z))
= \lambda^{-1}T(\phi_1(z)) + \lambda^{-2}T(\phi_2(z)) \tag{7.25}
\]
as required.

7.2.2 The case $1 < |\lambda| \leq \omega^{-1}$

We now prove a weaker version of this result for $1 < |\lambda| \leq \omega^{-1}$. We shall use this weaker result to construct eigenfunctions in the case $\lambda = -\omega^{-1}$ and a generalised eigenfunction in the case $\lambda = \omega^{-1}$.

**Theorem 5.** Let $\lambda \in \mathbb{R}$ with $1 < |\lambda| \leq \omega^{-1}$, let
\[
\sigma = \frac{\log |\lambda|}{\log \omega},
\tag{7.26}
\]
and let $1 < r < \omega^{-1}$ and $z_0 \in \Delta(0, r) \setminus \{1\}$. Then there exists a solution $\tilde{T}$ of the equation
\[
\tilde{T}(z) = \lambda^{-1}(\tilde{T}(\phi_1(z)) - \tilde{T}(\phi_1(z_0))) + \lambda^{-2}(\tilde{T}(\phi_2(z)) - \tilde{T}(\phi_2(z_0))).
\tag{7.27}
\]
This solution is given by
\[
\tilde{T}(z) = (1 - z)^\sigma - (1 - z_0)^\sigma + \tilde{S}(z),
\tag{7.28}
\]
where $\tilde{S}$ is analytic on $\Delta(0, r)$ and is given by
\[
\tilde{S}(z) = \sum_{k=1}^{\infty} \sum_{i_1, \ldots, i_k=1}^\infty \frac{1}{\lambda^{2i_1}} ((1 - \phi_{i_1} \circ \cdots \circ \phi_{i_k}(z))^\sigma - (1 - \phi_{i_1} \circ \cdots \circ \phi_{i_k}(z_0))^\sigma).
\tag{7.29}
\]
Moreover, if $\lambda \neq \omega^{-1}$ then there exists $K \in \mathbb{R}$ such that
\[
T(z) = \tilde{T}(z) - K
\tag{7.30}
\]
satisfies (7.2).

**Proof.** Setting $\lambda_1 = -\lambda/\omega$ and applying theorem 4 we obtain a solution
\[
T_1(z) = \sigma(1 - z)^{\sigma_1} + \sigma S_1(z)
\tag{7.31}
\]
of the equation
\[
T_1(z) = \lambda_1^{-1}T_1(\phi_1(z)) + \lambda_1^{-2}T_1(\phi_2(z)),
\tag{7.32}
\]
where
\[
\sigma_1 = \frac{\log |\lambda_1|}{\log \omega} = \sigma - 1,
\tag{7.33}
\]
and where
\[
S_1(z) = \sum_{k=1}^{\infty} \sum_{i_1, \ldots, i_k=1}^\infty \frac{1}{\lambda_1^{2i_1}} ((1 - \phi_{i_1} \circ \cdots \circ \phi_{i_k}(z))^{\sigma_1} - (1 - \phi_{i_1} \circ \cdots \circ \phi_{i_k}(z_0))^{\sigma_1})
\tag{7.34}
\]
is analytic on $\Delta(0, r)$. Moreover, the series for $S_1(z)$ converges uniformly on $\Delta(0, r) \supseteq \Delta(0, 1)$. (Note that we have introduced the constant $\sigma$ in (7.31) merely for convenience in what follows.)

Integrating on a contour from $z_0$ to $z$ in $\Delta(0, r)$ we may define
\[
\tilde{S}(z) = \sum_{k=1}^{\infty} \sum_{i_1, \ldots, i_k=1}^\infty \frac{1}{\lambda^{2i_1}} ((1 - \phi_{i_1} \circ \cdots \circ \phi_{i_k}(z))^\sigma - (1 - \phi_{i_1} \circ \cdots \circ \phi_{i_k}(z_0))^\sigma),
\tag{7.35}
\]
since the series converges uniformly on $\Delta(0, r)$. 

For $z \in \Delta(0,1)$ we may define
\[
\tilde{T}(z) = \int_{z_0}^z T_1(w)dw = \int_{z_0}^z \sigma(1-w)^{\gamma-1}dw + \tilde{S}(z) = (1-z)^\gamma - (1-z_0)^\gamma + \tilde{S}(z),
\] (7.36)
and, in view of (7.32), we have
\[
\tilde{T}(z) = \lambda_1^{-1}(-\omega^{-1}) \left( \tilde{T}(\phi_1(z)) - \tilde{T}(\phi_1(z_0)) \right) + \lambda_1^{-2} \omega^{-2} \left( \tilde{T}(\phi_2(z)) - \tilde{T}(\phi_2(z_0)) \right)
\]
\[
= \lambda^{-1} \left( \tilde{T}(\phi_1(z)) - \tilde{T}(\phi_1(z_0)) \right) + \lambda^{-2} \left( \tilde{T}(\phi_2(z)) - \tilde{T}(\phi_2(z_0)) \right),
\] (7.37)
as required.

