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SELF-TWISTING BLADES FOR PASSIVE REGULATION OF A SMALL WIND-TURBINE

by

Julian Feuchtwang

A Doctoral Thesis

submitted in partial fulfilment of the requirements for the award of
Doctor of Philosophy of Loughborough University

on the 30th September 1998

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ABSTRACT

It is widely recognised that for small stand-alone battery charging wind-turbines, below 10kW in rated power, the issue of regulating the rotor speed is one that has not been resolved definitively, especially with regard to the runaway condition. Current methods of regulation work, but are prone to problems. It is desirable to find methods of regulation with reduced mechanical complexity.

Building on earlier work at Reading University, a method of blade construction is examined which involves asymmetric lay-up of fibre-composite materials. This may be done in a manner which causes the blade to twist in response to centrifugal loads and thus the rotor speed of the turbine. There is a need to optimise the blade design to exploit this effect to the full. The equations describing the blade twist are developed and expressed simply and explicitly in terms of blade shape design parameters. Along with an established aerodynamic model, a system is developed for examining the expected steady state behaviour of a wide range of rotors of this type in order to find the most appropriate configuration.

A range of existing aerofoil profiles is also examined in order to select the most appropriate choice and a shortlist of these is modelled numerically in 2D negative incident flow using an established model.

In recognition of the inevitable flexibility of self-twisting designs of blade, a start is made on dynamic modelling of the rotor. A Rayleigh-Ritz approach is used in order to find both static blade deflections under loading and the blade’s natural modal frequencies and shapes.

A prototype design is developed, commissioned and tested under forced-rotation conditions to validate the twisting and bending models. The predicted modal frequencies are also compared with the results of vibration tests on the blades.

Recommendations are made for further improvements.
ACKNOWLEDGEMENTS

Many people have helped me in different ways in the course of my research and I am grateful to them all, but I would like to single out certain people without whom my work would not have been possible at all.

Dr. David Infield, my supervisor, has supported me tirelessly throughout and shown me enormous patience especially in the last few days. David Sharpe, who has also been supervising my work, has given me invaluable advice on the technical aspects of the project, particularly on aerodynamics and mechanical dynamics.

I would like to thank the late Mr. John Fawkes and Paul Fitches of Marlec Engineering Co. Ltd who have been closely involved with the project before me and who have provided me with financial support. Paul helped enormously during the laboratory testing of the blades and has continued to monitor the prototype turbine.

Professor George Jeronimidis has given me much advice on materials aspects of my work and his assistants at Reading, Axel Schmeer, Dr. Alison Hunt and Dr. Dan Davies have all contributed much to the project through their finite element modelling of the self-twisting blade. Axel also did most of the work on experimental validation of the work on blade twisting. George also determined the normal modes of vibration of the blades.

I would also like to thank Professor Vic Middleton for his work on the blade manufacture and our discussions on the hub design.

Finally I would like to thank my sister, Dr Nicola Feuchtwang and Shashi Patel just for being there.

This thesis is dedicated to my late mother and father, Edith and Bill Feuchtwang
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<tr>
<td>$a$</td>
<td>Axial induction factor (rotor aerodynamics)</td>
</tr>
<tr>
<td>$a'$</td>
<td>Tangential induction factor (rotor aerodynamics)</td>
</tr>
<tr>
<td>$A$</td>
<td>Enclosed area of blade section (in materials contexts) ($m^2$)</td>
</tr>
<tr>
<td>$A'$</td>
<td>Swept area of rotor (in aerodynamic contexts) ($m^2$)</td>
</tr>
<tr>
<td>$A''$</td>
<td>Non-dim. enclosed area of blade section (normalised with respect to chord squared)</td>
</tr>
<tr>
<td>$a_j$</td>
<td>Coefficient of $j$th shape function in Rayleigh-Ritz description of blade deflection</td>
</tr>
<tr>
<td>$a$</td>
<td>Column vector of Rayleigh-Ritz shape function coefficients</td>
</tr>
<tr>
<td>$A$</td>
<td>Matrix of Rayleigh-Ritz shape function coefficients for unit solutions</td>
</tr>
<tr>
<td>$c$</td>
<td>Chord width of blade (m)</td>
</tr>
<tr>
<td>$c'$</td>
<td>Chord taper parameter</td>
</tr>
<tr>
<td>$c_0$</td>
<td>Chord width at the blade root (m)</td>
</tr>
<tr>
<td>$c_1$</td>
<td>Chord width at the blade tip (m)</td>
</tr>
<tr>
<td>$c_o(x)$</td>
<td>Ideal chord distribution (m)</td>
</tr>
<tr>
<td>$c_{ref}$</td>
<td>Chord width at the reference point (see $x_{ref}$) (m)</td>
</tr>
<tr>
<td>$C_{D_{min}}$</td>
<td>Minimum drag coefficient of an aerofoil</td>
</tr>
<tr>
<td>$C_L$</td>
<td>Lift coefficient of an aerofoil</td>
</tr>
<tr>
<td>$C_{L'\alpha}$</td>
<td>Gradient of lift coefficient with respect to angle of attack ($\text{deg.}^{-1}$)</td>
</tr>
<tr>
<td>$C_{L_{max}}$</td>
<td>Maximum lift coefficient of an aerofoil</td>
</tr>
</tbody>
</table>
$C_p$  Power coefficient of a rotor

$C_Q$  Torque coefficient of a rotor

$C_T$  Thrust coefficient of a rotor

$D$  Drag force (N)

$dD$  Drag force density (N/m)

ds  A differential element on the blade section perimeter (m)

e  Column vector of polynomial coefficients for elastic stiffness distribution

$E_{11}, E_{22}$  Young's modulus of a material respectively parallel & perpendicular to the principal fibre direction (Pa, GPa)

$E_a$  Young's modulus of material referred to blade axis (Pa, GPa)

$EI_0$  The root bending stiffness of a blade section (Pa, MPa)

$(EI)_F, (EI)_L$  Flapwise and lead-lag bending stiffnesses of a blade (Nm², MNm²)

$(EI)_F^*, (EI)_L^*$  Reduced flapwise and lead-lag bending stiffnesses of a blade (Pa, MPa)

$(ESt)$  Tensile stiffness of a blade (N, MN)

$(ESt)^*$  Reduced tensile stiffness of a blade (Pa, MPa)

$E_{xx}, E_{yy}, G_{xy}, v_{xy}$  As above, but referred to axes at an angle $\theta$ to the principal fibre direction in the laminar plane. (Pa, GPa)

$F_{tip}$  Tensile load on a blade (section) (N, kN)

g  Column vector of polynomial coefficients for the centrifugal field distribution

$gg$  Column vector of polynomial coefficients for the accumulated centrifugal field distribution
$G$ Shear modulus of material referred to blade axes (Pa, GPa)

$G(x)$ Tension at $x$ in blade due to centrifugal load on its own mass

$G_{12}$ Shear modulus of a material parallel to the principal fibre direction in the laminar plane (Pa, GPa)

$G_{tip}(x)$ Tension at $x$ in blade due to centrifugal load on tip-mass

$(GJ)$ Torsional stiffness of a blade (Nm², MNm²)

$(GJ)^*$ Reduced torsional stiffness of a blade (Pa, MPa)

$h$ Non-dimensional hub radius

$[H]$ Stiffness matrix of a blade section (mixed units)

$[H']$ Reduced stiffness matrix of a blade section (Pa, GPa)

$H_E$ Non-dimensional elastic matrix in Rayleigh-Ritz calculation

$H_G$ Non-dimensional centrifugal-field matrix in Rayleigh-Ritz calculation

$H_M$ Non-dimensional mass matrix in Rayleigh-Ritz calculation

$h_Q$ Non-dimensional external work vector in Rayleigh-Ritz calculation

$H_Q$ Non-dimensional external work matrix for unit solution in Rayleigh-Ritz calculation

$I_{az}$ Non-dim. 2nd moment of area of the enclosed area of a blade section

$I_{sz}$ Non-dim. 2nd moment of area of the perimeter of a blade section

$I_{yy}$ Non-dim. 2nd moment of area of a blade section in flapping, lead-lag bending

$k$ The bending stiffness of a blade (N/m, kN/m)
\( k_0 \)  
'Basic' (elastic) stiffness of a blade \((\text{N/m, kN/m})\)

\( k_G \)  
Centrifugal stiffness of a blade \((\text{N/m, kN/m})\)

\( k_0, k_1, k_2 \)  
Coefficients for quadratic fit to drag curve (in spin-test aerodynamics)

\( k_d \)  
Curvature of drag curve \((\text{deg}^{-2})\)

\( k_{tip}, k_{base} \)  
Twist factors for calculating twist distribution due to tip-mass

\( K_{Vbl} \)  
Blade twist constant for twist due to skin mass \((\text{s}^2)\)

\( K_{Vtip} \)  
Blade twist constant for twist due to tip-mass \((\text{s}^2)\)

\( K_W \)  
Blade untwist constant due to propeller effect \((\text{s}^2)\)

\( k_{wtip} \)  
Untwist factor for propeller effect on tip-mass

\( L \)  
Lagrangian \((\text{J})\)

\( L \)  
Blade length \((\text{m})\)

\( L \)  
Aerodynamic Lift force \((\text{N})\)

\( dL \)  
Lift force density \((\text{N/m})\)

\( M \)  
Bending moment on a blade (section) \((\text{Nm, kNm})\)

\( M_E(x) \)  
Bending moment distribution acting on blade elasticity \((\text{Nm, kNm})\)

\( M_G(x) \)  
Bending moment distribution on a blade due to centrifugal load \((\text{Nm, kNm})\)

\( M_Q(x) \)  
Bending moment distribution on a blade due to external load \((\text{Nm, kNm})\)

\( m_{\text{skin}} \)  
Mass of blade body \((\text{kg})\)

\( m_{tip} \)  
Mass of tip-mass \((\text{kg})\)

\( m \)  
Column vector of polynomial coefficients for the blade mass distribution
Non-dimensional centrifugal-field bending moment matrix for centrifugal-field in Rayleigh-Ritz calculation

Non-dimensional bending moment matrix for external load in Rayleigh-Ritz calculation

\( n_b \)  Number of blades

\( N_{rot} \)  Rotational speed of a rotor (rpm)

\( P \)  Power generated by a rotor (W, kW)

\( P \)  Load vector for a blade section (mixed units, N and N-m)

\( P(x) \)  Vector of successive powers of \( x \)

\( q \)  A bending load density on a blade (N/m, kN/m)

\( q \)  Column vector of polynomial coefficients for the bending load distribution

\( Q \)  A bending load on a blade (N, kN)

\( \dot{Q} \)  Torque developed by a rotor (Nm)

\( [Q] \)  Stiffness matrix for a lamina (Pa, GPa)

\( r \)  Radial position (of a blade section, relative to the rotor axis) (m)

\( R \)  Rotor radius (m)

\( R \)  Rotor radius (m)

\( \text{Re} \)  Reynold's number

\( \dot{\text{Re}} \)  Reduced Reynold's number

\( s \)  A dimension on the perimeter of a blade section (m)

\( S \)  Perimeter length of blade section (m)
$S'$  Non-dim. perimeter length of blade section (normalised with respect to chord)

$S_s$  Stretching strain – twisting curvature coupling constant for a blade section (rad,°)

$S_t$  Stretching load– twisting curvature coupling constant for a blade section (rad·Pa⁻¹,°·Pa⁻¹)

$[S]$  Compliance matrix for a lamina (Pa⁻¹, GPa⁻¹)

t  Thickness of blade skin (m)

$T$  Kinetic energy (J)

$T$  Tensile load on a blade (section) (N, kN)

$T$  Thrust load on rotor (N, kN)

$[T]$  Transformation matrix for properties of a lamina

$U_\infty$  Upstream, undisturbed wind speed (m/s)

$U,V$  Wind speed components seen by a rotating blade section respectively parallel and perpendicular to the undisturbed wind direction. (m/s)

$U_E$  Elastic strain energy (J)

$U_G$  Inertial potential energy (J)

$v$  The deflection of a blade under load

$V_{bd}(x)$  Shape function for blade twist distribution for twist due to skin mass

$V_{tip}(x)$  Shape function for blade twist distribution for twist due to tip-mass

$W$  Resultant wind speed seen by a rotating blade section (m/s)

$W$  Work done by external load (J)

$W(x)$  Shape function for blade untwist distribution due to propeller effect
Non-dimensional radial position relative to the blade root \( = r/L - h \)

Position on blade for reference chord (non-dim.)

Position on blade for start of tip-mass material

Displaced shape of a blade (non-dim.)

\( Z(j,x) \)  
\[ j \text{th Shape function for blade deflections} \]

\( Z(x) \)  
\[ A \text{ column vector of a finite series of the shape functions} \]

\( Z'(x) \)  
\[ A \text{ column vector of a finite series of the derivative of the shape functions} \]

\( Z''(x) \)  
\[ A \text{ column vector of a finite series of the 2nd derivative of the shape functions} \]

Angle of attack \( (^\circ) \)

Angle of attack corresponding to zero lift \( (^\circ) \)

Angle of attack corresponding to maximum lift \( (^\circ) \)

Pitch angle of blade (section, relative to the rotor plane) \( (^\circ) \)

Twist per unit length \( (^\circ/m) \)

Strain/deflection vector for a blade section

Strain vector for a lamina

Normal strains respectively parallel & perpendicular to the principal fibre direction

As above but referred to axes at an angle \( \theta \) to the principal fibre direction in the laminar plane.

Flow angle (seen by a blade section, relative to the rotor plane) \( (^\circ) \)

Shear strain on principal axes
\( \eta \)  Lekhnitski coefficient of material referred to blade axes

\( \eta_{xy} \)  Lekhnitski coefficient of a material for shear strain in the laminar plane due to normal strains on the x axis (at an angle \( \theta \) to the principal fibre direction)

\( \eta_{x'y'} \)  Lekhnitski coefficient of a material for shear strain in the laminar plane due to normal strains on the y axis (at an angle \( \theta + \pi/2 \) to the principal fibre direction)

\( \varphi \)  Compliance factor of a blade

\( \mathbf{\vartheta} \)  Row vector of compliance factors of a blade in unit solution (Rayleigh-Ritz)

\( \kappa_y, \kappa_z \)  Bending curvature of a blade (m\(^{-1}\))

\( \kappa_{yz} \)  Twisting curvature of a blade (rad-m\(^{-1}\), o-m\(^{-1}\))

\( \lambda \)  Tip speed ratio of rotor. (Sometimes local speed ratio of a blade section)

\( \lambda_j \)  The jth non-dimensional modal frequency of a blade

\( \Lambda \)  Bending-centrifugal ratio

\( \mu_0 \)  Mass per unit length of the blade skin at the root end (kg/m)

\( \mu_{tip} \)  Equivalent mass per unit length of the blade tip-mass (kg/m)

\( v_{12} \)  Poisson's ratio in the laminar plane

\( \theta \)  Fibre angle (relative to the blade's spanwise axis) (°)

\( \rho \)  Density. Air density or material density according to context. (kg-m\(^{-3}\))

\( \sigma_1 \)  Stress vector for a lamina (Pa, MPa)

\( \sigma_{11}, \sigma_{22} \)  Normal stresses respectively parallel & perpendicular to the principal fibre direction (Pa, MPa)
\( \tau_{12} \quad \text{Shear stress on principal axes (Pa, MPa)} \)

\( \omega \quad \text{Angular frequency (of vibration) (rad s}^{-1}) \)

\( \omega_j \quad \text{The } j \text{th modal frequency of a blade (rad s}^{-1}) \)

\( \omega_0 \quad \text{'Basic' frequency of a blade (rad s}^{-1}) \)

\( \Omega \quad \text{Rotational speed of rotor (rad s}^{-1}) \)
1. INTRODUCTION

The wind has been exploited as a source of energy for hundreds of years, but its use for generating electricity is a more recent phenomenon. The first electricity generating windmill is believed to be one build in Denmark towards the end of the last century. At the close of the millennium, wind power could now be said to be coming of age. This is most obvious in the rapid and prominent expansion in ‘wind farming’, the use of arrays of large turbines for generating power of the order of megawatts, mainly feeding into a grid, predominantly in Western Europe, North America and in India.

However, there is also a significant demand world-wide for ‘stand-alone’ power on a smaller scale in places where there is no grid supply. This is often in remote areas, whether for electrification of farms and villages and even nomadic encampments remote from the grid, or for ‘technical’ uses such as radio and telecommunication repeater stations, weather stations and research stations. In all these situations, there is a need for reliability, but it is particularly so in the technical uses where the site is unattended. Even routine maintenance may be difficult and expensive, and attendance to repair a failure may not be possible at all without a significant delay.

When, in the course of conversation I tell people that my research concerns wind turbines, they frequently suppose that I am trying to improve their efficiency. I usually reply that, on the contrary, I'm working on making them less efficient. This may seem somewhat facetious but is nevertheless to some extent true, though perhaps a more direct answer would be to say that my aim is to improve reliability.

For two reasons, my work is not particularly concerned with efficiency. In gross terms, the question of efficiency is mostly quite well understood and has been since the days of Froude, Betz and Glauert. There has been important work in recent years on sustaining efficiency throughout operation by the design of special aerofoil profiles which are insensitive in their performance to the roughness that builds up through their life. An example is the work of Tangler et al. [43].
Secondly, it is important to note that efficiency is not an end in itself in wind-turbine design, though it may well be a means to more important end goals. The 'fuel' is free and cannot be stored, so far more of it goes to 'waste' than can ever be captured. The main driving force for development is economics and it is the cost of energy (per unit) that is crucial.

People often mistakenly assume that wind power should be free or at least cheap because the wind is free. In reality, because air has a low density, the energy carried is very diffuse and the equipment required to capture it has to be relatively large. The energy cost thus tends to be dominated by the requirement for high levels of capital investment.

The efficiency of the equipment does not, therefore, matter inherently, but of course an inefficient machine has to be larger for a given power capture than a more efficient one. This may in theory be a cost effective solution, but in practice on the large commercial scale, upwards of 100kW and into the 1MW range, efficiency has to be pursued. An unnecessarily large machine would contain more material (which is a major part of the cost), run slower and thus need a more expensive gearbox or a more expensive generator. The alternative would be to have narrower blades in order to run at the same speed, but these would then need to have a higher mass or strength of material.

On the smaller scale, efficiency is not such an important issue as small increases in size do not make as great an impact on the total cost. Material costs are rarely as dominant as manufacturing costs and there is less of a problem with regard to matching speed and torque between the turbine rotor and the generator.

However ensuring that wind turbines survive high wind speeds, generally referred to as regulation, is a problem that is far from having been solved definitively.

The need to regulate wind turbines becomes clear when one looks at the gross properties of any wind-powered device. Simple aerodynamics and its inherent scaling laws arrived at by dimensional analysis tell us that if the aerodynamics remain unchanged, the speed of such a device goes up in proportion to incident wind speed.
Meanwhile, the loads on the device go up with the square of wind speed and the intercepted power throughput rises with the cube.

Most wind turbines of any scale are designed to start operating in winds of 2-4 m/s and are designed to reach their rated power at 7-14 m/s. Most will probably need to survive winds of up to 60 m/s on an occasional basis. At 60 m/s, a given area of wind turbine is capable of intercepting 216 times as much power as at 10 m/s and that power is just as capable of being destructive as of being useful. The loads are 36 times as great as at 10 m/s. It makes no sense to rate them at such a speed as these winds may only return once every few decades. A machine rated at 60m/s would be need to be incredibly strong and thus expensive and would have difficulty starting in low winds. It would be unlikely to capture much energy in the more common moderate wind speed range and, averaged over a period of time, it would actually capture considerably less energy than one rated at a lower speed.

It is also important to note that, small wind turbines generally have a different role from large machines, often being the only source of power at a remote site. Such sites are chosen for other reasons than ideal wind conditions. In order to maintain continuity of supply and to reduce the requirements for back up and storage, it is generally essential that small machines are able to start up in much lower wind speeds than large machines.

As a rule of thumb, for the most common pattern of wind speed distribution, the rated power of a wind turbine should be that available at roughly twice the mean wind speed (Freris [17]). Whilst it makes sense to design them to be as efficient as possible up to their rated speed, thereafter (although it is rarely expressed in this way,) they should become progressively less efficient. The challenge, then, is to avoid intercepting most or any of the power available at wind speeds over the rated value.

1.1 Wind-turbine speed regulation

How is this done? There are many approaches to regulating and controlling wind turbines and they vary according to the scale of the machine.
Stall control

At the large scale, most machines in existence are connected to a grid which can be allowed to hold them at constant or nearly constant speed. Many machines are designed so that the blades undergo aerodynamic stall at wind speeds above the rated value. This largely limits the increase in power captured. Beyond a figure 2-3 times the rated speed, the machine is shut down to allow it to survive. Increasingly, machines are run at variable speed and have servo systems which control the blade pitch in order to limit power.

At the small scale, stall control is not an option as most machines are not grid connected. It is not an option to use complex active control systems, with transducers to measure conditions, processors and actuators to effect movement of control surfaces, partly because it would be too expensive but also for the simple reason that there may not be room.

No regulation

The smallest machines, up to about 70-80W, may get away with no regulation system. They may simply be sturdy enough to withstand the strongest winds. Just having blades the right shape should mean that they have adequate strength for the loads they are subjected to. For similar shapes of blades and similar aerodynamics, the maximum stress in a blade does not change with scale, but, as mentioned earlier, small machines can run at relatively low tip-speed ratios without incurring gearbox or generator problems and can thus have a relatively high solidity. For a given loading, the stress is much lower in a wide blade than in a narrow one. Power can be controlled with a relatively simple charge regulator to prevent overcharging of the battery.

Furling

Up to about 1kW, it is common practice to have a system which furls the wind turbine. In high winds it is turned sideways or tipped back so that it faces progressively more edge on to the wind and intercepts less. See Figure 1.1.1
Figure 1.1.1 Rotor Furling

Such systems are relatively simple and can work reasonably well. If designed well, the furling system limits the power output correctly as well as the rotor speed. However, spending a large proportion of the time with the rotor facing partially off axis (whether yawed or tilted), the blades are, in effect, alternately advancing into and receding from the wind at the opposite sides of the rotor plane. As a result, they are subjected to loads with a large once-per-revolution cyclic component, and this can give rise to problems with fatigue. This is less of an issue at the small end of the scale, when blades are unlikely to be close to strength limits, but tends to become more of a problem as size increases.

Spoilers

A few machines employ spoilers (air brakes). These are surfaces which deploy at high rotor speeds, usually to a position perpendicular to the rotor plane and thus create high levels of drag to prevent further increases in rotor speed. These may be located at the blade tips or may be separate surfaces close to the hub but are almost invariably centrifugally operated. Like furling, such systems can be effective but tend to suffer from fatigue related problems as well as simple failure of hinges in extreme conditions.
Blade pitching

The commonest alternative is to employ a system which pitches the blades passively, using only the existing loads on the rotor and no separate power source. See Figure 1.1.2.

Figure 1.1.2 Blade Pitching

Blade pitching will be explained in more detail in the chapter on rotor aerodynamics, but a brief explanation is warranted here. Any rotor is designed to operate at a particular tip-speed ratio. Its blades are constructed with a particular distribution of chord-width and twist appropriate for these aerodynamic conditions and are mounted at a particular set angle. If the blades are pitched in a direction to increase their angle relative to the rotor plane (the so-called ‘pitch angle’, in an analogy with the pitch of a screw, as propellers and windmills were formerly regarded as airscrews) the angle of attack, and thus the lift, are reduced. There is thus less torque generated and the rotor speed is less likely to rise. Alternatively, the pitch angle may be decreased, thereby increasing the angle of attack. This takes it beyond the maximum value of lift into a region where the blade is stalled. The lift remains high but does not rise further and the drag rises considerably. It is the drag which limits the rotor speed.

There are many successful designs for such systems but all of them suffer from a number of related problems. They are always relatively complex. They require the blade to be mounted on a bearing situated at just the point (the blade root) which suffers the most severe bending moments. They have large numbers of movable parts which are mostly stationary and which move at low speeds when they do move. Low speed bearings present many challenges to the designer as hydrodynamic lubrication is not possible and the challenge is all the greater when the movement is only occasional. Yet, they must able to operate in all conditions, and especially in the worst conditions. A single part jamming may result in the whole turbine being
destroyed within a fraction of a second. They are expensive to manufacture and require regular maintenance.

**Twisting**

There are also a few examples of wind turbines in which the blade or the blade mounting twists in order to achieve aerodynamic regulation. In the case of the UTRC Composite Bearingless Rotor which was never commercialised, the blade itself is of quite a conventional glass-fibre construction but is mounted on a torsion tube which replaces the hub mounting, bearing and spring of a more conventional pitching system. The pitch actuator is a small pendulum mounted on the hub and connected to the blade’s leading edge by a flexible strap. The centrifugal load on this device results in pitching motion. In the case of the Bergey BWC series turbines, the whole blade is mounted rigidly on the hub and is torsionally flexible. A pitching weight fixed on the leading edge of the blade near the tip experiences a load due to the propeller effect which gives rise to the required twisting action. These turbines also employ a furling mechanism in high winds.

Unfortunately, almost any moving part is a potential site for failure, particularly when exposed to harsh weather conditions, and it is not unreasonable to say that, as a rule of thumb, the risk of a machine failing within a given time increases with the number of moving parts. According to this dictum, the obvious way to increase reliability is to reduce the part-count.

**Self-twisting**

Clearly, a wind turbine needs to rotate and, if it is a horizontal axis machine, it needs to yaw to face the oncoming wind. Could a wind turbine be regulated with no moving parts other than these? This thesis and the project it describes are about one approach which seeks to achieve just that. It describes an approach whereby a wind turbine is designed with blades which twist themselves in order to regulate in response to the loads on them, with no additional parts. The blades are constructed out of a fibre-reinforced resin composite, in this case largely carbon fibre - epoxy. The carbon fibre is laid-up with its fibres running at an angle to the axis of the blade in a helical
pattern. (See Figure 1.1.3) As the wind turbine speeds up, the blades experience stretching loads due to centrifugal force. The asymmetric lay-up of the composite gives rise to twist in the blade which acts as an aerodynamic control in an analogous manner to the whole-blade pitching movement of more conventional systems. In simplistic terms, the lay-up can be imagined to act like the turns of a helical spring which unwinds to some extent when it is extended. For a more detailed explanation see chapter 3.

![Figure 1.1.3 Helical Lay-up leading to twist](image)

Earlier work was carried out on the project at Reading University by N.M. Karaulis \[ 31 \] in which he developed a theoretical description of how blades would behave. The results of optimising the material lay-up were implemented in the manufacture of blades whose planform was identical to an existing design. Whilst these blades resulted in a demonstrable effect on the rotor's behaviour, it was clear that a greater degree of speed regulation would be required for a functioning wind turbine and that it was unlikely that enough twist could be induced with any foreseeable materials to achieve control of power.

### 1.2 Aims and objectives

The aim of the project recounted in this thesis was to develop a wind-turbine blade design which achieves a useful degree of speed regulation by exploiting the self-twisting approach.

To this end, a number of objectives were set.

The first principal objective was to develop methods of predicting the behaviour of a wind-turbine with self-twisting blades. It was also part this objective to be able to
predict the effects of a range of possible changes in design parameters. This phase is largely covered by chapters 2-5.

The second objective was to be able to infer design guidelines from the predictions of the model, and to implement the design lessons in a prototype design for manufacture. This phase is covered by chapters 6 and 7.

The third objective was to test the manufactured rotor against the theoretical predictions of its behaviour, which is covered by chapter 8.

It is recognised by the author that it is normal practice to test theoretical models repeatedly before implementing them. However it became clear early on that manufacturing self-twisting blades would be quite expensive, whereas funding was only available for one, possibly two sets of blades. It was thus not feasible, however desirable, to manufacture and test a variety of blade designs in order to test the theory. It is for this reason that manufacture and testing come late on in the list of objectives.

1.3 Chapter summary

Chapter 2 of this thesis sets out a brief overview of rotor aerodynamic theory with the emphasis on understanding (as far as is possible) the rotor runaway condition and on those aspects needed to understand how blade pitching and twisting can bring about aerodynamic regulation.

Chapter 3 reviews the theory of stretch-twist coupling in composite tubes and sets out the simplifications and development of this theory to give equations describing the induced twist distribution in a tube experiencing centrifugal loading. This leads directly to the requirements for maximising twist. The chapter also covers the first stage of experimental validation of the theory and a brief look at the parallel work at Reading using Finite Element methods to model blade twist.

Chapter 4 sets out the model of how the twisting equations are used in association with rotor aerodynamic software to make predictions of behaviour, performance and loading for a rotor incorporating self-twisting blades. It goes on to describe the
predictions made by the model for comparing the effects of each of a number of design parameters which are varied. Some design lessons are drawn from these results.

**Chapter 5** describes the development of a model describing the blade’s static blade bending deflections under loading and it’s modes and frequencies of vibration, based on the Rayleigh-Ritz method. This model is then used to make predictions of the actual deflections in a small range of scenarios.

**Chapter 6** examines those characteristics which are required of aerofoils for the prototype blade and assesses a shortlist of candidate profiles to give a final selection. 2-D aerofoil aerodynamics simulation is also used for two purposes: to extend the range of available data into the area which are required for predicting rotor behaviour, and to assess the impact of departing from prescribed aerodynamic profiles, in particular their manufacture with a blunt trailing edge. A choice is made as to the best way of modifying the profile in order to provide the thickness.

**Chapter 7** sets out the finalisation of the blade design in some detail, taking into account both the specific requirements for a self-twisting blade design and the constraints of cost, availability, handling of material, and the manufacturing process as well as safety. Predictions of the performance of the prototype rotor design were also made based on the model developed earlier.

**Chapter 8** describes the design of the experimental set-up for testing the twisting response of the manufactured blades. It also sets out the results from these tests and compares them to predictions, and does the same for further tests which were performed to find the blade’s vibrational modes and frequencies.

**Chapter 9** Draws some conclusions from the work and sets out some recommendations for future work on the project.
2. AERODYNAMIC MODELLING

2.1 Basic Blade element theory

At the heart of any wind turbine is its aerodynamics. It generates torque and power aerodynamically. In the case of the wind turbine which is the subject of this project, aerodynamics is also central to its unique feature, its means of overspeed control. In order to understand how it is expected to work, and in order to model its behaviour, some method of aerodynamic modelling must be applied.

There is a wide spectrum of models that can be used to describe wind turbine aerodynamics, which ranges from the earliest, simplest models to ones which mostly come later, are considerably more complex and sophisticated and it is hoped more accurate. To some extent they represent a continuum, though there are also qualitative differences.

For all approaches to modelling, the critical elements that distinguish models, and which determine how well they work, are the methods used a) to determine the flow-field at the rotor, as it is influenced by and influences the flow both upstream and downstream and b) how the forces on the rotor blades are determined.

(Chassapoyannis et al [6]).

The model that has been employed here, is called ‘Blade Element - Momentum Theory’, or BE-MT, or sometimes just blade element theory. A number of texts, (Glauert [20], Jansen and Smulders [27], Wilson Lissaman and Walker [46], Lysen [34] Sharpe in Freris [17], Wilson in Spera [42], and Sharpe [41]) give accounts of this model and most of what follows in this section is excerpted from these. It represents quite a standard approach amongst many wind turbine designers and has the advantages of relative simplicity, speed of computation and availability as off-the-shelf operational and validated software packages. A commercial code, Garrad Hassan’s ‘Blades’ has been used here.
BE-MT is characterised by a number of features and assumptions:

1. There is assumed to be non-frictional loss.
2. The flow through the rotor is treated as being constant around the rotor plane and is assumed to flow within a streamtube and not to mix with the air outside this.
3. The forces on the rotor as a whole are calculated by a momentum balance between the flows upstream and downstream of the rotor.
4. The blade forces are calculated (iteratively) strip by strip from the flow angle calculated at that blade station and a knowledge of the aerofoil characteristics of that section of blade.
5. It is further assumed that no significant interaction occurs between the streamtubes passing through adjacent sections of blade.

Some terms should be defined at this stage. The most important term to define here is the wind speed, $U_\infty$, which is always refers to the value far upstream, before the rotor has had any influence. In all the discussion which follows, the wind direction is assumed to be perpendicular to the rotor plane.

The rotor speed, $\Omega$, is often referred to in its non-dimensional form, the tip-speed ratio, $\lambda$. This is the ratio between the speed of the blade tip and the undisturbed wind speed, $U_\infty$. If the rotor radius is $R$, then the tip-speed ratio is given by

$$\lambda = \frac{R \cdot \Omega}{U_\infty}$$

As with most other aerodynamic forces, the thrust on the rotor, $T$, which is the total force on the rotor acting parallel to the rotor axis, is expressed in non-dimensional form as the thrust coefficient, $C_T$, where the normalising factor is the total momentum flux of an airstream of the same cross-section as the rotor, far upstream.

$$C_T = \frac{T}{\frac{1}{2} \rho \cdot A \cdot U_\infty^2}$$

where $A$ is the swept area of the rotor and $\rho$ is the air density.

The power, $P$, and torque, $Q$, are also normalised with respect to appropriate parameters.
\[ C_P = \frac{P}{\frac{1}{2} \rho A U_{\infty}^3} \quad C_q = \frac{Q}{\frac{1}{2} \rho A R U_{\infty}^2} \]

The retardation of the flow at the rotor is expressed as a proportion of the undisturbed flow, termed the axial induction (or influence) factor (or coefficient), \( a \). Thus the axial air speed at the rotor is given by

\[ U_1 = U_{\infty} (1 - a) \]

An important result of BE-MT is that there is a maximum possible value of \( C_P \) of 16/27 or 0.59 (known as the Betz limit) which occurs when the axial induction factor is 1/3.

The rotation of the flow at the rotor, \( \omega \), is also expressed as a proportion of the rotor's rotational speed, giving the tangential induction factor, \( a' \). Thus

\[ a' = \frac{\omega}{\Omega} \]

The flow experienced by an element of the blade consists of the wind which results from its own rotation and the approaching axial wind. It can thus be represented as a velocity triangle. (Figure 2.1.1)

\[ W = \sqrt{U_{\infty}^2 \cdot (1 - a)^2 + r^2 \Omega^2 \cdot (1 + a')^2} \]

**Figure 2.1.1 Flow triangle due to wind and blade rotation**

The resultant is at a shallow angle, \( \phi \), to the rotor plane.
\[ W = r\Omega \cdot \left( 1 + \frac{(1-a)^2}{\lambda^2} \right) \]

\[ \phi = \tan^{-1} \left( \frac{U}{r\Omega \cdot (1 + a')} \right) = \frac{1-a}{\lambda} \text{ for small angles} \]

The blade is set at a pitch angle, \( \beta \), relative to the rotor plane and thus sees a wind at an angle of attack, \( \alpha \), which is the difference between the pitch angle and the flow angle.

\[ \alpha = \beta - \phi \]

The lift and drag forces on the section, \( dL \) and \( dD \), are calculated from the angle of attack, \( \alpha \), the resultant wind, \( W \) and the (known) aerofoil characteristics, \( C_l(\alpha) \) and \( C_d(\alpha) \).

Lift: \[ dL = \frac{1}{2} \rho n c(r) W^2 C_l(\alpha) \]

Drag: \[ dD = \frac{1}{2} \rho n c(r) W^2 C_d(\alpha) \]

Once these are resolved onto axes perpendicular to and parallel to the rotor plane they can be recognised as making up thrust forces bending the blades and tower and torque providing power (Figure 2.1.2).
2.2 Rotor Runaway Condition

A brief word here is necessary about the intended method of controlling the wind turbine. It has to be recognised that the self-twisting mechanism will not be sufficient to control the power output of the turbine. It will merely allow the rotor to protect itself in the event of overspeed by limiting the degree of overspeed. In general, the power control will be managed electronically by controlling the excitation of the generator. The turbine would be made to follow its optimum tip-speed-ratio up to the generator's rated power, and after that the power would be capped. The latter regime would result in a falling torque demand curve and for most wind speeds over rated, the turbine would be running at or close to a runaway state. It is thus clear that characterising the runaway state is more important for the model than any others.

What is meant here by the runaway state? In essence it is a condition in which the turbine neither produces nor demands torque or power. (In reality it must provide for bearing friction). This is equivalent to the 'auto-giro', 'auto-rotation', or 'power-off' condition for lifting-rotor aircraft. (See Johnson [29])
According to blade element theory, for a uniformly loaded rotor (a convenient fiction), runaway occurs when the in-rotor-plane components of lift and drag are exactly equal and opposite to each other. The rotor speed will adjust itself until this is the case. For any practical low solidity rotor, the force vector triangle which represents this situation has a lot of uncertainty in it, the flow angle being very shallow. The angle itself is dependent on the axial induction and thus on the thrust. Whilst the dependence of lift on angle of attack is quite well characterised and not very dependent on Reynold’s number, drag is highly dependent on Reynold’s number and also on many other factors such as blade roughness, radial flow, angle-of-attack history. There are many other ways in which this picture is over-simplistic, nevertheless it is useful.

Figure 2.2.1 Velocity and Force Diagram for ‘Pure’ Runaway
Even if blade-element theory is assumed to apply, it is unlikely that all blade stations would reach a zero-torque condition at the same speed. In reality, at any given speed, some stations will still generate torque, whilst others will demand torque. The runaway condition is then the speed at which the net torque, integrated over the whole rotor, is zero.

More serious deviations occur; the runaway rotor is in a condition generally described as the ‘turbulent-wake’ state. Despite there being no generation of useful motor power, a substantial amount of usable kinetic energy is still removed from the airstream - indeed more than in the generating state. However, most (if not all) of this
is returned to the airstream in the form of turbulent eddies which have disordered and unusable kinetic energy. The air pressure in this wake region is consequently low, giving rise to a high thrust load on the rotor disc.

Unlike the low pressure in ordered streamtubes, the pressure in a turbulent wake does not recover to its original value as it slows down and expands in the far wake. It can only recover significantly through turbulent mixing with the relatively undisturbed air outside the wake.

The two main assumptions underlying momentum theory can therefore be seen to have broken down, firstly, that Bernoulli's equation can be applied to the wake behind the rotor, and secondly, that the momentum equations can be applied within given sets of streamtubes. (To some extent, the former problem is covered by Glauert's empirical curve-fit for high rotor loading. See Sharpe in Freris [17])

Given the potential shortcomings of Blade-Element Momentum theory (BE-MT), is it appropriate to apply a more detailed model, based on the wake structure of the rotor? Much work has gone into developing such models, of varying degrees of complexity, based on lifting lines or lifting surfaces (panels) and with varying degrees of prescription or freedom for the behaviour of the vortices in the wake. Up to now, virtually all this work has been concerned with improving predictions of power generation at design wind speeds and the associated fatigue loads.

The runaway condition is recognised to be a greater challenge and only Wood and co-workers [47] & [48] have paid much attention to it. They have made adjustments to their wake models to take account of the ways in which a heavily loaded rotor differs from one in a more normal running condition, but they still assume a uniform loading. This is not even approximately true for the flow state of the self-twisting bladed rotor once the blades have twisted and the rotor is regulating itself.

All is not hopeless, however. As has already been mentioned, even before twisting, it is likely that the runaway state involves some blade stations generating torque and some demanding it. If this variation is only moderate, then it is probably not too unreasonable to approximate this with a uniform loading and zero torque throughout. However, the self twisting blade acts to decrease the angle of attack of the outermost
portions of the blade by quite large angles. This reduces the lift to the point where it crosses over into negative values in order to produce a braking torque. This portion of the blade is then acting as a propeller.

Meanwhile, if the rotor has slowed down, the angle-of-attack increases at the untwisted, innermost, portions of the blade which act as a windmill. In between, there may be a region in something close to the 'zero slip' condition with angle of attack close to zero and no significant lift or drag. It is clear that, at some point on the blade, the direction of the induced flow will change, implying a type of vortex ring imposed on the axial flow. This is clearly not the kind of flow state that Wood is modelling.

Once the rotor is in the regulating state, because it very successfully reduces the runaway tip-speed ratio, the loading again becomes quite light, with low values of axial induction factor. If it is considered reasonable to neglect the inaccuracies that will arise from the normal BE-MT assumption that the flow 'annuli' are independent of each other, then momentum theory can again be applied with reasonable confidence. For the most part the rotor is not even in the 'turbulent wake' state which would require Glauert's empirical fit.

BE-MT is inappropriate, due to a high disc loading when the rotor runs away with little twist. However, this should only occur at low wind speeds, so any inaccuracies are unimportant. If the rotor can cope with running away at high wind speeds, it should be able to cope with lower ones. There is one exception to this, and some attention will have to be paid to it at a later stage; it is conceivable that with the rotor rotating slowly at low wind speeds, a sudden gust could be problematic. In the absence of significant centrifugal stiffening, blade deflections could actually be larger than at higher steady wind speeds and lead to damage.
The need for aerodynamic control and its implementation can easily be illustrated using BE-MT. It is clear how, with a low or zero power demand, a wind turbine rotor will tend to speed up until it reaches a runaway condition, which is a stable, steady state. The problems which arise from it are twofold. Firstly, the high speed itself may cause problems as even minor imbalances in the rotor may produce large, damaging out-of-balance forces which could destroy rotor, bearings, mountings or the tower. Secondly, high rotor speeds are generally associated with a large thrust loading which has to be borne by the blades, bearings and tower.

For a grid-connected wind turbine, the commonest method of control is to run the turbine at fixed speed or nearly fixed speed with an induction machine directly connected to the grid. The electrical machine characteristic is sufficiently stiff to maintain a nominally constant speed. Then, as the wind speed picks up, the flow angle is seen to increase which increases the angle-of-attack of the blade profile.
Beyond the linear range of the aerofoil characteristic, the lift curve levels off and then falls. At the same time, the drag curve starts to rise. Initially the torque characteristic of the wind turbine rises with increasing wind speed, but once the blade starts to stall, the torque and power generated level off and then fall.

By choosing suitable aerofoil characteristics and by appropriate design of the blade shape, this process can be made to happen smoothly and progressively along the blade. The rotor characteristic is then reasonably level at rated power.

In the absence of a grid connection and therefore without its the characteristic electrical stiffness, this methodology does not work. Provision has to be made in grid connected machines to prevent excessive speed-up in the event of the loss of electrical load. It is also not an appropriate method of control for a stand-alone machine, as the stiff load is not there at all, unless a very sophisticated electronic controller were used.

![Velocity triangles for stall limiting](image)

**Figure 2.2.6 Velocity triangles for stall limiting**

For a stand-alone machine, (and also for a grid-connected machine), there is the possibility of pitching the blades towards stall, which entails a reduction of blade pitch and thereby an increase in angle-of-attack. This depends on there being a rise in the drag coefficient as the angle-of-attack increases even before the lift levels off. In essence, for a given wind speed, the rotor speed is governed by the lift-to-drag ratio. This must fall for the runaway tip-speed ratio to fall. As the rotor tip-speed-ratio falls, there is a reduction in the rotor loading and in the axial induction factor. This
increases the flow angle further and pushes the blade further into stall, thus enhancing the effect already produced. The disadvantage of this approach is that the thrust load on the rotor may remain quite high.

Alternatively, the blade can be pitched towards 'feather', increasing the pitch angle and decreasing the blade angle-of-attack. In general, for the first stage of feathering, there is no rise in drag coefficient but there is a linear fall in lift coefficient. The resulting reduction in lift-to-drag ratio reduces the runaway tip-speed ratio and slows down the rotor. Unfortunately, the reduction in rotor loading and induction factor tend to reduce the flow angle and thereby slow down the fall in angle-of-attack. This reduces the effectiveness of the blade feathering action. The advantage of feathering is that the thrust on the blade is directly reduced with the decrease in lift. Feathering may continue to the stage where the blade angle-of-attack is sufficiently negative that negative lift is produced. This corresponds to the propeller condition mentioned earlier where both lift and drag work together to retard the rotor.

---

In the case of a blade which twists rather than pitching as a rigid body, there is little or no action at the blade root. Though it is clearly the case that any action at the tip, whether stalling or feathering, has a greater effect than a similar action at the root, it is likely that the tip region would have to twist through a larger angle than would be necessary for a whole blade when pitching. This is the basis for the employment of deployable tip-breaks and pitchable tips as mechanisms to prevent overspeed in otherwise stall-controlled rotors in the event of lost load.
3. MODELLING OF BLADE TWIST

Applying a tensile load to a sample of a material normally results in the sample stretching in the direction of the load and contracting in the orthogonal direction(s). This is also true of a unidirectional fibre composite if the load is applied parallel or perpendicular to the fibre direction. If it is applied at an angle to the fibre, there is, in addition, a tendency for the material to distort in such a manner that the fibres lie slightly closer in direction to that of the load. The material undergoes a shear as well as a stretch. (See Figure 3.1.1) If the material being stretched is wrapped up in a tube, still with the fibres at an angle, but now forming helical patterns, the shear of the material is expressed as a twist in the tube, somewhat analogous to the partial unwrapping of a helical spring as it is stretched. (See Figure 1.1.3)

3.1 The need for a simpler model

Karaolis' [31] results clearly show that there are limitations on the amount of twist developed when an asymmetric lay-up is applied to an existing blade design which is operated under normal conditions. Such an arrangement does not regulate the rotor's runaway speed sufficiently. There is obviously a need to optimise the blade design for self-twisting behaviour. A full analytical optimisation is not possible and some aspects of the design must remain a matter of judgement. Nevertheless, it should be possible to establish the sensitivities of various aspects of rotor behaviour and performance to a number of design parameters.

At the outset, it was expected that the following parameters would be critical in the blade design;

- Blade skin material
- Blade skin lay-up
Investigating all these parameters would clearly entail modelling a large number of rotor configurations. For this purpose a simplified approach was adopted to modelling the blade twisting behaviour.

Treating the blade as a thin-walled closed tube suggests that trends can be identified by using the Batho-Bredt equation in reverse. The equation is normally applied to a tube subjected to a torsional torque. The tube twists in response giving rise to shear strain. In this case the shear strain arises through coupling between normal stress and shear strain and the twist develops as a result. Nevertheless, the same formula applies. If a particular amount of twist is needed, then the shear strain needed in order to generate it is governed by the tube's section properties.

According to the aforementioned Batho-Bredt model, the twist per unit length, $d\beta/dl$, developed in a tube with constant skin-thickness is given by

$$\frac{d\beta}{dl} = \frac{1}{2A} \int \gamma(s) ds = \frac{S}{2A} \gamma$$

when $\gamma$ is constant around the section where $S$ is the section's perimeter length and $A$ is the enclosed area, whilst $\gamma$ is the shear strain. (Strictly speaking, $A$ should be the area enclosed by the mid-surface of the skin, but the difference in practice is small.)
Given that $S$ varies little between feasible aerofoil section shapes, it would seem that twist should be increased by reducing the enclosed area of the blade section. This could be achieved by choosing thin aerofoil sections and by designing narrow blades. Once the chord in the tip region is fixed by aerodynamic design considerations, a blade which is designed with little or no taper would tend to increase the twist that can be achieved compared to one with more taper.

However, the complete picture is more complex than this. The effect on the performance of the turbine of varying each blade parameter must be investigated and to this end, some means of incorporating a model of blade twisting into a rotor aerodynamics model must be found. It was decided to adopt a highly simplified approach to modelling blade twisting behaviour in order to allow rapid investigation of a wide range of rotor configurations.

Initially, the blade skin material and lay-up were treated as constant as much work had been completed by Karaolis [31]. Simplifying his approach considerably, it was appropriate to these assumptions from the start that only the helical lay-up approach would be used, meaning that the blades would twist in response to centrifugal rather than bending loads. Further simplifying assumptions can also be made, in particular that the lay-up is treated as constant throughout the blade and that the blade section
has a constant shape along the blade even as the section dimensions vary. The blade’s response to loads, most importantly the centrifugal stretching loads arising from rotation, can be expressed in terms of constant blade material constants, and section shape parameters, whilst all variation is wrapped up in a relatively simple distribution along the blade which is a function a only of the varying blade chord width.

3.2 Material properties of asymmetrical composite laminae

Basic material properties and stress strain relations

Most of the material concerning the transformation of material properties is based closely on Jones [30], Karaolis [31] and Vasiliev [45].

Taking a unidirectional lamina of fibre composite material as the starting point, the material can be regarded as specially orthotropic. The macroscopic properties can be summarised in the form of the longitudinal and transverse elastic moduli, $E_{11}$ and $E_{22}$ respectively, the shear modulus, $G_{12}$ and Poisson’s ratio, $\nu_{12}$.

The engineering constants $E_{11}$, $E_{22}$ and $G_{12}$ are expressed in units of Pa and therefore appear to be measures of stiffness. However, this is not strictly the case. They are defined in terms of measurements on samples under uniaxial loading, as this is much more practical to implement than any other load case, and they therefore relate most directly to the compliance matrix of the material rather than the stiffness matrix.

Loading a sample along one axis only, with all other degrees of freedom unconstrained, produces strains on that same axis and also on some or all of the other degrees of freedom, even in an isotropic material. The elastic modulus $E_{11}$ is the stress required to produce a unit strain on that axis. The normal strain produced on the orthogonal axis, even in isotropic materials, is represented by Poisson’s ratio $\nu_{12}$ which is the strain on the orthogonal axis when unit strain exists on the original axis. The compliance is defined as the strain per unit stress and may be defined both for stresses and strains on the same axis but may also be defined for the strain produced on one axis by a uniaxial stress on another by coupling.
Clearly the two representations, in terms of engineering constants and in terms of compliance, are related. Thus if a compliance matrix \([S]\) is defined, this relates the strain vector \(\varepsilon\) to the stress vector \(\sigma\) by \(\varepsilon = [S] \sigma\) and the compliance matrix is represented quite simply in terms of the engineering constants:

\[
\begin{bmatrix}
\varepsilon_{11} \\
\varepsilon_{22} \\
\gamma_{12}
\end{bmatrix} =
\begin{bmatrix}
1/E_{11} & -v_{12}/E_{11} & 0 \\
-v_{12}/E_{11} & 1/E_{22} & 0 \\
0 & 0 & 1/G_{12}
\end{bmatrix}
\begin{bmatrix}
\sigma_{11} \\
\sigma_{22} \\
\tau_{12}
\end{bmatrix}
\]

(3.2.1)

It should be noted that, as the material is being treated as a lamina, the stress and strain vectors have only three components, not six. The lamina is regarded as being very thin with respect to the cross-sectional dimensions of the blade. This is not to say that stresses and strains do not arise perpendicular to the plane of the lamina but that they have negligible effects on the macroscopic behaviour. The particular relationship shown is for a specially orthotropic material only, where the stress and strain axes are coincident with the symmetry axes of the material.

The same stress strain relationship may also be represented by stiffnesses, which are defined by imposing a strain on one degree of freedom of a material at a time, all other degrees of freedom being constrained. Stresses arise on the axis being strained and also in other degrees of freedom which would otherwise undergo a strain. The deflection on the loaded axis is thereby reduced, to a small degree. The stiffness is the stress per unit strain required to produce the uniaxial deflection case.

The stiffness matrix \([Q]\) relates the stress vector \(\sigma\) to the strain vector \(\varepsilon\) by \(\sigma = [Q] \varepsilon\) and like the compliance matrix, it generally includes off axis terms for the ‘coupling’ between different degrees of freedom. It should be clear, however, from the above explanation that despite the engineering constants having units of stress, the relationship with the stiffnesses is less simple than that with the compliances.

In matrix terms, the stiffness matrix is clearly just the inverse of the compliance matrix. As the laminate can be regarded as being in a situation of plain stress, the inversion is carried out on the 3x3 compliance matrix to yield a 3x3 stiffness matrix. The latter can be represented in terms of the engineering constants by
\[
\begin{bmatrix}
\sigma_{11} \\
\sigma_{22} \\
\tau_{12}
\end{bmatrix} =
\begin{bmatrix}
E_{11} / \kappa & v_{12} E_{22} / \kappa & 0 \\
v_{12} E_{22} / \kappa & E_{22} / \kappa & 0 \\
0 & 0 & G_{12}
\end{bmatrix}
\begin{bmatrix}
\varepsilon_{11} \\
\varepsilon_{22} \\
\gamma_{12}
\end{bmatrix}
\]

where \( \kappa = 1 - v_{12} \frac{E_{22}}{E_{11}} \)

However, as will be seen later on, once symmetry is lost and more of the terms of the matrices are non-zero, the relationship between the engineering constants and the stiffness matrix becomes much more complex, whilst that with the compliance matrix remains relatively simple.