Now if $\lambda \neq \omega^{-1}$ we set
\[
T(z) = \tilde{T}(z) - K,
\] (7.38)
for a constant $K$ to be determined. For $z \in \Delta(0,1)$ we have
\[
T(z) = \lambda^{-1}T(\phi_1(z)) + \lambda^{-2}T(\phi_2(z)) + \lambda^{-1}K + \lambda^{-2}K - K + C,
\] (7.39)
where
\[
C = -\lambda^{-1}\tilde{T}(\phi_1(z_0)) - \lambda^{-2}\tilde{T}(\phi_2(z_0)),
\] (7.40)
and so we choose
\[
K = \frac{C\lambda^2}{\lambda^2 - \lambda - 1}
\] (7.41)
thereby ensuring that $T$ satisfies (7.2).

8 Spectrum of the derivative of $\mathcal{R}$ at the fixed point

To analyse the spectrum of the renormalisation operator we write, as in [6], the original iteration (1.4) as an iteration for a pair of functions $(t_{n-1}(-\omega z), t_n(z)) = (u(z), t(z))$, say, thereby defining the renomalisation operator (2.15)
\[
\mathcal{R} : (u(z), t(z)) \mapsto (t(\theta_1(z)), t(\theta_1(z))u(\theta_2(z))),
\] (8.1)
where
\[
\theta_1(z) = -\omega z
\] (8.2)
\[
\theta_2(z) = -\omega z - 1.
\] (8.3)

We introduce the functions $\theta_1, \theta_2$ here for convenience only. They are related simply to the functions $\phi_1, \phi_2$ by $\phi_1 = \theta_1, \phi_2 = \theta_1 \circ \theta_2$.

We note that any solution of (4.1) gives a fixed point of $\mathcal{R}$ by setting $u(z) = t(\theta_1(z))$, and, vice versa, any fixed point of $\mathcal{R}$ gives, on eliminating $u$ using the fixed point equations, a solution of (4.1).

In [6], Ketoja and Satija made a conjecture for the spectrum of the derivative of $\mathcal{R}$ at the strong coupling fixed point. Theorem 3 is a more general rigorous statement which establishes their conjecture. In the next section we prove this theorem, after having set the scene in this section.

Let $n \in \mathbb{N}$ and let $t$ be the solution of (4.1) given by theorem 1. We set $u(z) = t(\theta_1(z))$. Then $(u, t)$ is a fixed point of $\mathcal{R}$. Formally the derivative of $\mathcal{R}$ at $(u, t)$ is the linear operator
\[
L = d\mathcal{R}_{(u, t)}
\] (8.4)
acting on tangent vectors $(\delta u, \delta t)$ given by
\[
L(\delta u, \delta t) = (\delta t(\theta_1(z)), \delta t(\theta_1(z))u(\theta_2(z)) + t(\theta_1(z))\delta u(\theta_2(z))).
\] (8.5)
To make this rigorous we specify a Banach space on which \( \mathcal{R} \) acts and which contains the fixed point \((u, t)\). \( L \) then also acts on this Banach space.

Although \( u \) and \( t \) are entire, it is convenient to work in a larger space of functions. Let us now fix \( 1 < r < \omega^{-1} \) and \( \omega r + 1 < r' < r\omega^{-1} \). (Note that there does exist such an \( r' \) since \( r > 1 \).) For a function \( f \) analytic on the disc \( \Delta(0, \tau) \), \( \tau > 0 \) we write

\[
\|f\|_{r, \infty} = \sup_{z \in \Delta(0, \tau)} |f(z)|
\]

for the supremum norm on \( \Delta(0, \tau) \). Let \( B \) denote the real Banach space of real analytic functions \( f \) on \( \Delta(0, r) \) with \( \|f\|_{r, \infty} < \infty \). Likewise, let \( B' \) denote the real Banach space of real analytic functions \( f \) on \( \Delta(0, r') \) with \( \|f\|_{r', \infty} < \infty \). We now define \( F \) to be the real Banach space \( B' \times B \) of pairs \((f, g)\) with \( f \in B' \), \( g \in B \), equipped with the norm

\[
\|(f, g)\| = \|f\|_{r', \infty} + \|g\|_{r, \infty}.
\]

Observe that if \( z \in \Delta(0, r') \) then

\[
|\theta_1(z)| = |z - \omega z| = |z| < \omega r' < r,
\]

and if \( z \in \Delta(0, r) \) then

\[
|\theta_1(z)| = |z - \omega z| = |z| < \omega r < r,
\]

\[
|\theta_2(z)| \leq |z| + 1 < \omega r + 1 < r'.
\]

Let us set \( \eta = \min\{\omega r', \omega r\} \) and \( \eta' = \omega r + 1 \). We now choose \( \rho \) and \( \rho' \) so that \( \eta < \rho < r \) and \( \eta' < \rho' < r' \). Then we have that

\[
\theta_1(\Delta(0, r')) \subseteq \Delta(0, \eta) \subseteq \Delta(0, \rho) \subseteq \Delta(0, r)
\]

\[
\theta_1(\Delta(0, r)) \subseteq \Delta(0, \eta) \subseteq \Delta(0, \rho) \subseteq \Delta(0, r)
\]

\[
\theta_2(\Delta(0, r)) \subseteq \Delta(0, \eta') \subseteq \Delta(0, \rho') \subseteq \Delta(0, r').
\]

The operator \( \mathcal{R} \) is then well defined on \( F \) and \( \mathcal{R}(F) \subseteq F \). For if \( z \in \Delta(0, r') \) then \( \theta_1(z) \in \Delta(0, r) \) and \( t(\theta_1(z)) \) is well defined. Likewise, if \( z \in \Delta(0, r) \) then \( \theta_1(z) \in \Delta(0, r) \) and \( \theta_2(z) \in \Delta(0, r') \) and \( t(\theta_1(z))u(\theta_2(z)) \) is well defined. Furthermore

\[
\|(t(\theta_1(z)), t(\theta_1(z))u(\theta_2(z)))\| \leq \|t\|_{r, \infty} + \|t\|_{r', \infty} \|u\|_{r', \infty} < \infty.
\]

By standard results on composition operators, we also have that \( \mathcal{R} \) is \( C^1 \) on \( F \) and its derivative is given by \( L \) as in equation (8.5) above.

We note that since the solution to equation (4.1), \( t \), is entire we have \( t \in B \) and \( u \in B' \). In particular \( \|t\|_{r, \infty} < \infty \) and \( \|u\|_{r', \infty} < \infty \).

### 9 Proof of Theorem 3

#### 9.1 Compactness

The compactness of \( L \) follows from the analyticity improving nature of the operator \( \mathcal{R} \), i.e., from relations (8.11)–(8.13). These conditions enable us to use Cauchy estimates to provide uniform bounds on derivatives and hence obtain equicontinuity. The compactness of \( L \) then follows from Ascoli’s theorem. Let us be more precise.

Recall that a linear operator \( L \) on \( F \) is compact if every bounded sequence is mapped to a sequence having a convergent subsequence. So, let \( \kappa > 0 \) and \((\delta u_n, \delta t_n) \in F \) be a sequence of tangent vectors in \( F \) with \( \|\delta u_n, \delta t_n\| \leq \kappa \) for all \( n \geq 1 \). Then \( \|\delta u_n\|_{r', \infty} \leq \kappa \) and \( \|\delta t_n\|_{r, \infty} \leq \kappa \). Let

\[
(\delta v_n, \delta u_n) = L(\delta u_n, \delta t_n) \in F.
\]

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We wish to show that $({\delta v}_n, {\delta w}_n)$ has a convergent subsequence. Now ${{\delta v}}_n(z) = {\delta t}_n(\theta_1(z))$, so, for $z \in \Delta(0, r')$, integrating around the circle $C(0, \rho)$ in $\Delta(0, r)$, we have

$$|{\delta v}_n'(z)| = \omega |{\delta t}_n'(\theta_1(z))| = \omega \left| \frac{1}{2{\pi i}} \int_{C(0, \rho)} \frac{{\delta t}_n(\xi)d\xi}{(\xi - \theta_1(z))^2} \right| \leq \frac{\omega \rho}{(\rho - \eta)^2} |{\delta t}_n|_{r, \infty} \leq \frac{\omega \rho \kappa}{(\rho - \eta)^2} = L_1,$$  

say. Thus

$$|{\delta v}_n'|_{r', \infty} \leq L_1. \quad (9.2)$$

The estimate for $|{\delta w}_n'|_{r, \infty}$ is similar but more complicated. Let $z \in \Delta(0, r)$. Then

$$|{\delta w}_n'(z)| \leq \omega (|{\delta t}_n'(\theta_1(z))u(\theta_2(z))| + |{\delta t}_n(\theta_1(z))u'(\theta_2(z))|) + |t'(\theta_1(z))\delta u_n(\theta_2(z))| + |t(\theta_1(z))\delta u_n'(\theta_2(z))|. \quad (9.4)$$

We may bound each of these terms using Cauchy estimates. Indeed

$$|{\delta t}_n'(\theta_1(z))u(\theta_2(z))| \leq \left| \frac{1}{2{\pi i}} \int_{C(0, \rho)} \frac{{\delta t}_n(\xi)d\xi}{(\xi - \theta_1(z))^2} \right| |u(\theta_2(z))| \leq \frac{\rho}{(\rho - \eta)^2} |{\delta t}_n|_{r, \infty} \|u\|_{r', \infty}. \quad (9.5)$$

Similarly we have

$$|{\delta t}_n(\theta_1(z))u'(\theta_2(z))| \leq \frac{\rho'}{(\rho' - \eta')^2} |{\delta t}_n|_{r, \infty} \|u\|_{r', \infty} \quad (9.6)$$

$$|t'(\theta_1(z))\delta u_n(\theta_2(z))| \leq \frac{\rho}{(\rho - \eta)^2} \|t\|_{r, \infty} \|\delta u_n\|_{r', \infty} \quad (9.7)$$

$$|t(\theta_1(z))\delta u_n'(\theta_2(z))| \leq \frac{\rho'}{(\rho' - \eta')^2} \|t\|_{r, \infty} \|\delta u_n\|_{r', \infty}. \quad (9.8)$$