**Angled fibres and stress and strain transformation and normal-shear coupling**

Once the material is laid-up with the fibres at an angle \( \theta \) to the load axes, calculation of the material's response requires the strains to be transformed onto the appropriate axes in order that the orthotropic properties should apply. The stresses are then transformed back onto the load axes in a similar manner.

The strain transformation is represented as \( \varepsilon = [T] \varepsilon_x \),

\[
\begin{bmatrix}
\varepsilon_{11} \\
\varepsilon_{22} \\
\gamma_{12}
\end{bmatrix} =
\begin{bmatrix}
\cos^2 \theta & \sin^2 \theta & \sin \theta \cos \theta \\
\sin^2 \theta & \cos^2 \theta & -\sin \theta \cos \theta \\
-2 \sin \theta \cos \theta & 2 \sin \theta \cos \theta & \cos^2 \theta - \sin^2 \theta
\end{bmatrix}
\begin{bmatrix}
\varepsilon_{xx} \\
\varepsilon_{yy} \\
\gamma_{xy}
\end{bmatrix}
\]

and that for the stresses as \( \sigma = [T]^{-1} \sigma_x \)

\[
\begin{bmatrix}
\sigma_{xx} \\
\sigma_{yy} \\
\tau_{xy}
\end{bmatrix} =
\begin{bmatrix}
\cos^2 \theta & \sin^2 \theta & 2 \sin \theta \cos \theta \\
\sin^2 \theta & \cos^2 \theta & -2 \sin \theta \cos \theta \\
-\sin \theta \cos \theta & \sin \theta \cos \theta & \cos^2 \theta - \sin^2 \theta
\end{bmatrix}
\begin{bmatrix}
\sigma_{11} \\
\sigma_{22} \\
\tau_{12}
\end{bmatrix}
\]

This process gives a new relationship between the stresses and strains on the new axes which, like before, can be summarised in a stiffness matrix (or a compliance matrix),

\[
\sigma = [Q'] \varepsilon_x,
\]

where \( [Q'] = [T]^{-1} [Q] [T]^T \)

28
Now, however, all 9 elements of the matrix are filled, though as the matrix remains symmetrical there are only 6 unique elements, compared to the 4 unique properties of the specially orthotropic lamina. The extra elements relate to coupling which, in addition to the usual coupling between normal strains on the x and y axes, is now also seen between normal stresses on the x and y axes and shear strain in the xy plane.

\[
\begin{bmatrix}
\sigma_{xx} \\
\sigma_{yy} \\
\tau_{xy}
\end{bmatrix} =
\begin{bmatrix}
Q_{11} & Q_{12} & Q_{16} \\
Q_{21} & Q_{22} & Q_{26} \\
Q_{61} & Q_{62} & Q_{66}
\end{bmatrix}
\begin{bmatrix}
\varepsilon_{xx} \\
\varepsilon_{yy} \\
\gamma_{xy}
\end{bmatrix}
\]

The required engineering constants are more easily obtained from the compliance matrix \([S']\)

\[\varepsilon = [S'] \sigma, \text{ where } [S'] = [Q^T]^{-1}\]

\[
\begin{bmatrix}
\varepsilon_{xx} \\
\varepsilon_{yy} \\
\gamma_{xy}
\end{bmatrix} =
\begin{bmatrix}
S'_{11} & S'_{12} & S'_{16} \\
S'_{21} & S'_{22} & S'_{26} \\
S'_{61} & S'_{62} & S'_{66}
\end{bmatrix}
\begin{bmatrix}
\sigma_{xx} \\
\sigma_{yy} \\
\tau_{xy}
\end{bmatrix}
\]

\[E_{xx} = 1/S'_{11}\]
\[E_{yy} = 1/S'_{22}\]
\[G_{xy} = 1/S'_{66}\]
\[\nu_{xy} = -S'_{12}/S'_{11}\]

Two more constants, normally applied to stress-strain relations in anisotropic materials, represent the rate at which shear strains develop in proportion to the normal strains on the x and y axes, as a result of the aforementioned coupling. They are analogous to Poisson’s ratio and are known as Lekhnitski coefficients or ‘coefficients of mutual influence’.

\[\eta_{xy,x} = S'_{16}/S'_{11}\]

where \[\gamma_y = \eta_{xy,x} \varepsilon_x\] under conditions of uniaxial stress,
such that \( \sigma_{xx} = \sigma, \sigma_{yy} = \tau_{xy} = 0 \)

Similarly,

\[ \eta_{xy} = S_{26} / S_{11} \]

where \( \gamma_{xy} = \eta_{xy,y} \varepsilon_y \)

when \( \sigma_{yy} = \sigma, \sigma_{xx} = \tau_{xy} = 0 \)

Relating the material axes to those of the blade skin, the \( x \) direction lies along the blade axis and the \( y \) direction lies around the blade section perimeter, perpendicular to the blade axis. Although some stresses do arise in the circumferential, \( y \), direction, due to constraint of the Poisson ratio type of coupling, they should be small enough to have a negligible effect on the macroscopic behaviour. Thus, for the purposes of this simplified model, only \( E_x, G_{xy}, \) and \( \eta_{xy,x} \) are of interest. For convenience, once the expressions for their transformed values have been derived, they will be referred to without subscripts as \( E, G, \) and \( \eta \).

For single laminae, they may be related directly back to the fibre-orientated engineering constants for the material.

\[
\frac{1}{E_x} = \frac{1}{E_1} \cos^4 \theta + \left( \frac{1}{G_{12}} - \frac{2v_{12}}{E_1} \right) \sin^2 \theta \cos^2 \theta + \frac{1}{E_2} \sin^4 \theta
\]

\[
\frac{1}{G_{xy}} = 2 \left( \frac{2}{E_1} + \frac{2}{E_2} + \frac{4v_{12}}{E_1} - \frac{1}{G_{12}} \right) \sin^2 \theta \cos^2 \theta + \frac{1}{G_{12}} \left( \sin^4 \theta + \cos^4 \theta \right)
\]

\[
\eta_{xy,x} = E_x \left[ \left( \frac{2}{E_1} + \frac{2v_{12}}{E_1} - \frac{1}{G_{12}} \right) \sin \theta \cos^3 \theta - \left( \frac{2}{E_2} + \frac{2v_{12}}{E_1} - \frac{1}{G_{12}} \right) \sin^3 \theta \cos \theta \right]
\]

(3.2.2)

Using standard trigonometric identities, these expressions can be recast in a different way to give a more transparent relationship to the fibre angle.
\[
\frac{1}{E_x} = \frac{3}{E_1} + \frac{1}{E_2} - \frac{2\nu_{12}}{G_{12}} + \frac{1}{G_{12}} \frac{1}{E_1} \cos 2\theta 
\]

\[
... + \frac{1}{E_1} + \frac{1}{E_2} + \frac{2\nu_{12}}{E_1} - \frac{1}{G_{12}} \cos 4\theta 
\]

\[
\frac{1}{G_{xy}} = \frac{1}{E_1} - \frac{1}{E_2} + \frac{1}{E_1} + \frac{2\nu_{12}}{E_1} - \frac{1}{G_{12}} \cos 4\theta 
\]

\[
\frac{\eta_{xy,x}}{E_x} = \frac{1}{E_1} - \frac{1}{E_2} + \frac{1}{E_1} + \frac{2\nu_{12}}{E_1} - \frac{1}{G_{12}} \sin 4\theta 
\]

It is thus relatively simple to find the lay-up angle which maximises the coupling shear strain per unit axial stress, \( \frac{\eta_{xy,x}}{E_x} \). If \( E_1 \) and \( E_2 \) are equal, such as in a balanced weave fabric, then the coupling will always be greatest when \( \sin 4\theta \) is at its maximum, which occurs at \( 22.5^\circ \). In other circumstances, it can be maximised by differentiating with respect to \( \theta \) and setting to zero.

\[
\frac{d}{d\theta} \left( \frac{\eta_{xy,x}}{E_x} \right) = \frac{1}{E_1} - \frac{1}{E_2} - \frac{1}{G_{12}} \cos 2\theta + \frac{1}{E_1} + \frac{1}{E_2} + \frac{2\nu_{12}}{E_1} - \frac{1}{G_{12}} \cos 4\theta = 0 
\]

or \( a(2\cos^2 2\theta + 4b \cos 2\theta - 1) = 0 \)

where \( a = \left( \frac{1}{E_1} + \frac{1}{E_2} + \frac{2\nu_{12}}{E_1} - \frac{1}{G_{12}} \right) \), \( b = \frac{1}{4} \left( 1 - \frac{R_E}{1 + R_E - R_\theta} \right) \), \( R_E = \frac{E_1}{E_2} \) and \( R_\theta = \frac{E_1}{G_{12}} - 2\nu_{12} \)

This can be solved as a quadratic, giving:

\[
\cos 2\theta_{\text{opt}} = -b \pm \sqrt{b^2 + \frac{1}{4}} 
\]

(3.2.3)

In order to maximise the coupling shear strain per unit axial strain, represented directly by the Lekhnitski coefficient, \( \eta \), it is necessary to differentiate the quotient of
the expressions in $\eta/E$ and $1/E$ which is somewhat cumbersome. It is simpler to find the maximum numerically or graphically.

At this point, it is worth examining which of the material properties are most important if the stretch-twist coupling is to be maximised. The earlier expression for $\eta$, (3.2.2) has a substantial imbalance in the two terms. At the small angles which are of interest, $\sin \theta \cos^3 \theta$ is much greater than $\sin^3 \theta \cos \theta$. Examining its coefficient, it is clear that coupling is most sensitive to the difference between the principal elastic modulus and the shear modulus.

In a unidirectional fibre composite, the shear modulus is dominated by that of the matrix material, so a material with a low shear modulus should be favourable for coupling. Unfortunately, without choosing exotic materials, there is little choice over this property. However, there is considerable variation in the principal elastic modulus between different fibre materials. Clearly, a high modulus fibre is desirable, indeed as high as possible.

The second point to note is that the second, smaller term acts to diminish coupling. This term approaches zero if the orthogonal direction modulus, $E_2$ is equal to roughly twice the shear modulus. Obviously, it will have quite a small value if there is no fibre reinforcement other than in the first direction. However a unidirectional composite may be undesirable from the point of view of practicality, in particular handling of the material during manufacture.

It should be noted that there are significant difficulties in handling and laying up fibre in an off-axis direction. In the case of a so-called unidirectional fabric, the small amount of weft (usually about 5%), is not enough to prevent the warp fibres of the fabric to part in places.

The alternative is to employ a filament-winding or tape-winding method. Here there might be difficulties winding around the relatively sharp trailing edge, especially with a stiff fibre such as carbon.
The other alternative is to use a fabric, preferably non-woven, with fibres running in two perpendicular directions. The properties of such a fabric can be closely approximated by the usual methods used for obtaining the properties of laminates.

Once different materials are being chosen for the different layers, it is clear from equation (3.2.2) that the most favourable choice would be for a material in the second direction which would set the second normal modulus to roughly twice the shear modulus or lower.

If a laminate made up from multiple layers at different orientations is considered, and the interlaminar properties are still ignored, then the properties of the laminate as a whole can be obtained from a weighted mean of the stiffness matrices of the individual laminae. (It is not correct to form the whole laminate engineering constants from those of the individual laminae by a similar weighted mean approach as this would not take account of the requirements of strain compatibility. Similarly, the compliance matrices can not be added)

However one short cut can be taken. From the points of view of obtaining and handling the materials, it is often more convenient to work with orthogonal weave materials or with non-woven material consisting of separate layers orthogonal to each other and stitched together. If the laminate consists of such an orthogonally orientated material, then the assembled laminate still has axes of symmetry (before it has been rotated), is still specially orthotropic and thus its properties can be represented by the usual four engineering constants.

If \( a \) is the proportion of material orientated in the principal direction and \((1 - a)\) lies in the orthogonal direction, then the assembled laminate has a stiffness matrix which can be represented as

\[
\begin{bmatrix}
    \frac{(aE_{11} + (1-a)E_{22})}{\kappa} & \nu_{12}E_{22}/\kappa & 0 \\
    \nu_{12}E_{22}/\kappa & \frac{(1-a)E_{11} + aE_{22}}{\kappa} & 0 \\
    0 & 0 & G_{12}
\end{bmatrix}
\]

where \( \kappa = 1 - \nu_{12}^2 \frac{E_{22}}{E_{11}} \)

The engineering constants are obtained from the new stiffness matrix by analogous relations to the relationship of the original values with the original matrix.
\[ \nu_{w12} = \frac{\nu_{12}}{\left( (1-a)(E_{11}/E_{22}) + a \right)} \]

\[ E_{w11} = \frac{(aE_{11} + (1-a)E_{22})}{((1-a)E_{11} + aE_{22})} \]

\[ \kappa = 1 - \nu_{12}^2 \frac{1}{a(E_{11}/E_{22}) + (1-a)((1-a)(E_{11}/E_{22}) + a)} \]

\[ E_{w11} = (aE_{11} + (1-a)E_{22}) \frac{\kappa}{\kappa} \]

\[ E_{w22} = ((1-a)E_{11} + aE_{22}) \frac{\kappa}{\kappa} \]

\[ G_{w12} = G_{12} \]

The advantage of deriving the quantities like this is that the explicit equations can be used for the variation of the engineering constants with orientation, which makes it simpler to generate plots using a spreadsheet such as Microsoft Excel. It is then a simple matter to compare different materials and to find the optimum layer orientations. Without the shortcut, separate transformations and inversions of the matrices would have to be performed at each angle.

As an example, a glass-epoxy composite having engineering constants as follows;

\[ E_{11} = 36 \text{ GPa} \quad E_{22} = 9.5 \text{ GPa} \quad G_{12} = 4.2 \text{ GPa} \quad \text{and} \quad \nu_{12} = 0.285 \]

A 50-50 orthogonal stitched fabric of such glass will form a composite with the following properties;

\[ E_{w11} = 22.9 \text{ GPa} \quad G_{w12} = 4.2 \text{ GPa} \quad \text{and} \quad \nu_{w12} = 0.119 \]

It is clear from equation (3.2.3) that the normal stress-shear strain coupling reaches a maximum at 22.5°.
If the orientation angle, $\theta$, is varied from 0 to 30°, the engineering constants vary as in the figure, below, where they have been normalised with respect to the fibre-wise modulus of the fabric.

At a fibre angle of 20°, the figures become:

\[
\begin{align*}
E_x &= 17.2 \text{ GPa} \\
G_{xy} &= 5.55 \text{ GPa} \\
\nu_{xy} &= 0.339 \\
\eta_{xy,x} &= -0.595 \\
\eta_{xy,x}/E_x &= 0.0346
\end{align*}
\]

![Graph showing variation of properties of glass-fibre with fibre angle](image)

**Figure 3.2.1** Variation of properties of glass-fibre with fibre angle

As has already been seen, there are two measures of the degree of coupling. The Lekhnitski coefficient, $\eta$, is a measure of the shear strain developed by coupling per unit direct strain. As the strength limits for fibre composite materials are often conveniently expressed in terms of maximum allowable strains, this coefficient is a good measure of how much twist can be achieved within strength limits. On the other hand, the off-diagonal term in the compliance matrix, $S'_{16}$, which equals $\eta/E_x$, gives a measure of the twist that can be achieved for a given stretching load. Maximising this quantity effectively minimises the required tip-mass. As $E_x$ diminishes with increasing $\theta$, it is not too surprising to find that $\eta$ and $S'_{16}$ reach their respective maxima at different fibre angles, 18° and 22.5° respectively.
Clearly, on grounds of strength alone, it would be preferable to lay the fibre up at the optimum angle for the Lekhnitski coefficient $\eta$, which is the smaller angle of the two. However, from the practical point of view, this may not be for the best. In practice, it is found that both very thin blade sections and quite a large additional mass in the blade-tip are required in order to develop adequate twist. Even with the fabric laid-up at the optimal angle for coupling per unit load, $S'_{16}$, a substantial length of the blade core needs to be taken up by added mass. For this reason, the lay-up angle is chosen as a compromise between the two criteria.

Similar calculations can be carried out for carbon fibre and for a fabric with carbon in the principal fibre direction and glass in the cross direction.

Table 3.2.1 Comparison of engineering constants for different weaves

<table>
<thead>
<tr>
<th></th>
<th>principal mod.</th>
<th>orthog. mod.</th>
<th>shear mod.</th>
<th>Poisson's ratio</th>
<th>Lekhn. coeff.</th>
<th>stretch-twist</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$E_x$ (GPa)</td>
<td>$E_y$ (GPa)</td>
<td>$G_{xy}$ (GPa)</td>
<td>$v_{xy}$</td>
<td>$\eta_x$</td>
<td>$-\eta_x/E_x$</td>
</tr>
<tr>
<td>uni-dir glass</td>
<td>36.0</td>
<td>9.5</td>
<td>4.2</td>
<td>0.285</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>50/50 glass</td>
<td>22.9</td>
<td>22.9</td>
<td>4.2</td>
<td>0.119</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>glass at 20°</td>
<td>17.2</td>
<td>5.55</td>
<td>0.339</td>
<td>-0.595</td>
<td>0.0346</td>
<td></td>
</tr>
<tr>
<td>uni-dir carbon</td>
<td>139</td>
<td>9.4</td>
<td>5.7</td>
<td>0.330</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>50/50 carbon</td>
<td>74.6</td>
<td>74.6</td>
<td>5.7</td>
<td>0.042</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>carbon at 20°</td>
<td>34.7</td>
<td>8.73</td>
<td>0.552</td>
<td>-1.268</td>
<td>0.0363</td>
<td></td>
</tr>
<tr>
<td>50/50 C-GI</td>
<td>72.05</td>
<td>13.76</td>
<td>4.95</td>
<td>0.162</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>C-GI at 20°</td>
<td>31.04</td>
<td>12.94</td>
<td>6.40</td>
<td>0.426</td>
<td>-1.434</td>
<td>0.0462</td>
</tr>
</tbody>
</table>
Figure 3.2.2 Variation of elastic modulus with fibre angle

Figure 3.2.3 Variation of Lekhnitski Coefficient (longitudinal-shear strain coupling) with fibre angle
It is clear from Figure 3.2.3 and Figure 3.2.4 that a material consisting of carbon and glass in orthogonal directions gives superior coupling properties when compared to either glass alone or carbon alone (whether on the criterion of coupling per unit stress or on coupling per unit load). It is less clear whether or not such a material would be easily available from suppliers, though it should be possible to employ separate layers of carbon and glass.

If only a single material could be used, there is only a small difference between carbon and glass on the basis of coupling per unit load, $S_{16}$, but apparently a substantial advantage in using carbon on the basis of Lekhnitski coefficient. However, as glass is capable of surviving much larger strains than carbon fibre, even the latter difference becomes insignificant. A choice of material would thus be made on some other basis. Glass is easier to handle during the manufacturing process and is considerably cheaper. On the other hand, if blade stiffness were an issue because of questions of blade deflections or modal frequencies, carbon would be the material of choice.
3.3 Blade section twisting behaviour

It is now possible to examine the properties of actual blade sections in terms of their geometry and the material properties. In Karaolis' [31] analysis of a blade section, he allowed the section four degrees of freedom. In the absence of any net hoop stresses due to pressurisation of the blade (which he looked at later as a separate possibility for speed regulation), the section can experience tensile loads, bending moments on the flapping and lead-lag axes and torsion, strains arise in these same four degrees of freedom. This requires a 4x4 stiffness matrix for the section in question. (as opposed to the stiffness matrix for the material or that for the whole blade).

\[
\begin{bmatrix}
T_x \\
M_y \\
M_z \\
M_{yz}
\end{bmatrix} = \begin{bmatrix}
\varepsilon_x \\
\kappa_y \\
\kappa_z \\
\kappa_{yz}
\end{bmatrix}
\]

\text{or} \quad P = [H] \varepsilon

where \( T \) and \( M \) are tensile and bending loads respectively, \( \varepsilon \) and \( \kappa \) are strains and curvatures. \( H_{ij} \) are elements of the stiffness matrix. (Note that the axis subscripts given here are according to my definition rather than Karaolis'.)

As for the material, this relationship can also be expressed in reverse as the compliance matrix which is the inverse of the stiffness matrix.

\[
\varepsilon = [h]P = [H]^{-1} P
\]

Unfortunately, the elements of these vectors and the matrix are not homogeneous in terms of their dimensions. It is, however, possible make them independent of the section dimensions by redefining the relevant quantities in an appropriate way. This yields a 'reduced' stiffness matrix, \([H']\), given by

\[
[H'] = \begin{bmatrix}
H_{11} / c t & H_{12} / c^2 t & H_{13} / c^2 t & H_{14} / c^2 t \\
H_{12} / c^2 t & H_{22} / c^3 t & H_{23} / c^3 t & H_{24} / c^3 t \\
H_{13} / c^2 t & H_{23} / c^3 t & H_{33} / c^3 t & H_{34} / c^3 t \\
H_{14} / c^2 t & H_{24} / c^3 t & H_{34} / c^3 t & H_{44} / c^3 t
\end{bmatrix}
\]
where $c$ and $t$ are the blade chord and the skin thickness, respectively. This is not dimensionless but is homogeneous in dimensions (unlike the original stiffness matrix) and has dimensions of stress.

For the helical lay-up which is of interest here, only the leading diagonal elements and element $H_{14}$ (and of course $H_{41}$ which is identical) are non-zero. The latter is the coupling stiffness term which represents the constant of proportionality between a tensile load and the twist per unit length which it induces.

$$
\begin{bmatrix}
H_{11}^* & 0 & 0 & H_{14}^* \\
0 & H_{22}^* & 0 & 0 \\
0 & 0 & H_{33}^* & 0 \\
H_{14}^* & 0 & 0 & H_{44}^*
\end{bmatrix}
$$

(3.3.1)

It is from the compliance matrix that the usual 'engineering theory of beams' properties can be obtained. The relevant properties are the longitudinal, tensile stiffness, $(ESI)$, the flapwise bending stiffness $(EI)_F$, the lead-lag bending stiffness $(EI)_L$, and the torsional stiffness $(GJ)$.

If $S$ is the perimeter dimension of the section and it has uniform skin thickness $t$, then the material cross sectional area is $St$. This is in distinction to $A$, which is defined as the area enclosed by the section perimeter line. (As already mentioned, this should strictly be the area enclosed by the mid-line of the skin material.)

For the sake of simplicity, the blade section shape and skin thickness will be kept constant throughout the blade's length, whilst the chord of the section varies in a linear taper. This allows the shape dependent properties to be expressed more conveniently in non-dimensional terms.

If $c$ is the chord width of the blade at the section being considered, then

$$
A^* = A/c^2 \quad \text{and} \quad S^* = S/c.
$$

The stiffnesses can also be given in their 'reduced' forms, $(ES)^*$, $(EI)_F^*$, $(EI)_L^*$, $(GJ)^*$, which are independent of the section dimensions, and which all have the dimensions of stress.
The coupling stiffness coefficient $S_t$ must also be considered, used by Karaolis as the principle measure of coupling between tensile loads and twist per unit length, where both are expressed in reduced form. The load is divided by the product of blade chord and skin thickness and has dimensions of stress, whilst the twisting curvature is multiplied by the chord to give a dimensionless quantity. $S_t$ thus has dimensions of reciprocal stress. He also employed a coefficient which he calls $S_r$, which is the twist developed per unit dimensionless length per unit longitudinal normal strain. This is dimensionless. The two properties are defined as follows and are derived from elements of the compliance or stiffness matrices.

Relating torsional strain (twist per unit length) to normal strain $\varepsilon_x$;

\[
\kappa_{yz} = \frac{\beta}{l} = \frac{S_r}{c} \cdot \varepsilon_x \\
S_r = \frac{h_{14}^*}{h_{11}^*} = -\frac{H_{14}^*}{H_{44}^*}
\]

(3.3.3)

and relating $\kappa_{yz}$ to the axial tensile load $T$;

\[
\kappa_{yz}. = \frac{\beta}{l} = \frac{S_r}{c^2t} \cdot T_x \\
S_r = h_{14}^* = -\frac{H_{14}^*}{H_{11}^*H_{44}^* - H_{14}^*H_{14}^*}
\]

(3.3.4)

The reduced stiffness matrix elements are obtained from expressions involving integrations of various material properties around the blade section.
Three such properties were enumerated by Karaolis [31] which are required in this work. He designated them as \( S_1 \), \( S_2 \), and \( S_3 \). Although he expressed them explicitly in terms of the elements of the stiffness matrix for the assembled composite, it is clear that they correspond closely to the so-called engineering quantities as defined in the previous section. Thus

\[
S_1 = \frac{Ea}{S_1} \quad S_2 = \eta \\
S_3 = \frac{1 - \eta^2}{G} = \frac{1}{G} \cdot (1 - R_\eta) \quad \text{where} \quad R_\eta = \frac{\eta^2 G}{Ea} \quad (3.3.6)
\]

It is now possible to take the integral expressions for the relevant stiffness matrix elements and simplify them considerably. Where the integral is of one of the three properties, it can be assumed that it remains constant around the section and can thus be taken outside the integral. In many cases, as long as the \( y \) and \( z \) co-ordinates of the shell are expressed relative to the tension centre, which in a uniform section should correspond to the centroid, this leaves a zero integral or one which can be expressed as a simple shape property.

\[
\int \frac{d s'}{s'} = S' = S/c \\
\int z' d s' = 0 \\
\int y' d s' = 0 \\
\int z'^2 d s' = I_\eta = I_{yy} / c^3 t \\
\int y'^2 d s' = I_{zz} = I_{zz} / c^3 t \quad (3.3.7)
\]
where $s'$ is a dimensionless length-variable around the perimeter of the section.