Putting these together we obtain

$$|{\delta v}_n'(z)| \leq \omega \left( \frac{\rho}{(\rho - \eta)^2} + \frac{\rho'}{(\rho' - \eta')^2} \right) \left( |{\delta t}_n|_{r, \infty} \|u\|_{r', \infty} + |t|_{r, \infty} \|\delta u_n\|_{r', \infty} \right) \leq L_2, \quad (9.9)$$

say, using $\|\delta u_n\|_{r', \infty} \leq \kappa$ and $|{\delta t}_n|_{r, \infty} \leq \kappa$. Thus

$$|{\delta w}_n'|_{r, \infty} \leq L_2. \quad (9.10)$$

Now let $z, z' \in \Delta(0, r')$. Then

$$|{\delta v}_n(z) - {\delta v}_n(z')| \leq |{\delta v}_n'(z)||z - z'| \leq L_1|z - z'|, \quad (9.11)$$

and so the sequence $\delta v_n$ is equicontinuous and bounded. It follows from Ascoli’s theorem that the sequence $\delta v_n$ has a convergent subsequence. Restricting to this subsequence, we now have, for $z, z' \in \Delta(0, r')$,

$$|{\delta v}_n(z) - {\delta v}_n(z')| \leq L_1|z - z'|, \quad (9.11)$$

and we may again invoke Ascoli’s theorem to obtain a subsequence for which both $\delta v_n$ and $\delta w_n$ converge. It follows that this subsequence of the sequence of pairs $(\delta v_n, \delta w_n) \in F$ converges and thus $L$ is compact.

### 9.2 Spectrum

From the compactness of $L$ it follows that the spectrum of $L$ consists of 0 together with discrete eigenvalues. Our study of the eigenvalues consists of the following steps.

Firstly we construct eigenfunctions and generalised eigenfunctions of $L$ using the results on the associated eigenproblem in section 7. Indeed, for $\lambda = \omega^{-1}, -1, \pm \omega, \pm \omega^2, \pm \omega^3, \ldots$, we use subsection 7.1 to construct
polynomial solutions, and, for \( \lambda = \pm \omega^{-2}, \ldots, \pm \omega^{-n} \), we may use theorem 4 of subsection 7.2 to obtain eigenfunctions. For \( \lambda = -\omega^{-1} \) we obtain an eigenfunction from theorem 5 of subsection 7.2, but for \( \lambda = \omega^{-1} \) we only obtain a generalised eigenfunction from theorem 5.

We then show that these values of \( \lambda \) are the only possible eigenvalues and that the eigenfunctions are unique (up to a multiple). Finally we show that, except when \( \lambda = -\omega^{-1} \), there are no generalised eigenfunctions.

Let \( (\delta u, \delta t) \in F \), which we may identify with the tangent space of \( F \) at \((u, t)\). We write

\[
\Delta T = \delta t/t.
\]  

Then \( \Delta T \) is analytic on the same domain as \( \delta t \), except at the zeros of \( t \), where it may have poles. In fact \( \Delta T \) is analytic on \( \Delta(0, r) \) except possibly at \( z = 1 \) which may be a pole of order at most \( n \). (Recall that \( t \) has a unique zero of order \( n \) at \( z = 1 \) in \( \Delta(0, r) \) for \( 1 < r < \omega^{-1} \).)

We now divide (9.16) by \( t(z) \), using the fixed point equation (4.1) to obtain

\[
\Delta T(z) = \lambda^{-1} \Delta T(\phi_1(z)) + \lambda^{-2} \Delta T(\phi_2(z)),
\]  

where \( \Delta T \) is a function analytic on \( \Delta(0, r) \) except for possibly a pole at \( z = 1 \). We recognise this as equation (7.2) which we analysed in section 7.

We note that any solution of (9.18) analytic on \( \Delta(0, r) \) except possibly for a pole of order \( k \leq n \) gives rise to an eigenfunction of \( L \) by setting

\[
\delta t(z) = t(z) \Delta T(z), \quad \delta u(z) = \lambda^{-1} \delta t(\theta_1(z)).
\]  

### 9.2.1 Construction of eigenfunctions

In subsection 7.1 we constructed solutions of (9.18) which were polynomials for \( \lambda = \omega^{-1}, -1, \pm \omega, \pm \omega^2, \pm \omega^3, \ldots \). Applying theorem 4 from subsection 7.2 for the cases \( \lambda = \pm \omega^{-k} \) for \( 2 \leq k \leq n \) we obtain solutions of (9.18) with a pole of order \( k \) at \( z = 1 \). Similarly, applying theorem 5 with \( \lambda = -\omega^{-1} \), we obtain a solution of (9.18) with a simple pole at \( z = 1 \). When \( \lambda = \omega^{-1} \) we note that theorem 5 does not give a solution of (9.18), but rather of (7.27) instead. Indeed there does not exist a solution of (9.18) that is analytic in \( \Delta(0, r) \) except for a simple pole at \( z = 1 \), as we now show.

Suppose \( \Delta T \) is a fixed point of the linear operator \( R_1 \) defined by

\[
R_1(f)(z) = \omega f(\phi_1(z)) + \omega^2 f(\phi_2(z)),
\]  

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with
\[ \Delta T(z) \sim \frac{A}{1 - z}, \]

as \( z \to 1 \), where \( A \neq 0 \). We define
\[ \Delta S(z) = \Delta T(z) - \frac{A}{1 - z}. \]

Then \( \Delta S \) is analytic on \( \Delta(0, r) \) with a removable singularity at \( z = 1 \). We have
\[
R_1(\Delta S)(z) - \Delta S(z) = R_1(\Delta T)(z) - \frac{\omega A}{1 - \phi_1(z)} - \frac{\omega^2 A}{1 - \phi_2(z)} - \Delta T(z) - \frac{A}{1 - z} = \frac{\omega A}{1 - \phi_1(z)},
\]
since \( 1 - \phi_2(z) = \omega^2(1 - z) \), and by iterating it follows that
\[
R^n_1(\Delta S)(z) - \Delta S(z) = A \sum_{k=1}^{n} \sum_{i_1, \ldots, i_k} \omega^{\Sigma j_i} (1 - \phi_{i_1} \circ \cdots \circ \phi_{i_k}(z))^{-1}.
\]

Now, as \( \Delta S \) is analytic in \( \Delta(0, r) \), we have, for \( z \in \Delta(0, 1) \),
\[
|A| \sum_{k=1}^{n} \sum_{i_1, \ldots, i_k} |\omega^{\Sigma j_i} (1 - \phi_{i_1} \circ \cdots \circ \phi_{i_k}(z))^{-1}| = |R^n_1(\Delta S)(z) - \Delta S(z)| \leq 2 ||\Delta S||_{1,\infty} < \infty,
\]
where we have used the fact that
\[
||R_1(f)||_{1,\infty} \leq \omega ||f||_{1,\infty} + \omega^2 ||f||_{1,\infty} = ||f||_{1,\infty}.
\]

Now if \( y \in [-\omega, 1] \) we have \( 0 < \omega \leq 1 - \phi_1(y) \leq 1 + \omega = \omega^{-1} \), so that
\[
\sum_{k=1}^{n} \sum_{i_1, \ldots, i_k} \omega^{\Sigma j_i} (1 - \phi_{i_1} \circ \cdots \circ \phi_{i_k}(z))^{-1} \geq \sum_{k=0}^{n-1} (\omega + \omega^2)^k \omega = n\omega,
\]
which contradicts (9.25). We conclude that no eigenfunctions exist with a simple pole at \( z = 1 \) and with \( \lambda = \omega^{-1} \). Thus in (7.27) we cannot have \( \lambda^{-1} T_\omega(\phi_1(z_0)) + \lambda^{-2} T_\omega(\phi_2(z_0)) = 0 \).

Remark. Indeed the case \( \lambda = \omega^{-1} \) our eigenvalue problem is merely the invariant density problem for the map with inverse branches given by the two functions \( \phi_1, \phi_2 \). The constant function is the only solution in this case.

We shall see below in subsection 9.2.4 that in this case, although there are no eigenfunctions of this type, there is in fact such a generalised eigenfunction of \( L \).

Having constructed these (generalised) eigenfunctions we now show that there are no other eigenvalues.

9.2.2 There are no other eigenvalues

In this subsection we determine the possible eigenvalues of \( L \). Our analysis consists of considering the possible analytic behaviour of \( \Delta T \) on \( \Delta(0, r) \).

Let \( (\delta u, \delta t) \) be an eigenfunction of \( L \) with eigenvalue \( \lambda \neq 0 \) such that \( \delta t \) is analytic on \( \Delta(0, r) \), with \( r > 1 \) and thus \( \Delta T \) is analytic on \( \Delta(0, r) \) except (possibly) at \( z = 1 \) at which we may have at most a pole of order \( n \). Then, as we saw above, \( \Delta T \) satisfies equation (9.18).

Firstly, if \( \Delta T \) is analytic at \( z = 1 \) (or has a removable singularity there) then let \( \ell \geq 0 \) be the smallest integer for which
\[
\frac{\omega^\ell}{|\lambda|} + \frac{\omega^{2\ell}}{|\lambda|^2} < 1
\]
(such an \( \ell \) clearly exists).
Differentiating (9.18) \( \ell \) times we obtain
\[
\Delta T^{(\ell)}(z) = (-\omega)^{\ell} \lambda^{-1} \Delta T^{(\ell)}(\phi_1(z)) + \omega^{2\ell} \lambda^{-2} \Delta T^{(\ell)}(\phi_2(z)),
\]
so we have
\[
0 \leq \|\Delta T^{(\ell)}\|_{1,\infty} \leq \left( \frac{\omega^\ell}{|\lambda|} + \frac{\omega^{2\ell}}{|\lambda|^2} \right) \|\Delta T^{(\ell)}\|_{1,\infty}.
\]
Now, since \( \Delta T \), and hence \( \Delta T^{(\ell)} \), is analytic on \( \Delta(0, r) \), we have \( \|\Delta T^{(\ell)}\|_{1,\infty} < \infty \), and so using equation (9.28) we must have \( \|\Delta T^{(\ell)}\|_{1,\infty} = 0 \) and hence \( \Delta T^{(\ell)}(z) = 0 \) for all \( z \in \Delta(0, r) \) since it is analytic. Thus \( \Delta T \) is a polynomial of degree \( k \leq \ell - 1 \). Consideration of the coefficient of \( z^k \) in equation (9.18) gives the equation
\[
1 = (-\omega)^k \lambda^{-1} + \omega^{2k} \lambda^{-2},
\]
which has solutions
\[
\lambda = (-\omega)^k \omega^{-1}, \quad \lambda = (-\omega)^k (-\omega),
\]
and from the definition of \( \ell \) we see that \( k = \ell - 1 \) or \( k = \ell - 3 \). Indeed it is now clear that \( \Delta T \) must be a multiple of the polynomial eigenfunction constructed in subsection 7.1.