Using the integrals from equation (3.3.7), the equivalences between the $S'$'s and the engineering properties in equations (3.3.6), and the fact that those properties remain invariant around the section, it is possible to make some simplifications.

\[
S_1 \int S' d's' = Ea \cdot S' \\
S_2 \int S' d's' = \frac{S'}{G} - \frac{\eta^2 S'}{G} = \frac{S'}{G} \cdot (1 - R) \quad (3.3.8) \\
S_2 \int S' d's' = S_2 \int S' d's' = 0 \\
\int S_2 \int S' d's' = S_2 \int S' d's' = 0
\]

It can now be seen that by substituting from equations (3.3.8) back into (3.3.5) the stiffness elements simplify to

\[
H_{11}^* = Ea \cdot S' + \frac{\eta^2 \cdot G}{(1 - R)} \cdot S' \\
H_{22}^* = Ea \cdot I_{yy} \\
H_{33}^* = Ea \cdot I_{zz} \\
H_{44}^* = 4 \cdot \frac{G \cdot A''}{(1 - R)} \cdot S' \\
H_{14}^* = - \frac{2 \eta \cdot G}{(1 - R)} \cdot A'
\]

It is now possible to return to the section engineering constants and to express them in simpler form.

\[
(ES)^* = H_{11}^* - \frac{H_{14}^*}{H_{44}^*} = Ea \cdot S' \\
(GJ)^* = H_{44}^* - \frac{H_{14}^*}{H_{11}^*} = 4 \cdot \frac{G \cdot A''}{S'} \\
(EL)^y = H_{22}^* = EI_{yy} \\
(EL)^z = H_{33}^* = EI_{zz} \quad (3.3.9)
\]
These simplified stiffness relationships can now be applied to derive the stresses and strains in a blade section to the applied loads and this, in turn can be related to the behaviour of the whole blade.

Assume that the only load is a tensile load $T$ acting along the blade axis, through the section’s tension centre. In the light of equations (3.3.9) it can be assumed that, for the most part, mean strains and mean stresses are in relatively simple relationships to the load.

The mean normal stress \[ \sigma = \frac{T}{S \cdot t} \]

The normal strain is related to this by the effective modulus $E_a$ so that \[ \varepsilon = \frac{T}{E_a \cdot S \cdot t} \]

The shear strain is related to this by the Lekhnitski coefficient $\eta$ so that \[ \gamma = \frac{\eta \cdot T}{E_a \cdot S \cdot t} \]

The expressions already derived for the development of twist in the blade can now be seen to be related to the Batho-Bredt equation commonly used to express shear strain to twist in a closed tube:

\[ M_{yz} = 2A \cdot \tau = 2G A \cdot t \gamma \quad \text{where } M_{yz} \text{ is a torsional moment.} \]

\[ \frac{\Delta \beta}{l} = M_{yz} \int \frac{ds}{G t} = 2GAt \gamma \int \frac{ds}{4A^2} = \frac{S}{2A} \gamma \]

since the section has uniform shear modulus and skin thickness.
Clearly this has been derived for the case of shear strains arising from a torsional load applied to the tube, but the relationship between shear strain on the micro scale and twist-per-unit-length on the macro scale is purely a question of strain compatibility and therefore geometry. Thus, for a given twist to develop, there must be a particular mean level of shear strain in the material.

Substituting for the shear strain arising from coupling.

$$\kappa_{yz} = \frac{\beta}{2A} = \frac{S}{2A} \cdot \gamma = \frac{S}{2A} \cdot \eta \cdot \varepsilon_x = \frac{S'}{2A'} \cdot \varepsilon_x = \frac{S'}{2A'} \cdot \varepsilon_x$$

where $S_s = \frac{S'}{2A'}$ as defined earlier.

This accords completely with the definition of $S_s$ in equation (3.3.3). The comparison also works for the load-twist coupling factor $S_t$.

$$\kappa_{yz} = \frac{\theta}{2A} = \frac{S}{2A} \cdot \frac{\eta}{Ea} \cdot \frac{T}{S} = \frac{\eta}{2Ea \cdot A' \cdot c^2 t} \cdot T = \frac{S_t}{c^2 t}$$

where $S_t = \frac{\eta}{Ea} \cdot \frac{1}{2A'}$ as defined earlier.

An example will illustrate the order of magnitude of the resulting twist. For simplicity, consider a uniform blade of elliptical cross section, with no mass in the blade itself but carrying a concentrated tip-mass. The blade spins about its root.
Table 3.3.1 Example twist calculation

<table>
<thead>
<tr>
<th>blade length</th>
<th>L</th>
<th>1.5 m</th>
</tr>
</thead>
<tbody>
<tr>
<td>chord</td>
<td>c</td>
<td>0.075 m</td>
</tr>
<tr>
<td>section ellipse</td>
<td>b/c</td>
<td>0.1</td>
</tr>
<tr>
<td>skin thickness</td>
<td>t</td>
<td>0.001 m</td>
</tr>
<tr>
<td>tip mass</td>
<td>m_t</td>
<td>0.5 kg</td>
</tr>
<tr>
<td>rotor speed</td>
<td>Ω</td>
<td>165 rad/s</td>
</tr>
<tr>
<td>effective modulus</td>
<td>E_a</td>
<td>20 GPa</td>
</tr>
<tr>
<td>Lekhnitski coefficient</td>
<td>η</td>
<td>0.45</td>
</tr>
<tr>
<td>tensile load</td>
<td>T</td>
<td>20 kN</td>
</tr>
<tr>
<td>perimeter coefficient</td>
<td>S'</td>
<td>2</td>
</tr>
<tr>
<td>area coefficient</td>
<td>A'</td>
<td>0.07</td>
</tr>
<tr>
<td>stretch-twist coefficient</td>
<td>S_t</td>
<td>0.15 GPa^{-1}</td>
</tr>
</tbody>
</table>

This degree of twist is of the right order for regulating the wind turbine.

3.4 Whole blade twisting behaviour

In the wind turbine blade being considered, the option of tapering the blade must be kept open. For a tapered blade, it is not strictly valid to treat section properties in exactly the same manner as for a prismatic element, but for a gentle rate of taper, the departure from such behaviour is negligible. (Gere and Timoshenko [18])

In most circumstances, the most obvious way to describe the blade dimensions would be to give the blade length and its chord width at both ends. Alternatively one could give the chord at one end and the ratio between the tip and root chords. In this case, however, the primary importance of the blade is aerodynamic and only secondarily structural. Whilst the length of the blade is important, more directly important is the rotor radius as this will govern the energy capture and the rotational speed. Whilst the size of the dead-zone in the hub will affect the energy capture in the second order, it is better to regard this as a variable with respect to a fixed radius rather than to a fixed blade length. It then becomes more relevant to see the blade chord in relation to rotor radius rather than to the blade length.
Following this logic, it is inappropriate to design the blade around a chord width chosen either at the hub or at the tip. The former region generates little power because of the smaller area of this region of the rotor disc, whilst the latter generates little power because of tip-loss effects, where the trailing vortices at the tip reduce the angle of attack in the outermost region of the blade. Given that the blade shape will depart considerably from so-called ideal planforms, it is best to fix the chord at a point where energy capture is at its greatest density. This is generally located at around 70-80% of the rotor radius.

However, it is simplest to describe the blade shape in terms of a co-ordinate $x$ running from 0 at the root of the blade to 1 at the tip, and $x_{\text{ref}}$ at the reference station mentioned above. Similarly, the chord can be defined as $c_0$ at the root, $c_I$ at the tip and $c_{\text{ref}}$ at the reference station. For most purposes, a taper rate $c'$ is defined such that

$$c(x) = c_0'(1 - c'x) \quad (3.4.1)$$

where $c' = 1 - \frac{c_I}{c_0}$ and $c_0' = \frac{c_{\text{ref}}}{(1 - c'x_{\text{ref}})}$.

![Diagram of blade dimensions](image)

Figure 3.4.1 Definitions of blade dimensions

### 3.4.1 Blade twist due to an applied tensile load

Let a tensile load $F_{\text{tip}}$ be applied to the blade tip. Assuming that stress is uniform, the stress is given by
and the axial strain is

$$\varepsilon = \frac{\sigma}{E_a} = \frac{F_{\text{tip}}}{E_a \cdot S' \cdot t \cdot c(x)}$$

The normal-shear strain coupling in the material gives rise to a shear strain governed by the Lekhnitski coefficient:

$$\gamma = \eta \cdot \varepsilon = \eta \cdot \frac{F_{\text{tip}}}{E_a \cdot S' \cdot t \cdot c(x)}$$

According to the Batho-Bredt equation, this can be related to the twist rate that develops

$$\frac{d}{dx} \beta = \frac{L}{2 \cdot A} \cdot \gamma (x) = \frac{L}{2 \cdot A'} \cdot \gamma (x)$$

Substituting for the shear strain,

$$\frac{d}{dx} \beta = F_{\text{tip}} \cdot \frac{S'}{2 \cdot A'} \cdot \frac{\eta \cdot L}{E_a \cdot S' \cdot c(x)^2}$$

The twist up to a point \(x\) is found by integrating the twist rate from the blade root to \(x\).
\[
\Delta \beta = F t \frac{\eta L}{2A^t E\alpha c} \cdot \frac{1}{2c^t} \cdot \frac{1}{(1 - c^t x^t - 1)}
\]

For the twist between the two ends, substitute for \( c' \) and set \( x = 1 \)

\[\Delta \beta = F t \frac{\eta L}{2A^t E\alpha c} \cdot \frac{1}{2c^t} \cdot \frac{1}{(1 - c^t)} \quad (3.4.2)\]

### 3.5 Experimental validation of Simple twisting model

In order to validate the simplified model, a 'dummy' test piece was constructed. This was carried out largely by Axel Schmeer, then of Reading University, though with some assistance from myself. The aim was to subject the test piece to a series of known tensile loads in a tensile testing machine in order to measure the twist developed. This is clearly a different load case from that experienced by the blades themselves, so the same equations do not apply. However, what this mode of testing should validate is the assumptions made about the twist rate developed at a particular section. There was no need to go to the trouble of forming the piece with an aerodynamic cross-section. Instead, the shape of the blade was approximated by employing an elliptical section, of approximately the same major-minor axis ratio as the aerofoil sections to be used on the wind turbine i.e. 9.5%. The test-piece had the same length, chord width and rate of taper as the true blade design but additionally had aluminium end pieces to allow it to be gripped by the jaws of the tensile tester without crushing.

Construction was as follows:

The core was manufactured on a CNC milling machine from a rectangular block of rigid PU foam. (Because of its thinness and flexibility, it had to be supported in a cradle machined out of high density fibreboard during machining of the second face.) The end pieces were similarly milled out of aluminium to have a matching section at the faces abutting the foam and having the same rate of taper. They were attached to
the foam with dowels and were also bonded to it with epoxy resin. The fibre layers were constructed from 75mm wide dry loose balanced weave glass-fibre tape. This was wound helically onto the foam core and the end-pieces at a mean angle of 20°. To keep this constant on the tapered core, it was necessary to stretch it more on one side of the tape than on the other. It was also not possible to achieve uniform coverage because of the taper, but an average of four layers were laid up, with epoxy resin being painted on and rolled at each layer. Four layers of dry cloth have a measured thickness of 0.557mm. The whole assembly was vacuum bagged and cold-cured. Samples of excess material were cut from the finished piece and were subjected to standard tests in the testing machine.

The tensile test was carried out by Mr. Schmeer alone. The testing machine was fitted with one fixed jaw and one free to rotate about the blade’s long axis. Loads were applied stepwise up to 10 kN. The angle of twist was measured manually with a thin bar fixed to the semi-free end. This was 914mm long and the distance of both ends of this bar from a datum line were measured for each value of load.

The equation for the twist of the test piece under an end load comes from the previous section.

\[ \Delta \beta = F \frac{\eta}{E a} \frac{L}{2 \cdot A'c \cdot q'c \cdot t} \]  

### 3.5.1 Initial validation of blade twisting theory

**Table 3.5.1 Blade dimensions**

<table>
<thead>
<tr>
<th>Blade length</th>
<th>Tip chord</th>
<th>Hub chord</th>
<th>Skin thickness</th>
<th>Section minor/major axis</th>
<th>Area coefficient</th>
</tr>
</thead>
<tbody>
<tr>
<td>L</td>
<td>C(T)</td>
<td>C(0)</td>
<td>t</td>
<td>b/c</td>
<td>A'</td>
</tr>
<tr>
<td>1.5 m</td>
<td>0.0768 m</td>
<td>0.1111 m</td>
<td>0.56 mm</td>
<td>9.5 %</td>
<td>0.0746 = \pi c b/4</td>
</tr>
</tbody>
</table>
Table 3.5.2 Test-blade material properties

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Principal modulus $E_1$</td>
<td>36.7 GPa</td>
</tr>
<tr>
<td>Orthogonal modulus $E_2$</td>
<td>8.5 GPa</td>
</tr>
<tr>
<td>Shear modulus $G_{12}$</td>
<td>3.64 GPa</td>
</tr>
<tr>
<td>Poisson ratio $\nu_{12}$</td>
<td>0.288</td>
</tr>
<tr>
<td>Weave</td>
<td>50/50</td>
</tr>
<tr>
<td>Fibre angles</td>
<td>$20^\circ$/$-70^\circ$</td>
</tr>
<tr>
<td>Number of cloth layers</td>
<td>4</td>
</tr>
<tr>
<td>Resultant axial modulus $E_a$</td>
<td>16.07 GPa</td>
</tr>
<tr>
<td>Resultant shear modulus $G$</td>
<td>4.96 GPa</td>
</tr>
<tr>
<td>Lekhnitski coefficient $\eta$</td>
<td>0.371</td>
</tr>
<tr>
<td>Stretch-twist couple $\eta/E_a$</td>
<td>0.0231 GPa$^{-1}$</td>
</tr>
</tbody>
</table>

Table 3.5.3 Tensile test set-up

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Load range</td>
<td>0 – 10 kN</td>
</tr>
<tr>
<td>Measuring bar length</td>
<td>914 mm</td>
</tr>
<tr>
<td>Test blade chord: thin end</td>
<td>77 mm</td>
</tr>
<tr>
<td>Test blade chord: thick end</td>
<td>111 mm</td>
</tr>
<tr>
<td>Predicted twist per unit load</td>
<td>2.60 °/kN</td>
</tr>
</tbody>
</table>

Table 3.5.4 Tensile test results

<table>
<thead>
<tr>
<th>Load (kN)</th>
<th>End 1</th>
<th>End 2</th>
<th>Twist (°)</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>517</td>
<td>626</td>
<td>6.83</td>
</tr>
<tr>
<td>5</td>
<td>485</td>
<td>653</td>
<td>10.53</td>
</tr>
<tr>
<td>6</td>
<td>470</td>
<td>665</td>
<td>12.22</td>
</tr>
<tr>
<td>8</td>
<td>448</td>
<td>682</td>
<td>14.67</td>
</tr>
<tr>
<td>9</td>
<td>415</td>
<td>705</td>
<td>18.18</td>
</tr>
<tr>
<td>10</td>
<td>400</td>
<td>717</td>
<td>19.87</td>
</tr>
</tbody>
</table>

| Twist per unit load (°/kN) | 1.99 |

Finally, the same parameters were used by Jeronimidis and co-workers at Reading University as the basis of an FE model in ALGOR. (Jeronimidis et al [28])
It can be seen from the results in Table 3.5.5 and Figure 3.5.2 that both the simple model and the FE model overestimate the twist somewhat, the former by approximately 30% and the latter by approximately 10%. These discrepancies may be due to the simplicity of the modelling methods, the fact that they fail to take full account of constraints. On the other hand, it is quite likely that the lack of precision in the hand lay-up of the test blade leaves uncertainties in the final construction of the blade.

**Table 3.5.5  Comparison of blade twist in simple model, FE model and experimental tensile test results**

<table>
<thead>
<tr>
<th>skin thick t (mm)</th>
<th>Simple model</th>
<th>FE</th>
<th>Tensile Test</th>
</tr>
</thead>
<tbody>
<tr>
<td>Load (kN)</td>
<td>0.56 0.6 0.7</td>
<td>0.6 0.7</td>
<td>0.6</td>
</tr>
<tr>
<td>0</td>
<td>0.00 0.00 0.00</td>
<td></td>
<td>6.83</td>
</tr>
<tr>
<td>3</td>
<td>8.34 7.79 6.68</td>
<td>10.53</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>13.91 12.98 11.13</td>
<td>12.22</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>16.69 15.58 13.35</td>
<td>14.67</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>22.25 20.77 17.80</td>
<td>18.18</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>25.03 23.36 20.03</td>
<td>24.30</td>
<td>19.87</td>
</tr>
<tr>
<td>10</td>
<td>27.82 25.96 22.25</td>
<td>21.18</td>
<td></td>
</tr>
<tr>
<td>15</td>
<td>41.72 38.94 33.38</td>
<td>42.13</td>
<td>37.80</td>
</tr>
<tr>
<td>20</td>
<td>55.63 51.92 44.50</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
3.6 Twist due to blade rotation

3.6.1 Blade twist due to centrifugal load on a point tip-mass

For a rotating blade, if the blade body is assumed to have no mass, but there is a concentrated mass $m_{tip}$ in the tip, then the centrifugal load acts as a pure end load on the blade. If the blade rotates at $\Omega$ rad/s and the hub radius is $h \cdot L$, then the load on the tip-mass is

$$F_{tip} = m_{tip} \cdot (1 + h) \cdot L \cdot \Omega^2$$

Substituting into (3.4.2) gives the twist over the whole blade:

$$\Delta \beta = F_{tip} \cdot \frac{\eta L}{Ea \cdot 2 \cdot A'c \cdot \theta'c \cdot I't} = \frac{m_{tip} \cdot (1 + h) \cdot L^2 \cdot \Omega^2}{Ea \cdot 2 \cdot A'c \cdot \theta'c \cdot I't}$$

For the whole twist distribution,
\[\Delta \beta (x) = \Omega \frac{2 \cdot \eta \cdot m_{\text{tip}} \cdot (1 + h) \cdot L^2}{Ea \cdot 2 \cdot A' \cdot t \cdot c_0^2} \cdot \frac{1}{c'} \left( \frac{1}{1 - c'*x} - 1 \right)\]

A numerical example will illustrate the order of magnitude of twist to be expected. All the parameters are, as far as possible, those of the real blade.

**Table 3.6.1 Blade parameters for numerical examples**

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>blade length (L)</td>
<td>1.302 m</td>
</tr>
<tr>
<td>hub ratio (h)</td>
<td>0.114</td>
</tr>
<tr>
<td>blade root chord (c_0)</td>
<td>0.109 m</td>
</tr>
<tr>
<td>blade tip chord (c_t)</td>
<td>0.073</td>
</tr>
<tr>
<td>skin thickness (t)</td>
<td>0.00132 m</td>
</tr>
<tr>
<td>set angle (\theta_0)</td>
<td>4°</td>
</tr>
<tr>
<td>area coefficient (A')</td>
<td>0.0620</td>
</tr>
<tr>
<td>perimeter coeff. (S')</td>
<td>2.035</td>
</tr>
<tr>
<td>second moment (non-dim) (I')</td>
<td>0.00263</td>
</tr>
<tr>
<td>elastic modulus (Ea)</td>
<td>28.02 GPa</td>
</tr>
<tr>
<td>shear modulus (G)</td>
<td>7.0 GPa</td>
</tr>
<tr>
<td>Lekhnitski Coeff (\eta)</td>
<td>1.334</td>
</tr>
<tr>
<td>density (\rho)</td>
<td>1560 kg m(^{-3})</td>
</tr>
<tr>
<td>tip mass density (\rho_{\text{tip}})</td>
<td>5650 kg m(^{-3})</td>
</tr>
<tr>
<td>tip mass length (L_{\text{tip}})</td>
<td>0.497 m</td>
</tr>
<tr>
<td>tip mass (m_{\text{tip}})</td>
<td>0.509 kg</td>
</tr>
<tr>
<td>rotor speed (\Omega)</td>
<td>1200 rpm</td>
</tr>
</tbody>
</table>

**Figure 3.6.1 Twist distribution due to an end load**

### 3.6.2 Blade twist due to centrifugal load on its own mass

The most important component of blade twist is that due to the action of centrifugal loads on the mass of the blade skin.

The mass of an elemental sectional slice of the blade is
\[
dm(x) = \rho \cdot S' \cdot t \cdot c'(x) \cdot L \cdot dx
\]

\[
\ldots = \mu \cdot 0' \cdot L \cdot (1 - c' \cdot x) \cdot dx
\]

where \( \mu = \rho \cdot S' \cdot t \cdot \left( \frac{c_{\text{ref}}}{1 - c' \cdot x_{\text{ref}}} \right) \)

The total mass of the blade skin is

\[
m_{\text{skin}} = \mu \cdot 0' \cdot L \cdot \left( 1 - \frac{c'}{2} \right)
\]

\[
m_{\text{skin}} = 0.502 \cdot \text{kg}
\]

The elemental centrifugal loading due to this section is

\[
dF(x) = \Omega \cdot L^2 \cdot r \cdot dm = \mu \cdot 0' \cdot \Omega \cdot L^2 \cdot (x + h) \cdot (1 - c' \cdot x) \cdot dx
\]

The accumulated load at this section is found by integration from \( x \) to the tip:

\[
F(x) = \int_x^{1} \cdot dF = \mu \cdot 0' \cdot \Omega \cdot L^2 \cdot (x + h) \cdot c(x) \cdot dx
\]

The axial stress is

\[
\sigma = \frac{F(x)}{S' \cdot t \cdot c(x)}
\]

and the axial strain is

\[
\varepsilon = \frac{\sigma}{Ea} = \frac{F(x)}{Ea \cdot S' \cdot t \cdot c(x)}
\]

Coupling gives rise to a shear strain of

\[
\gamma = \eta \cdot \varepsilon = \frac{\eta \cdot F(x)}{Ea \cdot S' \cdot t \cdot c(x)}
\]
which gives rise to a twist rate (according to Batho-Bredt) of

\[
\frac{d\beta}{dx} = L \cdot \frac{S}{2A} \cdot \gamma(x) = L \cdot \frac{S'}{2A'} \cdot \gamma(x) = \ldots
\]

\[
\frac{d\beta}{dx} = L \cdot \frac{S'}{2A'} \cdot \frac{1}{c(x)} \cdot \frac{F(x)}{Ea} \cdot S' \cdot c(x)
\]

(3.6.1)

\[
\ldots = \Omega \cdot 2 \cdot \rho \cdot \frac{1}{Ea} \cdot \frac{L^3}{c_{ref}} \cdot \frac{(1 - c'x_{ref})}{2A'} \cdot S' \cdot x (x + h) (1 - c'x) dx
\]

The twist at \( x \) is found by integration from the root to \( x \):

\[
\beta(x) = \int \frac{d\beta}{dx} dx = K V_{bl} \Omega \cdot 2 \cdot (x + h) (1 - c'x) dx
\]

This integral expression can be simplified considerably and assigned a name, so the twist distribution is

\[
\beta_{bl}(x) = \Omega \cdot 2 \cdot K V_{bl} V_{bl}(x)
\]

(3.6.2)

where 

\[
V_{bl}(x) = \frac{1}{6} x \cdot (3 + 6h - 2c' - 3h \cdot c') - 3h \cdot x - x^2
\]

\[
(1 - c'x)
\]

and 

\[
K V_{bl} = \rho \cdot \frac{\eta}{Ea} \cdot \frac{L^3}{c_{ref}} \cdot (1 - c'x_{ref}) \cdot \frac{S'}{2A'}
\]
3.6.3 Blade twist due to centrifugal load on added tip mass

A concentrated tip-mass has already been looked at, but in real situations, even using materials of high density, the tip mass may take up a considerable volume and therefore treating it as a point mass would be inaccurate.

Various schemes were considered for the construction of the tip-mass (always involving lead). These included a bag of lead shot, interleaving lead sheet into the blade material, strips of lead sheet. The final design incorporated a piece completely filling the tip region of the blade, consisting of lead powder set in a polyurethane matrix. However, the mass distribution of this arrangement is not straightforward, so for the sake of simplicity, an approximation is made. According to this, it is modelled as if it were as skin of high density coincident with the blade skin, and therefore having a linear distribution.

\[
\frac{dm}{dx}_{\text{tip}}(x) = \mu_{\text{tip}} \cdot L \cdot (1 - c \cdot x) \quad \text{for } x \geq x_{\text{tip}} \quad \text{and } = 0 \text{ for } x < x_{\text{tip}}.
\]

where \( \mu_{\text{tip}} = \rho_{\text{tip}} \cdot t_{\text{tip}} \cdot S \cdot c \cdot 0 \)

and \( t_{\text{tip}} \).
where \( K_{V_{tip}} = \frac{\rho}{\rho \cdot t} \cdot K_{V_{bl}} \)

Taking the two integrals separately,

\[
\int_0^x \frac{1}{(1 - c'x)^2} \, dx = \frac{x}{1 - c'x}
\]

and the second, being a constant can be given a label;

\[
k_{tip} = \int_{x_{tip}}^1 (x + h) \cdot (1 - c'x) \, dx
\]

\[
= \frac{1}{6} \left[ (3 - 3 \cdot h \cdot c' - 2 \cdot c' + 6 \cdot h) \ldots + 6 \cdot h \cdot x_{tip} + 3 \cdot (1 - h \cdot c') \cdot x_{tip}^2 + 2 \cdot c' \cdot x_{tip}^3 \right]
\]

The twist distribution can then be expressed as

\[
\beta_{tip}(x) = \Omega \cdot K_{V_{tip}} \cdot k_{tip} \cdot \frac{x}{1 - c'x} \tag{3.6.3}
\]

where, to recap, \( K_{V_{tip}} = \frac{\rho}{\rho \cdot t} \cdot K_{V_{bl}} \)

For values of \( x > x_{tip} \), returning to the variable expression for the twist rate

\[
\frac{d}{dx} \beta = \Omega \cdot 2 \cdot K_{V_{tip}} \cdot \frac{1}{(1 - c'x)^2} \int_{x_{tip}}^1 (x + h) \cdot (1 - c'x) \, dx
\]

the twist distribution is found from the pitch at \( x_{tip} \) and integrating the twist rate as far as \( x \)
where \( \rho_{\text{tip}} \) is the density of the added mass and \( x_{\text{tip}} \) is the value of \( x \) at which the added mass starts.