Next, if \( \Delta T \) has a pole of order \( k \) at \( z = 1 \), then \( \Delta T(\phi_1(z)) \) is analytic at \( z = 1 \) and so (9.18) shows that we must have
\[
1 = \lim_{z \to 1} \frac{\Delta T(z)}{\lambda^{-2} \Delta T(\phi_2(z))} = \lambda^2 \omega^{2k},
\]
and hence \( \lambda = \pm \omega^{-k} \).

### 9.2.3 Uniqueness of eigenfunctions

We now show that the (generalised) eigenfunctions are unique. We saw in section 9.2.2 that if \( \Delta T \) is analytic on \( \Delta(0, r) \) then the resulting polynomial eigenfunctions are unique. Now suppose \( \Delta T \) has a pole of order \( k \leq n \) at \( z = 1 \) and suppose that
\[
\Delta T(z) \sim \frac{K}{(1 - z)^k}
\]
as \( z \to 1 \), with \( K \neq 0 \).

Consider first the case \( k \geq 2 \), so that \( |\lambda| = |\omega^{-k}| > \omega^{-1} \), and let \( \Delta S \) be \( \Delta T \) with the eigenfunction constructed in theorem 4 multiplied by \( K \) subtracted. Then \( \Delta S \) satisfies (7.2) and does not have a pole of order \( k \) at \( z = 1 \).

If \( \Delta S \) has a pole of order \( \ell \leq k - 1 \) we have a contradiction with (9.33), and thus \( \Delta S \) is analytic on \( \Delta(0, r) \). We have
\[
0 \leq \|\Delta S\|_{1,\infty} \leq \left( \frac{1}{|\lambda|} + \frac{1}{|\lambda|^2} \right) \|\Delta S\|_{1,\infty} < \|\Delta S\|_{1,\infty},
\]
and, thus, for \( k \geq 2 \), we have \( \|\Delta S\|_{1,\infty} = 0 \) and thus \( \Delta S(z) = 0 \) for all \( z \in \Delta(0, r) \). Thus \( \Delta T \) is a multiple of the eigenfunction constructed by theorem 4.

We now consider the case when \( k = 1 \). Differentiating (9.18), we obtain
\[
0 \leq \|\Delta S^{(1)}\|_{1,\infty} \leq (\omega^2 + \lambda^4) \|\Delta S^{(1)}\|_{1,\infty} < \|\Delta S^{(1)}\|_{1,\infty} < \infty,
\]
thus \( \Delta S^{(1)}(z) = 0 \) for all \( z \in \Delta(0, 1) \) and so \( \Delta S \) is constant.

Furthermore, if \( \lambda = -\omega^{-1} \), we have that
\[
c = \Delta S(z) = -\omega \Delta S(\phi_1(z)) + \omega^2 \Delta S(\phi_2(z)) = (-\omega + \omega^2)c,
\]
which gives \( c = 0 \).

Now, if \( \lambda = \omega^{-1} \) we have already seen in subsection 9.2.1 that we do not have a solution of (7.2) with a simple pole at \( z = 1 \). However, from theorem 5 we do have a solution of (7.27). Let \( \Delta T \) be a solution of (7.27) analytic on \( \Delta(0, r) \) except for a simple pole at \( z = 1 \) with
\[
\Delta T(z) \sim \frac{K}{(1 - z)}
\]
as \( z \to 1 \), with \( K \neq 0 \). Let \( \Delta S \) be this solution with the solution constructed in theorem 5 multiplied by \( K \) subtracted. Then \( \Delta S \) has a removable singularity at \( z = 1 \), and satisfies

\[
\Delta S(z) = \omega \Delta S(\phi_1(z)) + \omega^2 \Delta S(\phi_2(z)).
\]  
\[ (9.39) \]

Differentiating, we obtain

\[
0 \leq \|\Delta S^{(1)}\|_{1,\infty} \leq (\omega^2 + \omega^4)\|\Delta S^{(1)}\|_{1,\infty} < \infty,
\]

and thus \( \Delta S \) is a constant. It follows that any solution of (9.39) is the solution constructed in section 7 up to a multiple and the addition of a constant.

### 9.2.4 Generalised eigenfunctions

We now consider the possibility of generalised eigenfunctions and show that (apart from one case) they do not exist. Suppose

\[
(L - \lambda I)(\delta u_1, \delta t_1) = (\delta u, \delta t),
\]

where \((\delta u, \delta t)\) is a nonzero eigenfunction satisfying

\[
L(\delta u, \delta t) = \lambda(\delta u, \delta t)
\]

with eigenvalue \( \lambda \neq 0 \). Then

\[
(L - \lambda I)(\delta u_1, \delta t_1)(z) = (\delta t_1(\theta_1(z)) - \lambda \delta u_1(z), \delta t_1(\theta_1(z)))u(\theta_2(z)) + t(\theta_1(z))\delta u_1(\theta_2(z)) - \lambda \delta t_1(z))
\]

\[
= (\delta u(z), \delta t(z))
\]

\[
= (\lambda^{-1} \delta t(\theta_1(z)), \lambda^{-1}(\delta t(\theta_1(z)))u(\theta_2(z)) + t(\theta_1(z))\delta u(\theta_2(z))).
\]

Writing

\[
\Delta U = \delta u/u, \quad \Delta T = \delta t/t, \quad \Delta U_1 = \delta u_1/u, \quad \Delta T_1 = \delta t_1/t,
\]

and using the fact that \((u, t)\) is a fixed point of \( \mathcal{R} \), we obtain

\[
(\Delta T_1(\theta_1(z)) - \lambda \Delta U_1(z), \Delta T_1(\theta_1(z)) + \Delta U_1(\theta_2(z)) - \lambda \Delta T_1(z))
\]

\[
= (\lambda^{-1} \Delta T(\theta_1(z)), \lambda^{-1}(\Delta T(\theta_1(z)) + \Delta U(\theta_2(z))).
\]

We now eliminate \( \Delta U \) and \( \Delta U_1 \) from these equations, using the first of these equations and the eigenvalue equation

\[
\Delta U(z) = \lambda^{-1} \Delta T(\theta_1(z)),
\]

\[ (9.46) \]

to obtain

\[
\lambda^{-1} \Delta T_1(\phi_1(z)) + \lambda^{-2} \Delta T_1(\phi_2(z)) = \Delta T_1(z) + \lambda^{-2} \Delta T(\phi_1(z)) + 2\lambda^{-3} \Delta T(\phi_2(z)).
\]

\[ (9.47) \]

As before, \( \Delta T \) can either be analytic on \( \Delta(0, r) \) or have a pole of order \( k \) at \( z = 1 \).

We first consider the case when \( \Delta T \) has a pole of order \( k \) at \( z = 1 \), so that \( \lambda^2 \omega^{2k} = 1 \). Then, in order to balance the last term on the right-hand side in (9.47), we must have that \( \Delta T_1 \) has a pole of order \( \ell \geq k \) at \( z = 1 \). If \( \ell < k \), then, dividing by \( \Delta T_1(z) \), we have

\[
1 < \frac{1}{\lambda^2 \omega^{2k}} = \lim_{z \to 1} \frac{\lambda^{-2} \Delta T_1(\phi_2(z))}{\Delta T_1(z)} = 1 + \lim_{z \to 1} \frac{2\lambda^{-3} \Delta T(\phi_2(z))}{\Delta T_1(z)} = 1,
\]

\[ (9.48) \]

a contradiction. (Here we have used the fact that the other terms are bounded as \( z \to 1 \).) Hence \( k = \ell \).

But then, taking the same limit in (9.47), we obtain

\[
1 = \frac{1}{\lambda^2 \omega^{2k}} = \lim_{z \to 1} \frac{\lambda^{-2} \Delta T_1(\phi_2(z))}{\Delta T_1(z)} = 1 + \frac{2}{\lambda^3 \omega^{2k}} \lim_{z \to 1} \frac{\Delta T(\phi_2(z))}{\Delta T_1(z)} \neq 1,
\]

\[ (9.49) \]
a contradiction. Hence no such pair \((\Delta U_1, \Delta T_1)\) exists, and so no generalised eigenfunctions of this type exist.

We now turn to the case when \(\Delta T\) is analytic on \(\Delta(0, r)\) for some \(r > 1\). Then we have that \(\Delta T\) is a polynomial of degree \(k\) where \(\lambda^2 = (-\omega)^k \lambda + \omega^{2k}\). (See equation (7.4).)

Now unless \(\lambda = \omega^{-1}\) and \(k = 0\) we see from (9.47) that \(\Delta T_1\) cannot have a pole at \(z = 1\) (for, as above, we need \(\lambda = \pm \omega^{-k}\) for a pole of order \(k\)). So, for \(\lambda \neq \omega^{-1}\), we have that \(\Delta T_1\) is analytic at \(z = 1\).

Differentiating equation (9.47) \(k - 1\) times we see that (using the norm argument used above) \(\Delta T_1^{(k-1)}(z) = 0\) for all \(z\), so that \(\Delta T_1\) is a polynomial of degree at most \(k\).

Let
\[
\Delta T_1(z) = b_0 + \cdots + b_k z^k, \tag{9.50}
\]
\[
\Delta T(z) = a_0 + \cdots + a_k z^k, \tag{9.51}
\]
where \(a_k \neq 0\). The leading term in (9.47) gives
\[
\lambda^{-1}(-\omega)^k b_k + \lambda^{-2} \omega^{2k} b_k = b_k + \lambda^{-2} (-\omega)^k a_k + 2\lambda^{-3} (-\omega)^{2k} a_k, \tag{9.52}
\]
which using equation (7.4) gives
\[
\lambda^{-2}(-\omega)^k + 2\lambda^{-3}(-\omega)^{2k} = 0. \tag{9.53}
\]
But equation (7.4) may be written
\[
\lambda^{-2}(-\omega)^k + \lambda^{-3}(-\omega)^{2k} = \lambda^{-1}, \tag{9.54}
\]
which is clearly a contradiction. Hence no generalised eigenfunctions exist in this case either.