For values of \( x > x_{\text{tip}} \), the load due to the tip mass follows the same pattern as that due to the blade skin.

\[
F(x) = \int_{x}^{1} \Omega \cdot 2 \cdot r(x) \, dm = \mu \cdot x_{\text{tip}} \cdot \Omega \cdot L^2 \cdot \left[ (x + h) \cdot (1 - c' \cdot x) \right] \, dx
\]

\[
F(x) = \mu \cdot x_{\text{tip}} \cdot \Omega \cdot L^2 \cdot \left[ \frac{3}{6} \cdot (1 - h \cdot c') + 6 \cdot h - 2 \cdot c' \ldots \right] + \frac{-6 \cdot h \cdot x - 3 \cdot (1 - h \cdot c') \cdot x^2 + 2 \cdot c' \cdot x^3}{1}
\]

For \( x \leq x_{\text{tip}} \), the load remains constant at its \( x_{\text{tip}} \) value.

\[
F_{\text{tip}} = \mu \cdot x_{\text{tip}} \cdot \Omega \cdot L^2 \cdot \left[ \frac{3}{6} \cdot (1 - h \cdot c') + 6 \cdot h - 2 \cdot c' \ldots \right] + \frac{-6 \cdot h \cdot x_{\text{tip}} - 3 \cdot (1 - h \cdot c') \cdot x_{\text{tip}}^2 + 2 \cdot c' \cdot x_{\text{tip}}^3}{1}
\]

Continuing with \( x \leq x_{\text{tip}} \), the twist rate, from Eqn. (3.6.1) is given by

\[
\frac{d \beta}{dx} = \frac{L}{2 \cdot A \cdot t \cdot E a} \cdot \frac{F_{\text{tip}}}{c(x)^2}
\]

Integrating from the root to \( x \) gives the twist as far as \( x \)

\[
\beta(x) = \int_{0}^{x} \frac{d \beta}{dx} \, dx = \frac{L}{2 \cdot A \cdot t \cdot E a} \cdot \frac{F_{\text{tip}}}{c(x)^2} \cdot \int_{0}^{x} \frac{1}{c(x)^2} \, dx = \ldots
\]

\[
\ldots = K \cdot V_{\text{tip}} \cdot \Omega^2 \cdot \left[ \int_{x_{\text{tip}}}^{1} (x + h) \cdot (1 - c' \cdot x) \, dx \right] \cdot \frac{1}{(1 - c' \cdot x)^2} \, dx
\]

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If the double integral expression is separated into two integrals, from 0 to \( x_{\text{tip}} \) and another from \( x_{\text{tip}} \) to \( x \) it can be recognised as the same expression which was used for the twist due to the blade’s own mass, and was defined for convenience (3.6.2) in that section as \( V(x) \). Therefore

\[
\int_{x_{\text{tip}}}^{x} \frac{1}{(x + h)(1 - c'x)} \, dx = V(x) - V(x_{\text{tip}})
\]

and

\[
\beta_{\text{tip}}(x) = \beta_{\text{tip}} + \Omega \frac{2}{\kappa_{\text{tip}}} \left( V(x) - V(x_{\text{tip}}) \right)
\]

From (3.6.3),

\[
\beta_{\text{tip}} = \Omega \frac{2}{\kappa_{\text{tip}}} \frac{x_{\text{tip}}}{1 - c'x_{\text{tip}}}
\]

so the twist distribution can be tidied up to give

\[
\beta(x) = \Omega \frac{2}{\kappa} \left( V(x) - k_{\text{base}} \right)
\]

where \( k_{\text{base}} = V(x_{\text{tip}}) - k_{\text{tip}} \frac{x_{\text{tip}}}{1 - c'x_{\text{tip}}} \)
Putting the two halves of the distribution together, define a new function, \( V_{tip}(x) \)

\[
V_{tip}(x) := \begin{cases} 
  \frac{x}{1 - c' \cdot x} 
  & \text{if } x \leq x_{tip} \text{ and } k_{tip} \cdot \frac{x}{1 - c' \cdot x} \cdot V(x) - k_{base} \\
  & \text{if } x > x_{tip}
\end{cases}
\]

(3.6.4)

Putting together the twist due to the tip-mass with that from the blade’s own weight as well as the set pitch, \( \theta_0 \),

\[
\beta_{tip}(x) := \Omega \cdot 2K \cdot V_{tip} \cdot V_{tip}(x)
\]

There are some additional effects on blade twist which need to be taken into account. The most significant of these is the so-called propeller effect. This arises when the mass of a blade or other object is distributed either side of the centre line and out of the rotor plane. The centrifugal forces on the distributed mass give rise to a couple which tends to flatten it out into the rotor plane. For a derivation of the method of estimating this effect, see Appendix A1.

To give an idea of the order of magnitude of this effect, based on the same parameters as in the rest of this section, at a rotor speed of 1200 rpm, the blade twists through 28.1° (from a set pitch of 4°) of which 8.7° comes from the blade’s own mass and 19.4° from the added tip-mass. The untwist is 5.8° to a first order approximation.
3.6.4 Influence of blade parameters and scaling on twist

Re-examination of the equations derived above which govern the development of blade twist, shows the effects that were expected. Looking at the inherent twist, due to the blade’s own mass, obviously twist is favoured by high density and a high Lekhnitski coefficient and disfavoured by a high effective modulus. It is favoured by a narrow and thin blade and by low or zero taper. The thickness of the blade skin has no effect as regards twist due to self-mass. Obviously, the stretching load increases as the skin is thickened, but the thicker skin also resists the stretch more and responds less.

It would appear from the equation that twist rises with the cube of blade length, but care should be taken in interpreting this – it is important to compare like with like.

If the blade is scaled up in a linear fashion across all dimensions, there would then only be a square-law rise in twist constant, because of the rise in chord. However, for the aerodynamics to remain as similar as possible (apart from Reynold’s number), the tip speed ratio would stay constant. This would entail the rotor speed lowering in
proportion to the reciprocal of blade length. As the rotor speed is present in the equation as a square, the end-to-end twist would remain the same.

\[ \beta_{bl}(x) = \Omega^2 K V_{bl} V_{bl}(x) \]  

(3.6.2)

where

\[ K V_{bl} = \rho \frac{\eta L^3}{Ea c_{ref}} \left(1 - c'x_{ref}\right) \frac{S'}{2A'} \]

\[ V_{bl}(x) = \frac{1}{6x} \left(3 + 6h - 2c' - 3h'c' - 3h'x - x^2 \right) \]

Regarding the twist due to added mass, it is not difficult to reverse the relationship between the amount of added mass and the degree of twist. If a particular degree of twist is desired, it is possible to calculate the amount of lead that is needed. As an example, Figure 3.6.4 shows the degree to which tapered blades need more lead in order to achieve adequate twist. In particular, the requirement climbs more steeply as more tapered blades are considered.

![Figure 3.6.4 added lead requirements for different blade taper rates.](image)

The equations for twist due to added mass are essentially the same as for the blade shell mass with regard to blade geometry, except that twist goes up with added tip mass and the skin thickness now has an inverse effect. The equations are set up to
represent the tip-mass in terms of material density and the dimensions of the blade cavity. Clearly, though, the intention is not to fill up the cavity with as much lead as possible. Rather, it is normal practice to calculate the amount of added mass required for a desired twist characteristic. The effect, however, of increasing skin thickness, for instance if greater bending stiffness were required, would be to restrict the blade cavity. The proportional increase in tip-mass would eventually become hard to accommodate and, as it extended further down the blade, would give progressively less effect for each increment in mass. Stations far from the blade tip experience a proportionally smaller centrifugal field.

A somewhat surprising inference is that with the effect of either blade mass or added mass, the twisting effect is unchanged by scaling up all dimensions of the blade. This only makes sense in the light of the scaling invariance being true for bending behaviour (see Chapter 5) in that loads rise with the square of scalar dimension, whilst the stiffness rises linearly. The elastic deflections thus tend to rise linearly and the stresses and strains remain constant. Similarly, the natural frequency falls with the reciprocal of linear dimension and therefore in proportion to rotational speed. Centrifugal stiffening thus follows exactly the same behaviour with scale as elastic stiffness.

The reality of scaling, however, is somewhat different. Over a small range of machine sizes, it is common to find similar blade construction. However, in reality, very few machines of greater power than a few kilowatts would have blades constructed as a simple shell. Generally, larger blades are made with a thin skin inside which is a spar which carries most of the load. There are too many reasons for this difference of construction to go into here, but one of the many is certainly economics: Laying-up a full scaled up thickness of load bearing composite over the entire blade surface and filling the cavity with foam would be uneconomical in terms of both material and labour.

Small blades are also not generally constructed as a shell but are often solid, manufactured by injection moulding, extrusion or pultrusion. If they have reinforcing fibre more likely to be strands rather than continuous fibre. A shell construction would be far too labour intensive for the volumes manufactured at the small scale.
Thus, although the equations governing twist are scale invariant, the practicalities of blade design mean that the particular approach adopted in this work would only be likely to apply over a limited range of rotor sizes. Having said this, other groups and companies are considering the principles of coupled twist to have potential. One group is looking at the potential for a separate, non-load bearing torsion tube to operate tip brakes in a particular wind turbine design (van den Berg et al. [31]).
4. Rotor Performance Simulation

4.1 Basis of the whole-rotor model

Some important guidelines for the design of a self-twisting rotor can be deduced from a separate understanding of the behaviour of the material, the relationship between loads and twist in the blades and of rotor aerodynamics. However, these can not be safely relied upon without some combination of the three into a simulation of how a wind turbine would behave with self-twisting blades. Karaolis [31] chose to simulate his wind turbine by writing a combined piece of software which simulated both the blade structure and the aerodynamics. I have chosen to do things differently. Even Blade-element momentum theory codes are not simple to write reliably as there are frequently difficulties in making the iterative procedures converge. Since there are numerous existing codes available either freely or commercially, I have employed one of these. Garrad Hassan kindly made available their Blades software at no cost. This code is relatively simple to use.

There are of course disadvantages in using a ready-made package. Clearly, such software cannot be 'called' as if it were a subroutine for another package. This ruled out the possibility of setting a particular set of wind conditions and finding the behaviour of the wind turbine in those conditions by iteration.

For the self-twisting bladed rotor, the principal variable governing blade twist is the rotor speed, though there is a small amount of twist due to aerodynamic loads. It is relatively simple to calculate the blade twist distribution for a given rotor speed. A simple routine was written using Matlab to assemble all the rotor design parameters and calculate the blade twist distribution for a series of rotor speed values. These were then used as the basis for rotor aerodynamic calculations over a range of wind
speeds, with the calculated blade twist distribution for each rotor speed input as the blade configuration.

Two slightly different approaches were used at different stages of the project. In the earlier stages, 'Blades' was used to calculate tables of power coefficient, torque coefficient and thrust coefficient against tip-speed ratio for a range of twist distributions. Linear interpolation was used to find the tip-speed ratio corresponding to zero power and this was taken to correspond to the runaway condition. This approach could also be used for finding maximum torque and maximum power coefficients.

Using the definition of tip-speed ratio, $\lambda = \frac{\Omega R}{U_\infty}$, and the fact that the rotor speed has already been set, gives $U_\infty = \frac{\Omega R}{\lambda}$. This then provides a table of wind speeds and runaway rotor speeds, unfortunately not, as might be desirable, with constant intervals in wind speed. Tables of maximum power and maximum torque against wind speed are generated in a similar manner from the corresponding dimensionless coefficients.

The alternative method, used in later stages of the study, was to calculate tables of rotor power for a given rotor speed over a range of wind speeds. Performing such calculations over a range of rotor speeds allows the creation of complete tables of rotor torque and power for the two variables. By linear interpolation, it becomes possible to find, for a given wind speed, the rotor speed at which no power or torque is generated i.e. the runaway speed. It is also possible to find for each wind speed the maximum power, the rotor speed at which it generates maximum power, and for wind speed over the rated value, the rotor speed at which it generates rated power.

With either approach, once the actual correspondence between wind speed and rotor speed is ascertained, whether for peak power, peak torque, rated power or runaway, the pair of values is used as the basis for new calculations of aerodynamic coefficients such as axial and tangential induction factors, flow angle, angle of attack and structural information such as blade bending moments in the flap and lead-lag directions.
These methods were used to test the soundness of previously gained design rules with regard to the blade shape. Thus, families of blades were generated to look at the effect of blade mean chord, blade taper, thickness to chord ratio.

Another part of the simulation procedure, required later on for Chapter 5, is to calculate the blade loading. This information can be obtained in two ways. The first is to calculate it from a reconstruction of the aerodynamic state - using the aerodynamic angle of attack at each blade station to reconstruct the lift-drag triangle of forces. Alternatively, it can be obtained directly from the 'Blades' software. The problem with the latter approach is that the software was designed for relatively large wind turbines and the loadings are quoted in kN/m. On this scale, the loadings on a 3m diameter machine do not register at all. In order to obtain any sensible values, the wind turbine dimensions are scaled up by a factor of 10× and the rotor speed is scaled down by a similar factor. The loading values obtained from the software appear scaled up by 10× relative to the actual machine, the shear forces appear scaled up by 100× and the bending moments by 1000×.

4.2 Blade design principles

Virtually any blade shape will work to some extent as a wind turbine blade if it is fixed on a hub at a suitable pitch angle and allowed to turn. However, it is also true that a systematic approach produces much better designs. Empirical rules of thumb were developed a long time ago by Smeaton, based on experiments on scale models. However the first approach based on a theory of how the wind turbine generates power was developed by Glauert [20]. He developed blade-element momentum theory to describe the performance of propellers and autogiros. (a type of helicopter in which the forward thrust comes from propellers and the lifting rotor runs as a windmill from the relative flow caused by forward motion.)

Just as Betz showed that for an actuator disc, the optimum power is produced at an induction factor of \( \frac{1}{3} \), which gives a power coefficient of \( \frac{16}{27} \), Glauert found a method for optimising the axial induction factor taking into account the tangential induction factor. This leads to a procedure for designing the chord and pitch distribution for an
ideal blade, which has been set out in slightly different ways with various modifications and refinements by a number of authors, especially Jansen and Smulders [27], Wilson Lissaman and Walker [46], Lysen [34] Sharpe in Freris [17], Sharpe [41] and Wilson in Spera [42]. I have predominantly used Lysen's and Jansen & Smulders' approaches.

The first parameter to be chosen is the design tip-speed ratio, \( \lambda_R \) though to some extent the choice of this depends on the type of aerofoil to be used and on how accurately it can be made, how sensitive it is to accumulation of dirt and the order of magnitude of the Reynold's number. All these factors influence the value of lift-to-drag ratio which can be relied upon, which in turn influences the tip-speed ratio choice. Because of the blade shape requirements already mentioned in the previous chapter, and the requirement for high centrifugal loads, I have opted to design the turbine for a relatively high tip-speed ratio of 8. Also, in order to allow a straight-through rotor design in a single piece, I have opted for a two blade design.

Once this has been decided, the speed ratio at every other station along the blade is fixed.

\[
\lambda_r = \lambda_R \cdot \frac{r}{R}
\]

There is then an optimum value of axial induction factor which varies from \( \frac{1}{4} \) to \( \frac{1}{3} \) (Sharpe, [41]). However, with a high tip-speed ratio, the variation is relatively small - in effect, the ideal value is close to \( \frac{1}{3} \) for all values of \( \lambda_r > 2.5 \) and even at \( \lambda_r = 0.75 \), \( a = 0.31 \). For the sake of simplicity, it can be set to \( \frac{1}{3} \) for all blade stations. The effect of this is to set the ideal flow angle \( \phi \) relative to the rotor plane to be

\[
\phi = \frac{2}{3} \tan^{-1} \left( \frac{1}{\lambda_r} \right).
\]

(The expression given in Jansen and Smulders [27], \( \lambda_r = \frac{\sin \phi \cdot (2 \cos \phi - 1)}{(1 - \cos \phi) \cdot (2 \cos \phi + 1)} \) is exactly equivalent to this, using trigonometric identities.)

From the design value of flow angle \( \phi \), the momentum theory dictates the total amount of lift for that section. This is equated to the product of the number of blades \( n_b \) with
the chord \( c \) and the design lift coefficient \( C_{ld} \). A choice must then be made of the design lift coefficient. The chord is calculated from the total lift.

In general most authors prescribe a strict choice of the highest lift-to-drag ratio available at the design Reynold's number. The angle of attack and lift coefficient corresponding to this are then selected as the design values. As Giguère and Selig [19] point out, and as set out in more detail in Chapter 6, this approach does not work for low Reynold's numbers as the drag varies to a great degree with Reynold's number. This means that the interdependence of the Reynold's number and the design lift must be taken into account. Unfortunately, their improved approach had not been published yet at the time of this phase of the project, and so the older approach was followed.

From Jansen and Smulders [27]

\[
c = \frac{8\pi r}{n_b C_{ld}} (1 - \cos \phi)
\]

If the design angle of attack is \( \alpha_d \), corresponding to the design lift, then the blade setting angle \( \beta \) at that station is given by \( \beta = \phi - \alpha_d \).

This procedure tends to produce a blade profile which is quite complex to manufacture, with non-linear taper and non-linear twist. It is common practice to 'linearise' this in order to ease manufacture, and in general this does not cause very serious loss in performance. In terms of power performance, even for a fully optimised blade, the power generated per unit length of blade is much greater near the tip than near the root - it is approximately proportional to radius. Clearly then, departing from optimal design near the root will cause less loss of performance than near the tip. To some extent, reducing both the chord and the set angle near the root in order to have a linear distribution in each actually compensate for each other up to a point - reducing the set angle increases the angle of attack and thus the lift coefficient (until stall) and reducing the chord compensates by reducing the area generating lift.

It is even possible to have an untwisted blade, at a constant setting angle, which has a chord distribution optimised for it (within the constraint of linear taper).
When performing such partial, constrained optimisations, it would of course be possible to minimise the actual performance loss by calculating this at each station. This is, however, somewhat involved and for the sake of simplicity I have adopted a more straightforward procedure.

Looking at plots of power density as a function of blade station, taking into account tip-loss factors, it makes sense to choose a station at approximately 80% radius as having the highest value. The chord and setting angle are fixed at the optimum values at this point, whilst at all other stations the angle of attack is then allowed to vary from the optimum. The optimum twist distribution is then calculated for a chosen linear chord distribution (taper) or the optimum chord distribution for a chosen linear twist distribution. These are, of course, non-linear. A straight line is then fitted to whichever of these was curved. This is constrained to pass through the optimal value at the reference station, and the squared error terms are minimised according to a weighting scheme, with the weighting proportional to radius. Of course, the weighting should really follow more closely a true power density distribution and, in particular, should diminish for stations outboard of the reference station as the proximity to the tip vortex reduces power generation. However, the reference station is quite close to the tip and so the errors are likely to be quite small anyway.

The derivation for this calculation is set out in Appendix A2.

Let the ideal chord distribution be designated \( C(x) \), whilst the fitted linear distribution is given by \( c(x) = C(1 - c'x) \) in the usual way. However, the chord is constrained to a value of \( c_{\text{ref}} \) at a blade position of \( x_{\text{ref}} \). The non-dimensional hub radius is given by \( h = r_{\text{h}}/L \).

The taper coefficient, \( c' \), is given by

\[
c' = \frac{c_{\text{ref}}}{2} \left[ \frac{(6h + 3)x_{\text{ref}} - (3h + 2)}{(6h + 4)x_{\text{ref}} - (4h + 3)} \right] - \frac{6}{12} \left[ \frac{h x_{\text{ref}} I_0 + (x_{\text{ref}} - h)I_1 - I_2}{x_{\text{ref}} I_0 + (x_{\text{ref}} - h)I_1 - I_2} \right]
\]

where \( I_0, I_1 \) and \( I_2 \) are the integrals involving the ideal chord distribution as given below. These need to be carried out numerically as there is no closed form solution.
\[
I_0 = \int_0^1 c_a(x) \, dx \\
I_1 = \int_0^1 x \cdot c_a(x) \, dx \\
I_2 = \int_0^1 x^2 \cdot c_a(x) \, dx
\]

The hub and tip chord values are easily calculated from \( c' \).

\[
c_0 = \frac{c_{\text{ref}}}{1 - c' x_{\text{ref}}} \quad \text{and} \quad c_1 = \frac{c_{\text{ref}}}{1 - c' x_{\text{ref}}} \cdot (1 - c')
\]

Further refinements are possible. For any imposed twist distribution which reduces the set angle in the root region (i.e. most likely choices of distribution), parts of the blade will be stalled to some extent even at the design tip-speed ratio. It is arguably preferable for these stations to be omitted from the line fit. This is relatively easily done, though the details will not be given here.

An example calculation is presented here, performed using Microsoft Excel. A blade twist of 5.5°/radius has been imposed. Figure 4.2.1 shows the ‘ideal’ shape and two linear tapered fits, one with all blade stations taken into account and one omitting the stalled stations.

The rotor diameter is 2.9m. Assume a chord of 8cm. If the design tip-speed ratio is 8 and the design wind speed is 10m/s, then the design Reynolds number is \( 8 \times 8 \times 0.1 / 1.5 \times 10^{-5} = 2 \times 10^5 \). For a NACA4412 aerofoil, using data from a Reynolds’s number of 3.3 \times 10^5, the highest L-D ratio is 64 at 6° angle of attack and a lift coefficient of 0.9. If the reference station is taken as 80% of the rotor radius (which is 1.164m), then the speed ratio here is \( \lambda_r = 8 \times 0.8 = 6.4 \). The design flow angle is \( \phi_d = \frac{2}{3} \tan^{-1} (1/6.4) = 5.9° \). The setting angle is \( \beta = \phi - \alpha = 5.9° - 6° = -0.1° \). If the rotor is to have 2 blades, then the chord is

\[
c = \frac{8\pi r}{n_b C_{l,d}} \cdot (1 - \cos \phi) = \frac{8 \times 3.14 \times 1.16}{2 \times 0.9} \cdot (1 - \cos 5.9°) = 0.086 \text{ m} = 8.6 \text{ cm}
\]

The remainder of the chord distributions have been calculated with an Excel spreadsheet giving the following results. The ideal shape tapers from 35cm at an angle of 28° at the hub to 7cm at -1° at the tip. Imposing a twist of 6°/radius gives
5.4° of twist over the length of the blade and requires an ideal taper (non-linear) from 24cm to 7cm. Linearised based on the whole blade, this becomes 18cm to 6cm and linearised only on the unstalled part of the blade gives a blade tapering from 16cm to 6.5cm. The full chord distributions can be seen in Figure 4.2.1.

If it is considered that the outer 65% of the blade sweeps out over 85% of the rotor area, it becomes clear that it is worth 'sacrificing' the stalled, inboard parts of the blade - they would tend to push up the amount of taper excessively. With the linear taper calculated without the stalled parts of the blade, the rest only deviates from the ideal chord distribution by at most 4.5mm, whereas basing the fit on the whole blade means that unstalled stations deviate by as much as 10mm in places.

![Figure 4.2.1](image_url)  
*Figure 4.2.1* Calculated blade chord distributions for Glauert ideal shape, constrained ideal shape, and two linear fits

For most purposes, in this project, the design has been relaxed further. The above procedure only takes aerodynamics into account, whereas the extent to which the blades successfully develop twist is just as important, if not more so. For much of the project, the effect of different design parameters on speed regulation was looked at. For these purposes, the chord and setting angle were calculated for the reference station by the above procedure but the dimensions of the rest of the blade were set by other criteria.
4.3 Aerofoil data sets for rotor simulation

Before any simulations could be made of any rotor configurations, it was necessary to find a suitable aerofoil data set. In order to allow for comparisons of different profiles one parameter at a time, it was decided to base the first blade designs and simulations on the so-called NACA four digit profiles. These were designed according to a consistent mathematical scheme as combinations of related camber and thickness distributions. They also have the advantage of having been widely used and of having been tested widely in wind tunnels. The later series of aerofoils produced by NACA, the 5 digit and 6 digit series were not considered as suitable. Their development was aimed at improving performance primarily in high Reynold’s number flows. Furthermore, the generation of the profile shapes followed a less clear-cut scheme, making it more difficult to compare directly the effects of different blade shape parameters.

The problem that arose was the lack of data on these (and most other) aerofoils covering angles of attack outside the normal aeronautical operating range. For aeronautical purposes, it is sufficient to know where stall begins to have an effect and to know the qualitative nature of that stalling behaviour - whether the aerofoil stalls suddenly and severely or gently and with warning and more quantitatively, the angle and lift at the beginning of incipient stall, when the lift slope loses linearity, and the angle and lift of maximum lift. These data are widely recorded. See Abbott and von Dönhoff [1], Miley [36], Anderson [2], Riegels [39] and others, especially NACA reports.

For many wind turbine applications, in particular when stall is employed as the means of power control and speed regulation, it is essential to have more detailed data regarding aerofoil behaviour in the stalled region, beyond maximum lift and much work in recent years has been concerned with such data. It has particularly been concerned with hysteresis-type behaviour and the effects of 3-D flow.

For any wind turbine simulation, there is a need for some data at very large angles of attack up to 90°, in order to examine the behaviour of the turbine when it is deeply stalled, at and immediately following start-up. These data exist only for a few profiles.
but it is known that most profiles behave in a similar manner so the existing data can be used with reasonable confidence.

For the purposes of this project, the need is for data covering lower Reynold’s numbers than usual and for data covering very low angles of attack including some negative values. This is generally not available as it is not generally of interest to aerofoil designers. The data sometimes exist to indicate the start of a trend and it is reasonable to assume that the behaviour at these angles of attack should resemble to some extent stalling behaviour at positive angles. However, more detail is needed than this. It is essential to know when the lift line departs from linearity, when it reaches a minimum, how it behaves after that point, and most importantly, the drag behaviour.

The only data I was able to find covering such a wide range of angles of attack for a cambered aerofoil, came from a draft copy of a RISØ report which gave no reference for the data. (Petersen [38]). No final versions of the report appear to exist, and so I can not vouch for the provenance of the data. However it is plausible, so I have used it for most of the simulations.

Some earlier simulations used data extracted from NACA reports by Miley [36], extended with existing data covering large angles which came with the BLADES software.

4.4 Effect of blade set angle on rotor behaviour

To start with, it is worth looking at a comparison between an unregulated rotor and the effect of pitching the blade. This can be done in either of two directions, ‘feathering’ and ‘stalling’ (corresponding respectively to increasing and decreasing the pitch angle relative to the rotor plane).

In these simulation runs, the aerofoil data used was that of a NACA4415 from RISØ, (Petersen [38]). An untwisted blade was used with a taper rate of $c' = 0.332$, with the chord distribution based on a design angle of attack of $6^\circ$. The 1.302m long blade tapered from 0.1095m at the root to 0.0731m at the tip, and was attached to a hub of
radius 0.148m. The ‘starting’ position was a pitch angle of 2°, which gives the optimum $C_p$.

Simulations were carried out at blade pitch angles ranging from -16° to 24°. Calculations were made solely to produce tables of power, torque and thrust coefficients as a function of tip-speed ratio. No effect of twist development was looked at. From the tables, it was possible to make estimates of maximum $C_p$ for each blade setting and of the runaway tip speed ratio and the corresponding thrust coefficient.

The changes in $C_Q$, $C_p$ and $C_T$ curves with changing blade set angle can be seen below in Figure 4.4.1, Figure 4.4.2 and Figure 4.4.3.

![Figure 4.4.1 Changes in torque coefficient curve with changes in pitch angle](image-url)
Figure 4.4.2 Changes in power coefficient curve with changes in pitch angle

The unregulated rotor can be assumed to have constant characteristics, apart from the effect of Reynold's number (which is ignored here for lack of data). What this means is that all the characteristics of the rotor scale appropriately with wind speed. For similar conditions, the rotor speed rises in proportion with wind speed, the loads, such as thrust, rise with the square of wind speed and power with the cube. The curves of $C_p$, $C_Q$ and $C_T$ against $\lambda$ remain constant.

Looking now at the effect of blade pitching action, the curves of $C_p$ against $\lambda$ in Figure 4.4.2 clearly show the maximum $C_p$ to be reduced by pitching the blade in either direction. However, as can be seen from the locus of maximum $C_p$, marked on the graph, they have differing effects on the rotor behaviour. Pitching towards feather reduces the tip-speed ratio at which $C_{p\text{max}}$ occurs, whereas it stays much more constant with pitching towards stall. It will also be noted that the most negative pitch angle shown is $-12.75^\circ$ rather than $-16^\circ$. This was the last pitch angle that showed any part of its $C_p$ curve to be positive. In the positive (feathering) direction, there was always some positive region on the curve.
Figure 4.4.3 Changes in thrust coefficient curve with changes in pitch angle

The curves of of $C_P$ against $\lambda$ in Figure 4.4.3 clearly show how positive pitch angles reduce the thrust coefficient, whereas negative angles increase it. Also marked on the graph is the locus of the runaway point, Ctr, showing how the thrust coefficient at runaway varies with pitch angle.

It is possible to see these effects more directly by plotting runaway tip-speed ratio and runaway thrust coefficient against pitch angle. (See Figure 4.4.4)
If it is assumed that the blade pitching action is actuated by a passive pitching mechanism, based somehow on centrifugal loads, then it is possible to estimate the behaviour of the wind turbine as wind speed varies for both the feathering and the stalling machine, and to compare them with an unregulated turbine. The pitching action has been assumed to be proportional to rotor speed squared and produces a pitch change of ±10° at 1265 rpm. The projected behaviour can be seen in Figure 4.4.5 and Figure 4.4.6.

Figure 4.4.4 Changes in runaway tip-speed ratio and runaway thrust coefficient with changes in pitch angle
Pitch-to-stall versus pitch-to-feather

Clearly both methods of pitching are able to regulate the rotor speed quite effectively. Initially, before much pitching action has taken place, both types of pitching rotor
allow the rotor speed to rise in a similar manner to that of the unregulated rotor. However, it soon levels off to the point where it increases only very gradually with further increases in wind speed. The feathering rotor only reaches 1390 rpm at a wind speed of 28 m/s compared to 3110 rpm for the unregulated rotor, whilst the stalling rotor reaches 1500 rpm at 27 m/s, a similar degree of regulation.

With the feathering rotor, the speed continues to increase monotonically with wind speed, though only gradually. Furthermore, at all wind speeds, it can be expected to be able to generate power. The thrust load is also dramatically reduced compared to that on the unregulated rotor, being 0.19 kN at 28 m/s compared with 3.96 kN, though it also continues to increase with wind speed in a roughly linear manner.

Some explanation is needed for the fact that the ‘stalling’ curve ends at quite a low wind speed. The last pitch angle for which positive values of $C_p$ and $C_Q$ exist is -12.75° and this is also, therefore, the last point for which a runaway speed can be calculated. On the pitch schedule used, this corresponds to a rotor speed of 1540 rpm and a wind speed of 32 m/s. Beyond this point, it is impossible for there to be a stable increase in rotor speed as there is no torque available and because any such increase would cause an even greater pitch into stall.

It is likely that at such higher wind speeds, the rotor tracks the other zero-crossing point of the $C_Q/\lambda$ curve, with the blade fully stalled. For an unregulated rotor, none of the left hand side of the curve, with a positive gradient, can ever be a stable operating point, as any deviation from the operating point is amplified. An increase in rotor speed results in a positive nett torque and leads to runaway acceleration. A decrease in rotor speed results in a negative torque and leads to the rotor slowing down further. For the pitch schedule being used, however, this lower crossing point does represent stable operation, as any slowing down causes a pitch change away from stall, resulting in an increase in torque, whilst any speeding up causes a decrease in torque.

The fully stalled condition has been plotted as a separate but similar curve marked ‘fully stalled’, continuing the ‘stall-regulated’ curve for both rotor speed in Figure 4.4.5 and for thrust load in Figure 4.4.6. It can be seen that, once fully stalled, the stall-regulated rotor gradually falls in rotor speed with increasing wind speed and the
thrust load also falls, though at all wind speeds it is higher than that for feathering, reaching a maximum of 4.1 kN compared to 0.44 kN for wind speeds up to 70 m/s—nearly ten times as great.

Some caution is required in comparing the feathering and stalling rotors and in generalising the comparison. Neither rotor has been optimised for its mode of operation. In particular, choice of aerofoil sections can make a big difference to the ability of stall controlled turbines to regulate effectively and this applies to those which pitch towards stall as well as those of fixed pitch. (Tangler and Somers [43]) A suitable section can help to reduce the blade and tower bending loads considerably. Nevertheless, it is still true to say that in general they suffer more severe loads in high winds than do feathering turbines.

A further problem is that the actual behaviour of stalled aerofoils is notoriously hard to predict and therefore hard to rely upon. In particular, aerofoils in the 3-D and rotating situation of a wind turbine tend to exhibit smoother stalling characteristics compared to the 2-D wind tunnel situation as well as both stall hysteresis and stall delay (a rise in the maximum $C_L$), though some progress has been made in predicting these (Butterfield et al [5], Eggleston and Stoddard [14]).

The same arguments should apply in comparing twisting towards stall and towards feather.

**Static Deflections (rough estimate)**

There are other potential problems which could arise in this project by twisting towards stall. It will be seen later on in this chapter and in Chapter 7 that thin and therefore highly flexible blades are required for twist development. In Chapter 5 it will be shown that the blades’ stiffness comes almost entirely from the centrifugal stretching load. The deflections can be shown to be of the order of the load divided by the blade mass and the rotor speed squared. This parameter, the ‘centrifugal deflection estimate’ is plotted against wind speed in Figure 4.4.7. Ironically, it may be that a pitch-to-stall rotor regulates rotor speed rather too well at high wind speeds and does not have enough centrifugal stiffness to keep the blade bending deflections safe.
Incidentally, the deflection curve for the unregulated rotor, in which the deflection estimate remains constant, also illustrates why it does not make sense to design a blade based on bending-twisting coupling alone. Such a blade would also need to be highly flexible and therefore reliant on centrifugal stiffening. The induced twist would then be related to the bending strains developed and therefore to the bending deflections. Clearly, when the blade twist is not dependent on rotor speed, any one pattern of blade twist would remain stable over a wide range of wind speeds, speeding up in proportion. The blade deflection would barely change and nor would the blade twist.

![Figure 4.4.7 Predicted deflection estimates for unregulated and pitching bladed rotors](image)

**Vibrations**

It is also widely recognised that wind turbines have very little structural damping compared to their excitation. Most of the damping they have is aerodynamic and arises from the fact that a blade’s lift coefficient rises with an increase in angle of attack (Eggleston and Stoddard [14]). However, this diminishes and may even be lost when the blades are deeply stalled, which may result in a large response to excitation or may even lead to self-sustaining stall-induced vibrations (Jamieson and Rawlinson-Smith [26]). Neither will tend to occur easily with stiff blades and tower
but, as has already been mentioned above, the self-twisting turbine blades will need to be flexible and may thus oscillate dangerously when stalled.

For all these reasons, it was decided early on not to pursue the option of twisting towards stall and instead to concentrate all investigations on blades which would twist towards the feathered condition.

Choosing the set-angle of the blade

It is quite clear from Figure 4.4.2 that the optimum pitch angle for the blade in order to produce maximum power would be $2^\circ$. However, as has been mentioned earlier, speed regulation is also important. It can be seen from Figure 4.4.4 that, as the setting angle is increased from this value, the runaway tip-speed ratio initially increases before starting to fall. It is precisely a reduction in the runaway tip-speed ratio that is required in order to regulate the runaway speed of the rotor. Although this will not be done by pitching the whole blade but by twisting it, it is clear that it would be undesirable for this speed to increase before falling.

Increasing the setting angle to $4^\circ$ would mean that the untwisted blade already had the highest runaway speed and any twist increase from there would work to regulate the speed. The maximum power coefficient would only fall from 0.422 to 0.403, a fall of 4.5%, whilst the runaway thrust coefficient drops from 1.27 to 0.96. The tip speed ratio for maximum power actually goes up from 8.9 to 9.5. In order to generate the same power for similar conditions, it would entail an increase in blade length from 1.302 m to 1.332 m, an increase of less than 2.5% in length and of roughly 7.5% in volume of material. Even so, despite the longer blades there would actually be a 4.3% increase in the design rotor speed.

To what extent does it make sense to compromise further the ideal design of the rotor and blades by departing from the ideal setting angle? Clearly to do so significantly would be undesirable. However, questions of efficiency should not be overemphasised. When one is generating power from a fuel then it is clearly a priority to minimise wastage. However, the ‘fuel’ here is to all intents and purposes in
unlimited supply. Efficiency may be an issue but only in so far as it impacts on the economics of the plant and there are many considerations here.

On large wind turbines, power coefficient and efficiency are highly critical as the material cost of the turbine, and particularly the blades, make up a large proportion of the overall cost per unit generated. On small scale machines, however, whilst a good power coefficient is desirable, it could be argued that having to increase the blade length by 3% would not affect the overall cost of the turbine by very much, as tooling and manufacturing costs are relatively more significant. For small turbines, often operated on remote sites, questions of reliability are of far more concern than aerodynamic efficiency.

It is also worth noting that users of small, stand-alone wind turbines are often highly concerned that the turbine should have a low cut-in wind speed (within reason). Even small amounts of continuous power in low winds can make a significant difference to the need for starting the back-up power unit. The increase in the set-pitch of the blade is likely to improve its starting torque considerably, and therefore also its cut-in speed.

4.5 Effects on rotor behaviour of blade parameters

Further tests were carried out to compare the effects on blade twist, rotor speed regulation and blade loads of a number of blade design parameters. These were the effects of blade chord, thickness, taper and twist as well as the addition of mass to the tip and the effect of changes in the aerofoil lift and drag characteristic.

4.5.1 Aerofoil lift and drag characteristics

Based on our understanding of how a twisting blade can limit increases in rotor speed we know that the rate of change of lift with angle of attack is an important factor. Clearly, for a given amount of twist, if the lift falls by a greater amount then the rotor speed would be better limited. However the lift slope varies very little between different profiles. Again given a certain degree of twist, we would expect to benefit
from differences in the shape of drag characteristic. Clearly, if the drag rises more for a given twist with one aerofoil than with another, we might expect this to limit rotor speed better. However, given similar drag characteristics, it might also be possible to benefit more from drag by using a highly cambered aerofoil.

The effects of camber on an aerofoil’s lift characteristic are to raise the lift at zero angle, to increase the maximum lift but to lower the angle of maximum lift. The lift slope tends to stay the same. The effects on drag are smaller. The drag tends to rise overall by a small amount and the angle of minimum drag as well as the whole of the drag curve tends to shift to a higher angle but only by a very small amount.

The overall effect of these two trends is that the minimum drag occurs at a higher value of lift and the values of lift at which a particular value of raised drag are reached tend to be more positive. Thus the effect when a blade is pitched or twisted towards stall is that there is further to go, when the action is in the direction of feathering, there is less far to go. To put this another way, there is a greater rise in drag for a given twist angle. In the case of the twisting blade, there could be a problematic effect of camber. As the rotor speed regulates, the root region of the blade is still in a windmill state, albeit a stalled one. With high camber, the blade would not stall as soon and would experience a higher value of lift once stalled.

An additional benefit should be gained from a highly cambered aerofoil; for a given design value of lift, the required set angle should be larger which, as mentioned in the previous section, would tend to give a slight improvement in starting torque. (In the deeply stalled state at start-up, most aerofoils show little difference in characteristic from a flat plate).

It was not possible to explore these effects by using a range of actual aerofoils, as it is too difficult to find any that are closely enough related to give a clear comparison. Instead, one aerofoil was employed and its lift and drag curves were, in turn, altered. In one case, this was done to give it higher lift, as if it had a higher percentage camber, and in the other case, it was changed to give it a narrower range of low drag.

The base case was a set of data for a NACA4415 aerofoil (Pederson [38]) which covered data for a full 180°. To mimic the effect of an increase in camber, the linear
portion of the lift curve as well as the reflex curved portions representing initial stall were boosted by a value of 0.2. The curve outside this region was kept unchanged, as in deep stall most aerofoils behave in a very similar manner to a flat plate. The ends of the boosted curve were blended with the unchanged portion curve.

To mimic the effect of early onset of the negative angle of attack drag rise, the drag curve for the same aerofoil was altered. The portion covering just the drag rise was shifted by 6° to a higher angle of attack. The gap was then filled by fitting a quadratic curve to the ‘deep stall’ curve at lower angles of attack, which was then extrapolated to cover the ‘vacated’ positions in the table. The effect on the lift and drag characteristics can be seen in Figure 4.5.1 below. The shifted drag curve, like the original from which it came, has a very harsh drag rise and it may be that it is not always as sharp as this on all aerofoils (Anderson [2]) so an alternative ‘altered’ characteristic was also created with a more gradual drag rise.

![Figure 4.5.1 Baseline aerofoil data curves with modified curves](image)

The result of simulating the wind turbine performance with these different aerofoil characteristics can be seen below. It is possible to break down the aerodynamic data from the rotor simulation in order to examine the mechanisms operating to limit rotor speed at runaway. These calculations are somewhat crude and therefore exhibit a number of anomalies, such as the failure in places for the torque contributions to sum
to zero. Nevertheless they do give a picture of the speed limiting mechanism and whether it is dominated by lift or drag. Figure 4.5.2 to Figure 4.5.3 clearly show the difference in this respect between the two modified aerofoil characteristics and the original.

The original characteristic (Figure 4.5.2) shows that at runaway with an untwisted blade, lift is balanced by drag, but that as the blade twists, it is the lift in the outer half of the blade which reverses and takes over the rôle of balancing the largely unchanged lift in the inner half. The drag plays little part in controlling the rotor. The boosted lift profile (Figure 4.5.3) acts in a largely similar manner, with the rotor speed again limited by negative lift. However a clear difference is observed in the case of the narrowed drag characteristic (Figure 4.5.4). As might be expected, drag here dominates the mechanism.

![Figure 4.5.2 Drag and lift contributions to torque coeff. from inner and outer rotor halves at runaway (original characteristic)](image-url)
The following figures (Figure 4.5.5 to Figure 4.5.7 ) illustrate the effects of the different aerofoil characteristics on the runaway behaviour of the turbine as a function of wind speed.

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**Figure 4.5.3**  Drag and lift contributions to torque coeff. from inner and outer rotor halves at runaway (boosted lift characteristic)

**Figure 4.5.4**  Drag and lift contributions to torque coeff. from inner and outer rotor halves at runaway(narrowed drag characteristic)
Figure 4.5.5  Effect on runaway speed curve of aerofoil curve modifications

Figure 4.5.6  Effect on tensile normal strain of aerofoil curve modifications
In Figure 4.5.5 it can clearly be seen that, on its own the 'narrow drag' rise has a greater effect on runaway speed than the 'boosted lift' but that together they produce a greater effect still. It would thus clearly be desirable to employ a highly cambered aerofoil with a narrow region of low drag. However, there are very few aerofoils existing with high levels of camber, particularly amongst those designed for low Reynold's numbers. In Selig et al [40], only 3 sections are listed with camber greater than 5% of chord, Miley M06-13-128, Wortmann's FX63-137 and the NACA6409. There are, on the other hand, several candidates which appear to have a suitable early drag rise as lift falls. It is, however difficult to be certain of their drag rise, as most of the data do not go very far into this region. See Chapter 6 for more detail on aerofoil selection and data curves.

The tensile strain in the blade (Figure 4.5.6), which is determined predominantly by the rotor speed squared, is similarly affected by the aerofoil characteristic changes, though the square law results in the curves being further separated.

The plot of root bending moments (Figure 4.5.7) is less clear. The curves are somewhat 'jerky'. This is due to inaccuracies in the aerodynamic calculations. In particular, the aerodynamic routine sometimes fails to converge in the tip region of the
blade. In general, the errors in the overall characteristics of the rotor, and therefore the runaway speed behaviour, are relatively small. However, the tip region has a disproportionately large effect on bending moment and therefore larger errors are produced. Having said this, clear trends do emerge. In particular, the original aerofoil tends to limit rotor speed with negative lift, as explained above. This would appear to cause the root bending moment to change in direction at some point as the thrust on the outer half of the rotor disc, which acts in this case as a propeller, is actually in the upwind direction. This may be a problem if the upwind deflections so caused are too great.

4.5.2 Section thickness

Because of the relationship between coupled twist and the Batho-Bredt equation, it is to be expected that thin sectioned blades would develop more twist than thick sections, all other parameters being equal. This was tested out by employing the same aerodynamic characteristics throughout, but varying the mechanical properties of the blade. The perimeter coefficient $S'$ and area coefficient $A'$ were taken from the series of aerofoils NACA 4409, 4412 and 4415, giving values of the ratio $A'/S'$ of approximately 3%, 4% and 5% respectively. The effect is quite clear, that there is a significant improvement in regulation to be seen by employing thin blades. Of course, in reality, it is not possible to separate the section thickness from the aerodynamic properties, but it will be seen in Chapter 6 that thin sections are favourable aerodynamically too.
The next question which arises is whether employing such a thin section has any structural benefit. In terms of static loading, the dominant stresses in the blade skin are the centrifugal tensile stresses and the bending normal and shear stresses. The centrifugal tensile stresses are assumed to be constant around the section and are found simply from the centrifugal load and the material cross-section. Since there is very little change in the section perimeters between different thicknesses of profile, we would expect the centrifugal stress to follow essentially the same pattern as the rotor speed and this appears to be the case. Of course, the centrifugal load is proportional to the square of rotor speed, so the regulation appears to be less effective in all cases but the differences between the cases is heightened.

The aerodynamic bending stresses are more difficult to calculate. It is simple enough, given the aerodynamic loading, to calculate the bending moment distribution, but with highly flexible rotating blades, much of the stiffness arises centrifugally. As the blade bends, the centrifugal loading exerts a restoring bending moment. Thus a relatively small part of the applied bending moment should appear as elastic stress. For calculation of this aspect of blade behaviour, for some of the cases, see chapter 5.
There is no standard way of defining a blade taper parameter but in this instance a value of 0 represents a prismatic blade (i.e. untapered) and a value of 1 represents a blade with its tip approximately half the width of the root. The width is fixed at
approximately 80% radius, so an increase in taper largely represents an increase in width (as well as depth and enclosed area) at the root. The twist per unit length at any section is inversely proportional to the chord width when all other parameters are constant, so it would be expected that speed regulation would be better with the least tapered blades. This is clearly born out by the results shown in Figure 4.5.11.

![Figure 4.5.11 Effect on runaway speed curve of blade taper](image)

**Figure 4.5.11 Effect on runaway speed curve of blade taper**

Figure 4.5.12 Effect on tensile normal strain of blade taper shows, however, that when the axial strain is examined, there is very little difference between the different degrees of taper. This can be explained by the fact that, to a first degree, the same angle of twist is required from all blade shapes to achieve a particular degree of regulation. With similar section thicknesses, a particular angle of twist also corresponds approximately to a particular level of shear strain, and thereby also to tensile strain.
4.5.4 Design lift coefficient

The choice of design lift coefficient as a parameter for investigation needs some explanation. Following the 'ideal blade' design procedure of Glauert and the related successor methods, one works on the assumption that every station of the blade is
designed for the optimum induction factor and at the optimum lift coefficient. Of course, this is to be preferred, but in practice, it may be beneficial for a number of reasons to deviate from either of these criteria as has been mentioned before.

If the design induction factor is kept constant, but a lift coefficient is chosen which is offset from the optimum, a number of things happen. In order to achieve the same induction factor, the blade section must generate the same lift at a given wind speed. If the lift coefficient is increased, then the chord must be reduced proportionately. This also entails a reduction in Reynold’s number. For the moment, the latter effect will be ignored.

Assuming the same lift curve as before, the increase in design lift means that a greater twist is required in order to reduce the lift to zero. However, the section is reduced in dimensions and therefore develops twist more quickly in exactly the same proportion.

\[
\beta(x) = \Omega^2 K V_{bl} V_{bf}(x) \tag{3.6.2}
\]

where \( K V_{bl} = \rho \frac{\eta}{E \epsilon_{ref}} \frac{L}{(1 - c' x_{ref})} \frac{S'}{2A'} \)

and \( V_{bf}(x) = \frac{1}{6} x \left( \frac{3 + 6 h - 2 c' - 3 h c'}{1 - c' x} \right) - x^2 \)

It is thus to be expected that, if the lift curve is dominant in the mechanism of limiting rotor speed, then a change in design lift should have no influence on rotor speed regulation. This would be expected also to be the case if, say, a certain amount of negative lift had to be generated as long as it could be achieved within the linear part of the lift curve.

If, on the other hand, the drag curve has a major effect on regulation, it would be expected that it would differentiate between the rotor speed curves. This would take place because the drag remains very low over a certain range and then rises at a rate of higher order than linear once it starts to rise. If drag dominates the regulating mechanism, it would be expected that, to a greater extent than with the lift mechanism, a particular value of lift coefficient would need to be reached, at least
approximately. If this value is positive, then configuration with a low design lift and wide chord would be expected to regulate best. If the drag dominates at a negative lift coefficient, then the configuration with high design lift and narrow chord would regulate best.

An alternative approach can be used to investigate only the effect of the changes in design lift. As design lift increases, the chord is adjusted downwards accordingly, but the percentage thickness of the section is also increased in proportionally in order to compensate, keeping the actual section thickness approximately constant. The twist-rate parameter is dependent on the solidity $c_{ref}/R$, the perimeter, $S'$, and the enclosed area $A'$. If it is assumed that the ratio $S'/A'$ is approximately inversely proportional to the percentage thickness of the section, then the twist rate parameter should remain constant. It would then be expected that, with a greater twist angle required in order to regulate, the configuration based on increased lift and a reduced chord should regulate poorly, whilst that based on decreased lift and increased chord should regulate well.

It should be noted here that the graphs of rotor speed against wind speed do not bear out these predictions (See Figure 4.5.15). It is the second case curves which are coincident, whilst the curves with varying section thickness and constant percentage thickness where the curves differ from each other. Furthermore, both the positive and negative deviations appear to regulate better than the base case. These results would appear to be anomalous.
Figure 4.5.14 Effect on runaway speed curve of design lift coefficient

Figure 4.5.15 Effect on runaway speed curve of design lift coefficient with compensatory change in percentage section thickness
4.5.5 Added tip-mass

An obvious way to increase the amount of twist at a given rotor speed is to increase the centrifugal load. This is achieved by putting additional mass in the tip of the blade. The added mass clearly improves speed regulation Figure 4.5.18.
Figure 4.5.18 Effect on runaway speed curve of added tip-mass

It is also obvious that the improved regulation is achieved at the expense of an increased load, and therefore axial strain, at a given rotor speed. If a particular value of twist could be identified as being needed to regulate at a given wind speed, then it would be expected that the two effects should counteract each other exactly and for the strain as a function of wind speed to remain the same. Thus, as can be seen in Figure 4.5.19, there is little benefit in terms of blade strains - indeed they may be worse.

However the improvement in speed regulation clearly carries benefits in itself. The aerodynamic loads on a wind turbine are highly dependent on the wind seen by the blade section as it cuts the air. This in turn is dominated by the rotor speed. Thus, even if the axial strains are largely unaffected, addition of tip masses serves to reduce the bending moments experienced by the blade as can be seen clearly in Figure 4.5.20.
Figure 4.5.19 Effect on tensile normal strain of added tip-mass

Figure 4.5.20 Effect on root bending moment of added tip-mass

4.5.6 Blade twist

A pre-set twist can be seen from Figure 4.5.21 to have virtually no effect on speed regulation. Figure 4.5.22 shows that it does appear to have some effect on the regulation of the thrust load though clearly not in terms of tensile strain which in this
case would follow the same pattern as rotor speed. Twist may thus be of some benefit, though there is a strong trade-off in terms of blade manufacture.

Figure 4.5.21 Effect on runaway of blade twist

Figure 4.5.22 Effect on thrust load of blade twist
Summary of design parameter effects

It is clear that with some parameters, such as the aerofoil characteristics, there is a clear benefit from some types of curve and that these should be selected without reservation. Unfortunately, the range of choice is relatively limited. The same is true to some extent with the percentage thickness of the section, though judgement would have to be reserved till the effects of bending strains has been taken into account.

With blade parameters such as additional tip-mass, blade twist and taper, there is considerable freedom of choice in blade design. However, additional mass has effects working in different directions for different loads and blade taper has a beneficial effect on loads which may be counterbalanced by the capacity of the blade to carry the load, and the design decisions therefore involve quite subtle trade-offs. Blade twist seems to have an overall beneficial effect but can make blade manufacture difficult.