Now if \(\lambda = \omega^{-1}\) and \(\Delta T\) is a polynomial, then \(\Delta T\) is a nonzero constant, and so (9.47) reduces to
\[
\omega \Delta T_1(\phi_1(z)) + \omega^2 \Delta T_1(\phi_2(z)) = \Delta T_1(z) + C, \tag{9.55}
\]
for \(C \neq 0\) a constant. Now suppose \(\Delta T_1\) is any solution of (9.55). If \(\Delta T_1\) is analytic at \(z = 1\) then (as above) \(\Delta T_1\) is a polynomial of degree \(k = 0\), i.e., a constant. It follows that \(C = 0\), a contradiction.

Otherwise \(\Delta T_1\) may have a simple pole at \(z = 1\). We have constructed such a solution in theorem 5, so in this case we have a generalised eigenfunction.

In general let
\[
\Delta T_1(z) \sim \frac{K}{(1 - z)}, \tag{9.56}
\]
with \(K \neq 0\) as \(z \to 1\). Then, subtracting off a multiple of the function constructed in theorem 5, we obtain a function \(\Delta S\) analytic on \(\Delta(0, r)\) satisfying (9.55) for some other constant \(C\), possibly zero. Differentiation gives \(\Delta S\) a constant (and indeed \(C = 0\) in this case) and thus \(\Delta T_1\) is a multiple of the solution constructed in theorem 5 plus a constant.

Thus we have shown that in the case \(\lambda = \omega^{-1}\) there is only one linearly independent generalised eigenfunction, which is that given by theorem 5.

The theorem is proved.

10 Connection with Abel and Schröder equations

It is interesting to note that the solutions of the multiplicative and additive equations (4.1) and (4.5) are intimately connected with the classical Schröder and Abel functional equations
\[
s(x) = a s(\phi_2(x)), \tag{10.1}
\]
and
\[
S(x) = S(\phi_2(x)) + A, \tag{10.2}
\]

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respectively. These equations have been extensively studied, and we refer the reader to the comprehensive book by Kuczma et al [8] for a review of the work done on these equations. Indeed the equations (4.1) and (4.5) studied in this paper fall under a class of functional equations briefly considered in chapter 6 of [8].

The connection is made as follows. Suppose $t$ satisfies (4.1) and $T$ satisfies (4.5), then setting

$$a = \lim_{x \to 1^-} \frac{t(\phi_2(x))}{t(x)}$$

$$A = \lim_{x \to 1^-} T(\phi_2(x)) - T(x),$$

and

$$s(x) = \lim_{n \to \infty} a^{-n} t(\phi_2^n(x))$$

$$S(x) = \lim_{n \to \infty} T(\phi_2^n(x)) - nA,$$

we have that $s$ and $S$ satisfy (10.1) and (10.2) respectively.

For the solutions $t_*$ and $T_*$ constructed in this paper we have $s(x) = 1 - x$ with $a = \omega^2$ and $S(x) = \log(1 - x)$ with $A = \log \omega^2$, which are easily seen to be solutions to (10.1) and (10.2) respectively. In fact, if we ignore the question of convergence, the solutions of (4.1) and (4.5) may be formally constructed from solutions $s$ and $S$ of (10.1) and (10.2) by setting

$$t(x) = \frac{s(x)}{s(\omega)} \prod_{k=1}^{\infty} \prod_{i_1, \ldots, i_k} \frac{s(\phi_{i_1} \circ \cdots \circ \phi_{i_k}(x))}{s(\phi_{i_1} \circ \cdots \circ \phi_{i_k}(\omega))},$$

and

$$T(x) = S(x) - S(\omega) + \sum_{k=1}^{\infty} \sum_{i_1, \ldots, i_k} S(\phi_{i_1} \circ \cdots \circ \phi_{i_k}(x)) - S(\phi_{i_1} \circ \cdots \circ \phi_{i_k}(\omega)).$$

11 Conclusions

We have rigorously examined the fixed point of a functional iteration (1.4) that explains the universality of the self-similar fluctuations in the the Harper equation in the strong coupling regime. Our results put the numerical results of Ketoja and Satija in [6] and [7] on a firm footing.

Of particular interest is the value of the universal scaling ratio $t(0)$, numerically calculated in [6] to be $0.172586410945$ with error bound one in the last digit. It remains to be seen whether we can evaluate our function $t_*$ at 0 (and hence $t(0)$) in closed form. Our solution does however provide a convergent series for it.

We anticipate that our analysis can be generalised to study the universal strange attractor found by Ketoja and Satija [6] in their analysis of the generalised Harper equation in which a next-nearest neighbour coupling is present.

In a forthcoming companion paper [10] we shall analyse piecewise constant periodic orbits of the recursion studied in this paper. Such orbits arise in a renormalisation analysis of the autocorrelation function of quasiperiodically forced systems. They have been witnessed in the analysis a strange nonchaotic attractor in a quasiperiodically forced system [2], and (in an additive version of the recursion) in that of a quasiperiodically forced two-level quantum system [3].

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References