On balance, it is to be expected that moderate degrees of tip loading such and of blade taper would represent suitable compromises for the prototype design and there should be no twist. However, these choices would need to be reviewed in the light of more detailed analysis and preferably on the basis of experiment.

As regards choice of design lift coefficient, the picture is unclear and probably needs further investigation.
5. Blade Bending Models

As was noted earlier, in chapter 4, estimates have been made of the aerodynamic performance of the various configurations of self-twisting bladed wind turbines based on a blade twisting model and an aerodynamic model. It also became clear that it was completely inappropriate to apply simple engineers' beam theory to the blades to predict their response to the aerodynamic loads. Any self-twisting blades which might achieve a respectable degree of self-regulation would have a thin section and would tend to be highly flexible. They would also require substantial additional tip-loading from lead weights in order to generate adequate twist.

For such blades, if their bending behaviour is modelled using engineers' (Euler) beam theory and resistance to the applied loads is based purely on their elastic stiffness alone, the condition of small deflections breaks down and the results become nonsensical. Yet, in order to check that the blades are strong enough and in order to rule out the possibility of tower interference, it is essential to make predictions of the deflections of the blades under load and in response to periodic excitation.

5.1 Simple illustrative cases

Case 1: Elastic stiffness only

To illustrate the unreality of such a simplistic calculation, consider a numerical example. The blade dimensions and properties are based as closely as possible on those of the prototype blade (see Chapter 7), including an approximate correspondence between the rotor speed and the load.

In this case, the blade either has no mass or is treated as if it were not rotating, thus benefiting from no centrifugal stiffening. It has uniform stiffness, $EI_0$, and the load,
$Q$, is concentrated at the tip. It is thus simply a standard, uniform, cantilever beam, of length $L$ subjected to an end load. The end deflection, is given by

$$v_E = \frac{QL^3}{3EI_0}$$

If $x$ is the normalised span-wise co-ordinate and $y_e$ is the normalised displacement function, such that $r = L(x+h)$ and $v(r) = v_E y_e(x)$ then

$$y_e = \frac{1}{2} \left(3x^2 - x^3\right)$$

The quantity $\frac{EI_0}{L^3}$ will appear repeatedly in these calculations and, as it has the dimensions of stiffness, it is worth assigning it an appropriate label.

The ‘basic stiffness’ is defined by $k_0 = \frac{EI_0}{L^3}$ and the ‘compliance factor’ is then

$$\varphi_E = \frac{v_E}{Q} \cdot k_0 = \frac{1}{3}$$

The end deflection is then expressed as $v_E = \varphi_E \cdot \frac{Q}{k_0}$.

A numerical example will serve to illustrate the order of magnitude of the deflection:

<table>
<thead>
<tr>
<th>blade length $L$</th>
<th>1.302 m</th>
<th>bending stiffness $EI_0$</th>
<th>51.9 Nm$^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>hub ratio $h$</td>
<td>0.114</td>
<td>basic stiffness $k_0$</td>
<td>23.5 N/m</td>
</tr>
<tr>
<td>blade chord $c_k$</td>
<td>0.0811 m</td>
<td>end load $Q$</td>
<td>45 N</td>
</tr>
<tr>
<td>skin thickness $t$</td>
<td>0.00132 m</td>
<td></td>
<td></td>
</tr>
<tr>
<td>second moment $l'$ (non-dim)</td>
<td>0.00263</td>
<td></td>
<td></td>
</tr>
<tr>
<td>tip mass $m_{tip}$</td>
<td>0.509 kg</td>
<td></td>
<td></td>
</tr>
<tr>
<td>rotor speed $\Omega$</td>
<td>68.0 rad/s</td>
<td></td>
<td></td>
</tr>
<tr>
<td>elastic modulus $E_a$</td>
<td>28.02 GPa</td>
<td></td>
<td></td>
</tr>
<tr>
<td>elastic deflection $v_e$</td>
<td>0.638 m</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Case 2 Elastic only, distributed load

As above but the load is uniform and distributed over the length of the blade.

The end deflection, is given by

\[ v_{Ed} = \frac{qLL^3}{8EI_0} \]

\( q \) is the distributed loading. \( \phi_{Ed} = 1/8 \)

The displacement function is given by \( y_{ed} = \frac{1}{3} (6x^2 - 4x^3 + x^4) \).

As mentioned before, for quite sensible values of aerodynamic load, the deflection calculated on the basis of elasticity alone become meaningless. No blade can be expected to deflect by half its length and such large deflections would require a non-linear analysis. Clearly, in the absence of centrifugal stiffening, the blade as currently designed would be completely incapable of resisting the magnitude of loads which routinely arise.

Case 3 Centrifugal stiffness only

As an alternative, it is also possible to model the blades as having mass but no elastic stiffness. Deflections are then calculated purely on the basis of the centrifugal stiffness.

The blade has no elastic stiffness, but has a tip-mass \( m_{tip} \) and an end load, \( Q \), as before and rotates at a speed \( \Omega \).

The end deflection comes about from a simple force equilibrium between the centrifugal load, the externally applied load and the blade tension.

\[ v_G = \frac{Q}{k_G} \]

\[ y_G = x \]

where \( k_G = m_{tip} \Omega^2 \), \( R = L(1+h) \) is the rotor radius, \( Lh \) is the hub radius.
Using the same numerical example as before,

<table>
<thead>
<tr>
<th>CF load</th>
<th>( G )</th>
<th>3413 N</th>
</tr>
</thead>
<tbody>
<tr>
<td>inertial stiffness</td>
<td>( k_r )</td>
<td>2621 N/m</td>
</tr>
<tr>
<td>inertial deflection</td>
<td>( v_G )</td>
<td>0.0172 m</td>
</tr>
</tbody>
</table>

This is clearly a more credible value.

**Case 4 Centrifugal stiffness, distributed load**

Equilibrium is now represented by

\[
\frac{q \cdot L^2}{2} \cdot (1 - x)^2 \cdot v_{Gd} \cdot G \cdot (1 - y) = G = m_{tip} \cdot L(1 + h) \cdot \Omega^2
\]

The deflection curve is given by

\[
v_{gd}(r) = v_{Gd} \cdot \gamma_{gd}
\]

where

\[
v_{Gd} = \frac{q \cdot L^2}{2G} = \frac{v_G}{2G} \quad \gamma_{gd} = 2x - x^2
\]

Returning to the same numerical example, but with the load distributed over the whole blade:

<table>
<thead>
<tr>
<th>aero. loading</th>
<th>( q )</th>
<th>345.6 N/m</th>
</tr>
</thead>
<tbody>
<tr>
<td>inertial deflection</td>
<td>( v_{Gd} )</td>
<td>0.0086 m</td>
</tr>
</tbody>
</table>

For flexible blades at high rotor speeds, this is closer to reality than the purely elastic description but also is not satisfactory. Clearly, no calculation of the stresses and strains in the blade are possible and anyway the deflected shape is likely to be very unrealistic. It does, however provide an upper bound to the results with both mass and elasticity taken into account and, because of the relatively small deflections, justifies the assumption made in the fuller treatment that the deflections are small.

**Case 5 Elastic and centrifugal stiffness**

The situation is better described by an equilibrium between three load systems; the aerodynamic loading, the elastic stiffness of the blade and additionally, the centrifugal stiffness acting as a result of restoring couples on the blade due to the action of centrifugal loads on the distributed mass. In the case of vibration analysis, the natural
modes are found from equilibrium between elastic stiffness, centrifugal stiffness and inertial loads.

First, it is worth examining the differential equation for the blade deflection. It is reasonable to neglect any terms due to shear deflection (Timoshenko beams) as the blade can justifiably be described as slender. Engineers’ (Euler) theory of beams is applied here.

If the deflected shape is described by the function $v(r)$ (where $r$ is the radial co-ordinate), then from Bramwell [4]

$$\frac{d^2}{dr^2} \left( EI(r) \frac{d^2 v}{dr^2} \right) - \frac{d}{dr} \left( G(r) \frac{dv}{dr} \right) + \mu(r) \frac{d^2 v}{dt^2} = \frac{dF}{dr} \tag{5.1.1}$$

where $G(r) = \int_r^R \Omega^2 \mu(r) r dr$ and is the centrifugal load

$EI(r)$ is the bending stiffness

$\mu(r)$ is the mass per unit length

$F(r)$ is the externally applied load

When the static deflections under load are calculated, the time dependent term vanishes, whereas for the modes of vibration, the external load is set to zero.

It is relatively simple to set up the differential equations describing these equilibria, but much more difficult to solve them. Apart from the simplest cases, it is not possible to obtain direct solutions.

To illustrate this, possibly the most complex case which is amenable to direct solution requires the blade, as before, to be uniform in section and to have its mass concentrated at the tip. The blade has the same uniform elastic stiffness as before and the same tip mass. Unlike the previous cases, this treatment is less straightforward and is not available as a known result from text books. However it is still amenable to standard methods of solving differential equations.

The differential equation for the static displacement of the blade is as follows, with an end load $Q$, a flexural rigidity $EI_0$ and a centrifugal end load $G = m_{tip} L(1+h) \Omega^2$. 

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The equation can be solved by an exact method, by direct solution of the differential equation:
\[ EI_0 \frac{d^2 v}{dr^2} + G_e (v_{EG} - v(r)) - Q (R - r) = 0 \]

In non-dimensional form, this is
\[ \frac{d^2 y}{dx^2} + \Lambda^2 (1 - y) - \frac{(1 - x)}{\varphi_{EG}} = 0 \]

where \( \Lambda^2 = m_{tip} (1 + h) \Omega^2 \frac{L^3}{EI_0} \) and \( \varphi_{EG} = \frac{Q \cdot L^3}{v_{EG} \cdot EI_0} \)

This has solutions of the form
\[ y_{eg} = \frac{\Lambda \left( 1 + \gamma^2 \right) x - \left( 1 - \gamma^2 \right) + \exp(-\Lambda \cdot x) - \gamma^2 \cdot \exp(\Lambda \cdot x)}{\Lambda \left( 1 + \gamma^2 \right) - \left( 1 - \gamma^2 \right)} \]

where \( \gamma = \exp(-\Lambda) \)

with the end displacement given by
\[ v_{EG} = \frac{Q}{k_{EG}} = \varphi_{EG} \cdot \frac{Q}{k_0} \]

where \( \varphi_{EG} = \frac{k_0}{k_{EG}} = \frac{1}{\Lambda^2} \left( 1 - \frac{\xi}{\Lambda} \right) = \frac{1}{\Lambda^2} - \frac{1}{\Lambda^3} \) and \( \xi = \frac{1 - \gamma^2}{1 + \gamma^2} \)

For the full derivation, see Appendix A3.

The quantities \( k_0 \) and \( k_{EG} \), which have already been defined, represent, in some manner, elastic and centrifugal stiffnesses. The ratio between them, \( \Lambda^2 \), clearly has a profound influence on the character of the solution and appears throughout the equations which give the displaced shapes of the blade. It is, of course, dependent on the rotational speed and the hub offset and can be expressed as a ratio between these two quantities and another quantity, the ‘basic frequency’, \( \omega_b \), which is a property of the blade and has the dimensions of angular frequency. It is not equal to the natural
frequency of the blade but it differs from it only by an, as yet unknown, dimensionless factor.

\[ \Lambda^2 = \frac{(1 + h) \Omega^2}{\omega_0^2} \quad \omega_0^2 = \frac{EI_0}{m_{tip} L^3} \]

Using the same numbers as before:

<table>
<thead>
<tr>
<th>bending-CF ratio $\Lambda$</th>
<th>compliance factor $\varphi_{EG}$</th>
<th>inertial deflection $v_G$</th>
<th>combined model deflection $v_{EG}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10.56</td>
<td>0.0082</td>
<td>0.0172 m</td>
<td>0.0155 m</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>exp(-$\Lambda$) $\gamma$</th>
<th>$\gamma_{pj}$, 0.0082</th>
<th>$\gamma_{pj}$, 0.0172 m</th>
<th>$\gamma_{pj}$, 0.0155 m</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.6 \times 10^{-5}</td>
<td>2.6 \times 10^{-5}</td>
<td>2.6 \times 10^{-5}</td>
<td>2.6 \times 10^{-5}</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>basic frequency $\omega_0$</th>
<th>combined model $v_{EG}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>6.80 rad/s</td>
<td>0.0155 m</td>
</tr>
</tbody>
</table>

### Case 6  Elastic and centrifugal stiffness, distributed load

The differential equation is now given by:

\[ EI_0 \frac{d^2 v}{dr^2} + G.(v_{EGd} - v(r)) - \frac{q}{2} (R - r)^2 = 0 \]

\[ \frac{d^2 y}{dx^2} + \Lambda^2 (1 - y) - \frac{(1 - x)^2}{2 \varphi_{EG}} = 0 \]

The solution to this gives the shape as:

\[ y_{Egd} = \frac{\Lambda^2 (2x - x^2) - 2.\gamma (1 - \exp(-\Lambda x)) - 2.\zeta (1 - \exp(\Lambda x))}{\Lambda^2 - 2\Lambda + 2 - 4\zeta} \]

where \( \zeta = \gamma \frac{1 - \gamma.\Lambda}{1 + \gamma^2} \)

and the end deflection is:

\[ v_{EGd} = \varphi_{Egd} \frac{q.L}{k_0} \]

As before, a fuller derivation is given in appendix A3
Again, the same numerical example will serve as an illustration.

<table>
<thead>
<tr>
<th>Compliance factor</th>
<th>$\varphi_{EGd}$</th>
<th>0.0037</th>
</tr>
</thead>
<tbody>
<tr>
<td>Inertial deflection</td>
<td>$\nu_{Gd}$</td>
<td>0.0086 m</td>
</tr>
<tr>
<td>Combined model deflection</td>
<td>$\nu_{EGd}$</td>
<td>0.0071 m</td>
</tr>
</tbody>
</table>

Figure 5.1.1  Comparison of blade deflected shapes for different simple models

In the two examples which take into account both elastic and inertial stiffness, it becomes clear that at realistic rotational speeds, the stiffness is dominated by the centrifugal contribution. It is clear, however, from the plot (Figure 5.1.1) of the actual deflected shapes, that there are important contributions from the elastic stiffness to the shape, namely that without it, there is no requirement for zero slope at the hub and therefore it contributes a whole extra degree of curvature.

5.1.2 Blade vibrations in simple cases

The natural frequencies of the blade are also an important consideration in analysing the expected behaviour of the rotor. The full differential equation has already been shown in Eqn (5.1.1) but the same simplified cases will also yield their estimates of natural frequency.
Only 3 cases will need to be considered, as the transverse load now comes purely from the inertial forces on the concentrated tip-mass and in the absence of distributed mass there can be no distributed load.

From the point-of-view of the tip-mass all three set-ups act simply as springs. It is well established that for a simple spring-mass system, the natural frequency is given by

\[ \omega_n = \sqrt{k/m} \quad \text{where } \omega_n \text{ is the natural frequency, } k \text{ the stiffness and } m \text{ is the mass.} \]

Like the ‘basic stiffness’, a ‘basic frequency’ has been defined:

\[ \omega_0 = \sqrt{\frac{EI_0}{m_{tip} \cdot L^3}} \]

For the elastic case, \( k_E = \frac{3EI_0}{L^3} \) so the frequency is

\[ \omega_E = \sqrt{\frac{3EI_0}{m_{tip} \cdot L^3}} = \sqrt{3} \cdot \omega_0 \]

For the case with only centrifugal stiffening, \( k_G = m_{tip} \cdot L(1+h) \cdot \Omega^2 \), so the frequency is

\[ \omega_G = \sqrt{\frac{m_{tip} \cdot (1+h) \cdot \Omega^2}{m_{tip}}} = \sqrt{1+h} \cdot \Omega \]

For the centrifugal and elastic case,

\[ k_{EG} = \frac{EI_0}{\Phi_{EG} \cdot L^3} \]

\[ \omega_{EG} = \frac{\omega_0}{\sqrt{\Phi_{EG}}} = \sqrt{\frac{\Lambda^3 \cdot EI_0}{(\Lambda - \xi) \cdot m_{tip} \cdot L^3}} = \sqrt{1 + \frac{h}{2 + \frac{1}{\Lambda}}} \cdot \Omega = \Omega \cdot \left(1 + \frac{h}{2} + \frac{1}{2 \cdot \Lambda} + \ldots\right) \]
Based on the same numerical example as before, the natural frequency estimates take on the following values.

Table 5.1.1 Natural frequencies:

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>bending-CF ratio $A$</td>
<td>10.56</td>
</tr>
<tr>
<td>rotational speed $\Omega$</td>
<td>68.0</td>
</tr>
<tr>
<td>'basic' frequency $\omega_0$</td>
<td>6.80</td>
</tr>
<tr>
<td>elastic only $\omega_e$</td>
<td>11.8</td>
</tr>
<tr>
<td>inertial only $\omega_i$</td>
<td>71.8</td>
</tr>
<tr>
<td>combined model $\omega_{ei}$</td>
<td>75.4</td>
</tr>
</tbody>
</table>

5.2 More complex blade constructions

For cases which are not quite so simple, direct solution of the differential equation becomes intractable. However, a number of well-established methods exist for its approximate solution and broadly, they fall into two classes.

5.2.1 Piece-wise or discretized methods

One approach is to analyse the blade piecewise, as if it were an assembly of parts with known masses and connecting stiffnesses, subject to a series of discrete loads applied to each part of the assembly. For the natural frequencies of a blade, the Mykelstad method (Mykelstad [37], Thomson [44]) may be used and can be extended to incorporate coupling between degrees of freedom. It is also possible to adapt the Mykelstad method, using the same mass and stiffness matrices, to calculate the blade's response to static loadings. More common these days is to employ finite element methods, as these are available as ready packages.
5.3 Rayleigh-Ritz model of blade bending

5.3.1 Principle of the Rayleigh Ritz method

However, for this project, the decision had already been made earlier to employ simple, whole blade analysis in order to predict the blade twist and so it was decided to employ whole-blade methods for the flapwise deflections, too. Of these methods, the Rayleigh-Ritz method has been chosen. This method has the advantage of relative simplicity and that the quantities being worked on are real-world quantities, amenable to visualisation. It is a variational method, based on the equivalence between an equilibrium description of the deflected state of a body or structure and one in which a minimum is found in the body’s potential energy.

The Rayleigh-Ritz method is itself based on the Rayleigh method. According to this, the deflected state of a body or system can be described in terms of a displacement function. Any function may be proposed and the resulting potential energy of the system calculated as the sum of external work done by the loads and internal work done in the structure, principally in the form of strain energy. Any departure from the correct deflected shape can be seen as being the result of an additional constraints which give rise to additional internal work. Thus, the correct displacement is characterised by having the minimum total potential energy (TPE). A further assumption of the method is that the closer the TPE is to the true value, the better the approximate description of the displacement, though it is hard precisely to define the meanings of ‘better’ and ‘worse’ in this context except in a rather circular manner.

The basis of the Rayleigh-Ritz method is that any number of displacement functions may be added together in a linear combination and that this may lead to an improvement over the individual functions. Furthermore, for any set of functions, the best linear combination can be found by varying their coefficients in order to minimise the TPE. It should be noted that this is best fit does not necessarily correspond to the correct displaced shape, as it might be improved by the inclusion of other functions not part of the set used.
Ideally, the displacement functions should satisfy the boundary conditions of the problem, though if they do not, it is possible to check the solution for its compliance.

In the case of a wind turbine blade deflecting under a static load, the significant contributions to the total potential energy, $U_{TOT}$, are the same forces that were considered as part of the equilibrium in the simple cases already seen and the differential equation already looked at. These are the work done by the external load, $W$, the strain energy, $U_E$, and the inertial potential energy, $U_G$. The latter results from the action of the centrifugal loading on the mass in the blade when it moves to a position of lower radius of rotation when deflected.

$$U_{TOT} = W + (U_E + U_G)$$

In a case in which the modes of vibration are sought, instead of an external load doing work on the system, there is periodic exchange of energy from kinetic energy, $T$, of vibration to potential energy $U$, (strain energy, $U_E$, and inertial PE, $U_G$) and vice versa. The maximum values of the two forms of energy must be equal and it is found that a term containing the square of natural frequency is contained in the expression for kinetic energy. Calculating the ratio of the expressions for KE and PE thus gives an expression for the natural frequency.

According to Rayleigh’s principle, as in the static case, any departure form the correct solution represents additional stiffness and thus raises the frequency, so the true frequency must be the minimum value. Whenever the ‘imposed’ mode shape has any degrees of freedom, such as the coefficients in a linear combination of functions, it becomes possible to seek mathematically the minimum value of the frequency. However this entails the differentiation of the ratio between two large expressions. It can be shown that it is equivalent to this to find stationary values of a quantity called the Lagrangian, $L$. This is simply the difference between the maximum kinetic energy and the maximum potential energy.

$$L = T - (U_E + U_G)$$

In either the static case or with free vibrations, a stationary value is sought in the quantity concerned, either $U_{TOT}$ or $L$. The stationary values are found with respect to
each of the degrees of freedom by differentiating with respect to its coefficient and setting to zero. In the static case, these yield a set of simultaneous equations which can be solved to find the displaced shape, by Gaussian elimination or by matrix inversion. In the vibration case, an eigenvalue problem results and this yields as many solutions as there are degrees of freedom, each solution representing one mode of vibration.

In calculating the required energy terms, it is assumed that the deflection of the blade, (as yet unknown) can be expressed in terms of the product of the tip deflection, $v_Z$, and a dimensionless displacement function, $y_Z(x)$, and, in the case of free vibrations, also a periodic time dependent function which is assumed to be harmonic.

$$v_Z(x, t) = v_Z \Psi(t) y_Z(x) \text{ where } \Psi_j(t) = \sin(\omega_j t + \phi_j)$$

When determining normal modes, the kinetic energy $T$ is found for the whole blade as the product of half the mass elements with the square of their velocities, integrated over the whole blade:

$$T = \frac{1}{2} \int_{r_h}^{R} \left( \frac{d}{dt} v_Z(r, t) \right)^2 dm = \frac{v_Z}{2} \int_{r_h}^{R} \left( \frac{d}{dt} \sin(\omega_j t + \phi_j) \right)^2 y_Z(x)^2 dm$$

where $v_Z$ is the tip-deflection

$y_Z(x)$ is the dimensionless displacement function

and $\sin(\omega_j t + \phi_j)$ is the time dependent function

For static deflections under load, the work done, $W$, by the external load $q(r)$ is found by forming half the product of the loading with the deflection, integrated over the whole blade.

$$W = \frac{1}{2} \int_{r_h}^{R} q(r) v(r) dr$$
The inertial potential energy can be explained in two equivalent ways. It can be seen as the work done by the centrifugal tension against the 'shortening' $\Delta r$ of each element of the blade due to its deflection, or as the work done by the centrifugal load acting as a restoring couple against the angular deflection.

$$U_G = \frac{1}{2} \int_{R}^{r_h} G(r) \left( \frac{d}{dr} v \right)^2 dr$$

Finally, the strain energy makes its contribution. According to elementary engineers' (Euler) beam theory, this is half the product of the bending stiffness with the square of the second derivative of the deflection.

$$U_E = \frac{1}{2} \int_{R}^{r_h} E I(r) \left( \frac{d^2}{dr^2} v \right)^2 dr$$

5.3.2 Shape functions

For the purpose of the Rayleigh-Ritz analysis, each of these energy terms has to be expressed in more detail in terms of the properties of the blade and in terms of the assumed shape functions and their coefficients. Concerning the displaced shape, it is possible to use simple functions such as powers of $x$ but in order to reach a good fit with the minimum number of terms, it is preferable to choose ones which will approximate the normal modes of the blade more closely. In this case, a set of shape functions, $Z(j,x)$ as defined below have been used.

$$Z(j,x) = \cos(j \cdot \pi \cdot x) - \cos \left( j + \frac{1}{2} \right) \pi \cdot x$$
The displacement function $y_z(x)$ is expressed as

$$y_z(x) = a_0 Z(0, x) + a_1 Z(1, x) + a_2 Z(2, x) + a_3 Z(3, x) + ... = \mathbf{a}^T \mathbf{Z}(x)$$

where $\mathbf{Z}(x)$ is a column vector consisting of a finite series of the shape functions such that

$$\mathbf{Z}(x)_j = Z(j, x)$$

and the as yet unknown coefficients $a_j$ make up the column vector $\mathbf{a}$.

For practical purposes, the representation has to be limited to a finite number of terms, $n + 1$, in the series. (The first term has an ‘index’, $j$, of zero)

The 1st and 2nd derivatives of the shape functions, the assembled column vectors and of the overall displacement function are defined in similar manner.

$$\frac{d}{dr} y_z = \frac{v Z}{L}$$

$$\frac{d}{dx} y_z = y'_z(x) = a_0 Z'(0, x) + a_1 Z'(1, x) + a_2 Z'(2, x) + a_3 Z'(3, x) + ...$$

Figure 5.3.1 First four shape functions
\[ T \cdot Z'(x) = a \cdot \frac{d}{dx} Z(x) \]

\[ Z'(x) = Z'(j, x) = \frac{d}{dx} Z(j, x) = \left( j + \frac{1}{2} \right) \pi \cdot \sin \left( \left( j + \frac{1}{2} \right) \pi \cdot x \right) - j \cdot \pi \cdot \sin(j \cdot \pi \cdot x) \]

Similarly,

\[ Z''(x) = Z''(j, x) = \left( j + \frac{1}{2} \right)^2 \pi^2 \cdot \cos \left( \left( j + \frac{1}{2} \right) \pi \cdot x \right) - j^2 \pi^2 \cdot \cos(j \cdot \pi \cdot x) \]

Returning to the various contributions to the potential energy and the Lagrangian, it is necessary to examine the shape of the blade and how it influences its mass distribution and its stiffness distribution.

**5.3.3 Kinetic energy calculation — mass matrix**

For the kinetic energy, it is the mass distribution which must be considered. Assume that the blade has its mass distributed over its length and that it tapers in a linear manner. Its root is offset from the rotational axis. The chord can be represented as a function of \( x \) which is non-dimensional and runs from 0 at the blade root to 1 at the tip:

\[ c(x) = c_0 \cdot (1 - c' \cdot x) \]

As the section is hollow, the mass of an element of the blade is proportional to the chord.

\[ dm = \mu \cdot 0 \cdot (1 - c' \cdot x) \cdot L \cdot dx \quad \text{where } L \text{ is the blade length,} \]

\[ \mu \cdot 0 = \rho \cdot S' \cdot c_0 \cdot t \quad \text{and is the mass per unit length of the blade at its root.} \]

\( c_0 \) is the chord at the blade root,

\( t \) is the skin thickness and \( \rho \) is the material density.

\( S' \) is the non-dimensional measure of section perimeter.

The offset at the hub can be defined relative to the blade length as \( h \cdot L \).
Due to the linear taper of the blade, its mass and elastic properties can readily be expressed as cubic (or lower order) distributions in $x$. For this purpose a column vector $\mathbf{m}$ can represent the coefficients of the mass distribution; whilst $\mathbf{P}(x)$ is a column vector operator of powers of $x$. A mass element then consists of the product of the mass per unit length at the root end with the blade length and the dimensionless polynomial.

$$
\mathbf{m} = \begin{bmatrix}
1 \\
-c' \\
0 \\
0
\end{bmatrix}, \quad \mathbf{P}(x) = \begin{bmatrix}
1 \\
x \\
x^2 \\
x^3
\end{bmatrix}
$$

$$
dm = \mu \cdot (\mathbf{1} - c' \cdot x) \cdot L \cdot dx = \mu \cdot L \cdot \mathbf{P}(x)^T \cdot \mathbf{m} \cdot dx
$$

Examining the terms in the expression for the kinetic energy,

$$
T = \frac{1}{2} \int_{r_h}^R \left( \frac{d}{dt} \mathbf{v} \right)^2 \frac{dm}{2} = \frac{v Z}{2} \left( \frac{d}{dt} \sin(\omega \cdot j \cdot t + \phi) \right)^2 \cdot \int_0^1 y_z(x)^2 \, dm
$$

Time component:

$$
\frac{d}{dt} \sin(\omega \cdot j \cdot t + \phi) = \omega \cdot j \cdot \cos(\omega \cdot j \cdot t + \phi)
$$

The maximum KE occurs when the cosine-time term equals one so the expression as a whole takes the value $\omega \cdot j$.

Space component:

$$
\int_{0}^{1} y_z(x)^2 \, dm = \int_{0}^{1} \mathbf{a}^T \cdot \mathbf{Z}(x) \cdot \mathbf{Z}(x) \cdot \mathbf{a} \cdot \mathbf{P}(x)^T \cdot \mathbf{m} \, dx
$$

so the maximum KE is

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\[ T_{\text{max}} = \frac{1}{2} \mu \cdot 0 \cdot L \cdot v \cdot Z^2 \cdot \omega \cdot \int_0^1 a^T \cdot Z(x) \cdot Z(x)^T \cdot a \cdot \mathbf{P}(x) \cdot m^T \, dx \]

Whilst the full integral expression is a scalar, the coefficient column vectors are not dependent on \( x \) and thus can be taken outside the integral. The product of the shape function column vectors, within the integral, then becomes a square matrix. The product of this with the polynomial column vector does not work according to the standard conventions of matrix multiplication. However it is possible to create a square matrix \( \mathbf{H}_M \) from which the KE can be calculated in a convenient manner, separating the dimensioned quantities as scalars from dimensionless column vectors and a matrix.

\[ \mathbf{H}_M = \int_0^1 Z(x) \cdot Z(x)^T \cdot \mathbf{P}(x) \cdot m^T \, dx \]

The elements of \( \mathbf{H}_M \) are defined by the following:

\[ \mathbf{H}_{M_{j,k}} = \sum_{l=0}^{3} m_l \int_0^1 x^l \cdot Z(j,x) \cdot Z(k,x) \, dx \]

where \( j := 0 \ldots n \) and \( k := 0 \ldots n \)

There is also a contribution to the kinetic energy from the motion of the added tip-mass. The tip-mass runs a certain length of the blade-tip and fills the void left by the blade skin. An adequate approximation to this can be achieved by representing it as a sheet having a notional constant thickness \( t_{\text{tip}} \) and of high density material \( \rho_{\text{tip}} \) coincident with the blade-skin and thus having the same distribution. This allows one to use the same integrals as for the mass of the skin.

\[ dm''_{\text{tip}}(x) = \mu \cdot t_{\text{tip}} \cdot L \cdot (1 - c^\prime \cdot x) \quad \text{for} \ x \geq x_{\text{tip}} \quad \text{and} \ 0 \text{ for} \ x < x_{\text{tip}}. \]
where \( \mu_{\text{tip}} = \rho \cdot \text{tip} \cdot \text{tip} \cdot S \cdot c \cdot 0 \)

and \( t_{\text{tip}} = \frac{c \cdot 0 \cdot A \cdot \left[ 1 - c \cdot \left( \frac{1 + x_{\text{tip}}}{2} \right) \right]}{S} - t \)

Using this for the KE calculation,

\[
\int_{x}^{1} y_{z}(x)^{2} \cdot (1 - c \cdot x) \, dx = \left[ \int_{x}^{1} a^{T} \cdot Z(x) \cdot Z(x) - a \cdot P(x) \cdot m \, dx \right]^{x_{\text{tip}}}_{x}
\]

Thus \( T_{\text{tip.max}} = \frac{1}{2} \mu \cdot \text{tip} \cdot \text{tip} \cdot 2 \cdot \omega \cdot j \cdot a^{T} \cdot Z(x) \cdot Z(x) - a \cdot P(x) \cdot m \, dx \)

The whole matrix can then be defined element by element as

\[
H_{M_{j,k}} = \sum_{l=0}^{3} m_{l} \left( C(j, k, l, 0) + C(j, k, l, x_{\text{tip}}) \right)
\]

where \( C(j, k, l, x) = \int_{x}^{1} x^{T} \cdot Z(j, x) \cdot Z(k, x) \, dx \) (see Appendix A3)

Giving the kinetic energy as

\[
T_{\text{max}} = \frac{1}{2} \mu \cdot 0 \cdot \omega \cdot j \cdot a^{T} \cdot H_{M} \cdot a
\]

Additionally, the integrations can easily be adapted in order to calculate the blade mass and added tip mass. (See appendix A3)
In the case of the prototype blade (see chapter 7) the blade parameters are as follows:

### Table 5.3.1 Prototype blade parameters (see Chapter 7)

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Blade length ($L$)</td>
<td>1.3021 m</td>
</tr>
<tr>
<td>Root chord ($c_0$)</td>
<td>0.1095 m</td>
</tr>
<tr>
<td>Skin thickness ($t$)</td>
<td>0.00132 m</td>
</tr>
<tr>
<td>Blade taper ($c'$)</td>
<td>0.333</td>
</tr>
<tr>
<td>Blade material density ($\rho$)</td>
<td>1687 kg/m^3</td>
</tr>
<tr>
<td>Mass per unit length at root ($\mu_0$)</td>
<td>0.496 kg/m</td>
</tr>
<tr>
<td>Blade mass ($m_{bl}$)</td>
<td>0.5385 kg</td>
</tr>
<tr>
<td>Tip mass ($m''_{tip}$)</td>
<td>0.5095 kg</td>
</tr>
</tbody>
</table>

### 5.3.4 External work calculation — work vector

For the purpose of calculating static deflections, instead of the kinetic energy, it is necessary to calculate the work done by the externally applied loads. As in the case of the mass distribution in the KE calculation, these loads are expressed as polynomials in $x$.

An example loading is given here for the purposes of illustration, based on the simplified propeller aerodynamics derived for the 'spin-test' (see Chapter 8 and Appendix A4)

$$q(x) = \frac{Q}{L} (p(x)^T \cdot q)$$

where $Q = 45$-N and $q = [0.082, -2.0921, 6.2015, -0.4819]$
Figure 5.3.2  Load distribution used in example calculation

The load's contribution to potential energy, the work done, is given by

\[ W = \frac{1}{2} \int_{r_h}^{R} q(r) \cdot v(r) \, dr = \frac{1}{2} v \int_{0}^{1} Z^{T} \cdot a \cdot T \cdot Z(x) \cdot P(x) \cdot T \cdot q \, dx \]

It is convenient to define the work vector, \( \mathbf{h}_Q \), to separate the dimensioned quantities from the dimensionless vectors which represent the spatial distribution. The external work done is then given by:

\[ W = \frac{k_0}{2 \cdot \varphi \cdot Z} v Z^{2} \left( a^{T} \cdot \mathbf{h}_Q \right) \]

where \( \varphi_Z = v Z \cdot \frac{k_0}{Q} \)

and its differential with respect to the shape function coefficients is

\[ \frac{dW}{da} = \frac{k_0}{2 \cdot \varphi \cdot Z} v Z^{2} \cdot \mathbf{h}_Q \]

The elements of \( \mathbf{h}_Q \) are therefore given by

\[ \mathbf{h}_Q = \sum_{l=0}^{3} q_l \cdot \left( B(j,l,0) - B\left(j + \frac{1}{2},l,0\right) \right) \]

where \( B(j,l,x) = \int_{x}^{1} x^l \cdot \cos(j \cdot \pi \cdot x) \, dx \)  (see Appendix A3)
The problem with this approach is that the model becomes cumbersome, as the load vector has to be recalculated for every new loading distribution. It is clearly preferable to have it in a form in which the calculation is performed once in a form which can be applied without change to any load distribution. Fortunately, this is possible, as the entire method as applied here is based on the assumption that the system is linear. (Rayleigh-Ritz can be applied to non-linear systems.)

Work vectors can be calculated for a series of unit vectors, one for each order of the polynomial:

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$  

The resulting column vectors can be assembled into a (non-square) matrix $HQ$:

$$HQ_{j,l} = B(j, l, 0) - B\left(j + \frac{1}{2}, l, 0\right) \quad \text{where } j = 0..n \text{ and } l = 0..3$$

Another version of this may also be necessary to allow for loads which only apply for part of the blade. For example, if the blade is operated horizontally there is a contribution to the loading from the tip-mass.

$$HQ_{\text{tip}, j,l} = B\left(j, l, x_{\text{tip}}\right) - B\left(j + \frac{1}{2}, l, x_{\text{tip}}\right)$$

Using these unit load matrices, a universal solution is calculated by whatever method is preferred, matrix inversion or Gaussian elimination. The result is a four column matrix having the full length of the coefficient vector for each unit vector.

### 5.3.5 Inertial potential energy calculation — centrifugal matrix

The centrifugal loading makes contributions to the potential energy for both static and dynamic calculations.
The elemental centrifugal load due to the mass and acceleration of a section is given by

\[ dG(r) = \Omega^2 \cdot r \cdot dm = \Omega^2 \cdot L \cdot (h + x) \cdot \mu \cdot \left(1 - c' \cdot x \right) \cdot L \cdot dx \]

\[ = \mu \cdot 0 \cdot L^2 \cdot \Omega^2 \cdot \left[ h + (1 - h \cdot c') \cdot x - c' \cdot x^2 \right] \]

As before, this can be expressed as a polynomial consisting of a coefficient vector \( \mathbf{g} \) and the 'power of x' vector \( \mathbf{P}(x) \).

\[ dG(x) = \mu \cdot 0 \cdot L^2 \cdot \Omega^2 \cdot \mathbf{P}(x)^T \cdot dx \cdot \mathbf{g} \]

\[ \mathbf{g} := \begin{bmatrix} h \\ 1 - h \cdot c' \\ -c' \\ 0 \end{bmatrix} \]

The accumulated centrifugal load acting at a particular section can be found by integrating from there to the blade tip.

\[ G(x) = \mu \cdot 0 \cdot L^2 \cdot \Omega^2 \cdot \int_x^1 \mathbf{P}(x)^T \cdot dx \cdot \mathbf{g} = \mu \cdot 0 \cdot L^2 \cdot \Omega^2 \cdot \mathbf{P}(x)^T \cdot \mathbf{g} \]

where \( \mathbf{P}(x) = \int_x^1 \mathbf{P}(x) \cdot dx \)

This product, \( \mathbf{P}(x)^T \cdot \mathbf{g} \) is probably more usefully expressed in terms of the original \( \mathbf{P}(x) \) and a modified \( \mathbf{g} \) as \( \mathbf{P}(x)^T \cdot \mathbf{gg} \)

\[ \mathbf{gg} := \begin{bmatrix} h + \frac{1}{2} \cdot (1 - h \cdot c') - \frac{c'}{3} \\ -h \\ \frac{1}{2} \cdot (h \cdot c' - 1) \\ c' \\ \frac{3}{3} \end{bmatrix} \]
At this point the centrifugal tension caused by the tip-mass should be added in:

\[ G_{\text{tip}}(x) = \mu_{\text{tip}} L^2 \cdot \Omega^2 \cdot \mathbf{P}(x)^T \cdot \mathbf{g} \]

where \( x > x_{\text{tip}} \)

and

\[ G_{\text{tip}}(x) = \mu_{\text{tip}} L^2 \cdot \Omega^2 \cdot \mathbf{P}(x_{\text{tip}})^T \cdot \mathbf{g} \]

where \( x \leq x_{\text{tip}} \)

\[ G(x) = \mu \cdot L^2 \cdot \Omega^2 \cdot \left( P(x)^T + \frac{\mu}{\Omega} \cdot \mathbf{P}(x_{\text{tip}}) \cdot \mathbf{I}_{\text{tip}} \cdot \mathbf{P}(x_{\text{tip}})^T \right) \cdot \mathbf{g} \]

At a rotational speed of 550 rpm, the centrifugal tension at the blade root is

\[ G(0) = 3374.6 \cdot \text{N} \]

\[ G(x) 
\begin{array}{c}
0 \\ 2000 \\ 4000 \\
\end{array} \\
\begin{array}{c}
0 \\ 0.2 \\ 0.4 \\ 0.6 \\ 0.8 \\ 1 \\ 1.2 \\ 1.4 \\ 1.6 \\
\end{array}
\]

Figure 5.3.3 Centrifugal load distribution

As mentioned before, the inertial potential energy is obtained from the gradient of the deflected shape.

\[ U = \frac{1}{2} \int_0^R \left[ G(r) \left( \frac{d}{dr} \right)^2 \right] dr \]

Using the expressions already developed for the displacement function in terms of the shape functions and that for the centrifugal tension in terms of the polynomial,

\[ U = \mu \cdot L^2 \cdot \Omega^2 \cdot \frac{1}{2} \int_0^1 a^T \cdot \mathbf{Z}'(x) \cdot \mathbf{Z}'(x)^T \cdot a \cdot \mathbf{P}(x)^T \cdot \mathbf{g} dx \]

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As before, there is no way of expressing this in standard matrix notation, but it is possible to formulate from it a combined centrifugal matrix, $H_G$.

![Image](https://via.placeholder.com/150)

Then, consistent with earlier notation, the potential energy can be expressed as dimensioned scalar parameters multiplied by a non-dimensional matrix-vector product.

$$U_G = \frac{k_0}{2} \nu Z^2 \Lambda^2 \left( a^T H_G a \right)$$

where $k_0 = \frac{EI_0}{L^3}$ and $\Lambda = \frac{\Omega}{\omega} \quad \text{and} \quad \omega = \sqrt{\frac{EI_0}{\mu \cdot L^4}} = \sqrt{\frac{k_0}{\mu \cdot L}}$

$$\mu \cdot L = 0.646 \cdot \text{kg} \quad k_0 = 57.827 \cdot \text{N} \cdot \text{m}^{-1} \quad \text{and} \quad \omega \cdot L = 9.4617 \cdot \text{rad} \cdot \text{sec}^{-1}$$

The centrifugal matrix, $H_G$, is then calculated element by element as

$$H_{G,j,k} = \sum_{l=0}^{3} gg_l \begin{bmatrix} C'(j,k,l,0) \\ \mu \cdot \text{tip} \cdot C'(j,k,l,x_{\text{tip}}) \\ \mu \cdot \text{tip} \cdot x_{\text{tip}} \cdot (C'(j,k,0,0) - C'(j,k,0,x_{\text{tip}})) \end{bmatrix}$$

where $C'(j,k,l,x) = \int_{x_{\text{tip}}}^{1} x^l Z'(j,x) Z'(k,x) dx \quad \text{(See Appendix A3)}$
5.3.6 Strain energy calculation — elastic matrix

It is now necessary to calculate the strain energy, the elastic contribution to the potential energy. The strain energy must also take into account the spanwise variation of the bending resistance. Being a shell of constant skin thickness, this varies with the cube of the chord width. Expressing $EI$ as a function of $x$,

$$EI(x) = EI_0 (1 - c' x)^3$$

where $EI_0 = E a \cdot I' \cdot c \cdot t$

For the prototype blade, the parameters are as follows;

<table>
<thead>
<tr>
<th>Blade elastic modulus</th>
<th>$Ea$</th>
<th>28.02 GPa</th>
<th>2nd moment coefficient $I'$</th>
<th>0.00263</th>
</tr>
</thead>
<tbody>
<tr>
<td>Root stiffness $EI_0$</td>
<td>127.7 Nm$^2$</td>
<td>'Basic' stiffness $k_0$</td>
<td>57.83 N m$^{-1}$</td>
<td></td>
</tr>
</tbody>
</table>

As before the distribution is expressed in vector form as a polynomial

$$EI(x) = EI_0(P(x) \cdot e)$$

where $e = \begin{bmatrix} 1 \\ -3 \cdot c' \\ 3 \cdot c'^2 \\ -c'^3 \end{bmatrix}$

As stated before, the elastic strain energy is given by half the product of stress and strain.
Again, a matrix $H_E$ is defined to correspond with earlier definitions, such that

$$U_E = \frac{k}{2} \nu Z^2 \left( a^T H_E a \right)$$

It is then possible to calculate the elastic matrix element by element.

$$H_{E_{j,k}} = \sum_{l=0}^{3} e_l C''(j,k,l,0)$$

where $C''(j,k,l,x) = \int_0^1 x^l Z''(j,x) Z''(k,x) dx$ (See Appendix A3)

### 5.4 Blade vibration model using Rayleigh-Ritz

In order to determine the mode shapes and frequencies of free vibration, it is now necessary to assemble an expression for the Lagrangian from those already derived for the different energy contributions.
A stationary value of the Lagrangian can be found with respect to the coefficients in the vector \( \mathbf{a} \) by differentiating and equating to zero.

\[
\frac{d}{d \mathbf{a}} L = \frac{EI}{2L^3} \left( \omega_j \mathbf{a}^T - \mathbf{H}_M \mathbf{a} \right) = 0
\]  

(5.4.3)

Note here that the quantity \( \omega_j \) is not an actual natural frequency of the blade - it is merely a property having the dimensions of frequency based on the mass and stiffness properties at the root of the blade.

### 5.4.1 Non-rotating mode shapes and frequencies

It is worthwhile solving this first for the non-rotating frequencies and modes of vibration by setting the rotational speed, \( \Omega \), to zero. Pre-multiplication by the inverse of the mass matrix gives an eigenvalue problem.

\[
\mathbf{H}_M^{-1} \mathbf{H}_E \mathbf{a} = \frac{\omega_j^2}{\omega} \mathbf{a} \quad \text{or} \quad \left( \mathbf{H}_M \mathbf{E} - \lambda \mathbf{I} \right) \mathbf{a} = 0
\]  

(5.4.3)
where $H_{M'E} = H_M^{-1} H_E$ and $\lambda_j = \frac{\omega_j}{\omega_0}$

**Modal frequencies**

The eigenvalues are equivalent to the normalised modal frequencies squared

$$\lambda = \sqrt{\text{eigenvals}(H_{M'E})}$$

This vector of non-dimensional values of the modal frequencies can be converted to frequencies or angular frequencies by multiplying by the basic frequency, $\omega_0$, in Hertz or in rad./sec. respectively.

$$f = \frac{\omega_0 \lambda}{2\pi} \quad f_j = \frac{\omega_0 \lambda_j}{2\pi}$$

$$\omega = \omega_0 \lambda \quad \omega_j = \omega_0 \lambda_j$$

Figure 5.4.1 below shows for the first 4 modes how the frequencies converge with increasing numbers of terms in the calculation. (This is no guarantee of final accuracy, however.)
5.4.2 Validation of modal frequency calculations

The first few calculated modal frequencies can be seen in Table 5.4.1 compared with experimental data that were obtained from vibration tests conducted at Reading University on the blade, clamped at the root end. A finite element model of the blade was also set up at Reading and the results from modal frequency calculations are also set out here for comparison. The Rayleigh-Ritz results can be seen to agree quite well with the experimental values, especially for the first mode, and somewhat better than the F-E results.
Table 5.4.1  Comparison of measured and calculated modal frequencies

<table>
<thead>
<tr>
<th>Mode number (and identification)</th>
<th>Measured frequency (Hz)</th>
<th>Rayleigh-Ritz frequency (Hz)</th>
<th>Finite element frequency (Hz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1st flapwise bending</td>
<td>2.7</td>
<td>2.98 (+10%)</td>
<td>3.66 (+36%)</td>
</tr>
<tr>
<td>(? )</td>
<td>5.4</td>
<td>—</td>
<td></td>
</tr>
<tr>
<td>2nd flapwise bending</td>
<td>17.6</td>
<td>21.18 (+20%)</td>
<td>26.42 (+50%)</td>
</tr>
<tr>
<td>1st edgewise bending (+ some torsion)</td>
<td>23.2</td>
<td>—</td>
<td>31.40 (+35%)</td>
</tr>
<tr>
<td>3rd flapwise bending</td>
<td>48.1</td>
<td>59.3 (+23%)</td>
<td>71.67 (+49%)</td>
</tr>
<tr>
<td>(?)</td>
<td>52.7</td>
<td>—</td>
<td></td>
</tr>
<tr>
<td>1st torsional + some 4th flapwise bending</td>
<td>86.4</td>
<td>—</td>
<td>97.94 (+13%)</td>
</tr>
<tr>
<td>4th flapwise bending + some 1st torsional</td>
<td>93.7</td>
<td>115.16 (+23%)</td>
<td>141.13 (+51%)</td>
</tr>
<tr>
<td>5th flapwise bending</td>
<td>149.5-153.6</td>
<td>186.85 (+25%)</td>
<td></td>
</tr>
<tr>
<td>2nd edgewise bending</td>
<td></td>
<td></td>
<td>214.04</td>
</tr>
<tr>
<td>(?)</td>
<td>225.1</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

5.4.3 Mode shapes

The displaced shape is found by calculating the coefficients for the shape function series, and these are obtained directly as the eigenvectors for the eigenvalue equation corresponding to each of the modal frequencies/eigenvalues. By convention, the coefficients are normalised to give a value of unity to the displacement function at the tip.

\[
\varphi \cdot \mathbf{a}(j) = eigenvectors \left[ \mathbf{H} \mathbf{M}^{-1} \mathbf{E} \cdot (\lambda_j)^2 \right]
\]

\[
\mathbf{a}(j) = \frac{(\varphi \cdot \mathbf{a}(j))}{(\varphi \cdot \mathbf{a}(j)) \cdot Z(1)}
\]

(Here, \( \varphi \), has no physical significance and is merely an arbitrary product of the eigenvalue solution)
Each mode shape is then found from the product of the appropriate coefficient vector with the vector of shape functions.

\[ y_z(j, x) = a(j) \cdot Z(x) \]

Figure 5.4.2 below illustrates the first 4 mode shapes.

![Mode Shapes Graph](image)

Figure 5.4.2 First four mode shapes calculated for the blade

The following are the normalised coefficient vectors for the solutions of the first 4 modes based on calculations involving 7 terms.

\[
\begin{align*}
a(l, 0) &= \begin{bmatrix} 1.0103 \\ 0.0077 \\ -0.0014 \\ 0.0009 \\ -0.0001 \\ 0.0001 \\ -0.0001 \end{bmatrix} \\
\mathbf{a}(l, 1) &= \begin{bmatrix} 0.1441 \\ -1.0786 \\ 0.0387 \\ -0.0283 \\ -0.0038 \\ -0.0021 \\ 0.0000 \end{bmatrix} \\
\mathbf{a}(l, 2) &= \begin{bmatrix} 0.835 \\ 1.415 \\ 1.434 \\ 0.069 \\ 0.081 \\ 0.013 \\ 0.009 \end{bmatrix} \\
\mathbf{a}(l, 3) &= \begin{bmatrix} 0.039 \\ -0.786 \\ -1.193 \\ -1.045 \\ 0.18 \\ -0.134 \\ 0.009 \end{bmatrix}
\end{align*}
\]

It can be seen from these that, for the first few modes at least, the displacement function in each case is dominated by one or two shape functions and that the
coefficients for the other functions are quite small. Thus, with good shape functions not very many terms are required before a good fit is found and the solutions converge.

5.4.4 Rotating mode shapes and frequencies — full solution

Having calculated the non-rotating mode shapes and frequencies, the next step is to calculate them for the rotating blade, when centrifugal stiffening alters their character.

Returning to the equation for the Lagrangian, (5.4.1), this time the eigenvalue problem is set up with the centrifugal matrix included.

\[
\begin{pmatrix}
H^{-1}M^{-1}E + \Lambda^2 H^{-1}G - \lambda^2 I
\end{pmatrix}a'(j) = 0
\]

or

\[
\begin{pmatrix}
H' M' E + \Lambda^2 H' M' G - \lambda^2 I
\end{pmatrix}a'(j) = 0
\]

(5.4.4)

where

\[
H' M' G = H^{-1}M^{-1}G
\]

\[
\Lambda = \frac{\Omega}{\omega} \quad \lambda = \frac{\omega}{\omega} j
\]

Again, the eigenvalues relate to the modal frequencies, while the eigenvectors relate to the description of the mode shapes.

Modal frequencies

\[
\lambda'(\Lambda) = \sqrt{eigenvals(H' M' E + \Lambda^2 H' M' G)}
\]

Modal shape coefficients (for the \(j\)th mode)

\[
a'(\Lambda) = \frac{1}{\varphi} \cdot vec \left( H' M' E + \Lambda^2 H' M' G, \lambda'(\Lambda) \right)
\]

Mode shape for the \(j\)th mode

\[
y_j(\Lambda, x) = a'(\Lambda)_j^T \cdot Z(x)
\]

Figure 5.4.3 illustrates the extent to which the mode shapes (in this case the 1st, 2nd and 3rd modes) change with rotational speed. Clearly, though there is a change, it is
not that great, bearing in mind that the curves covers a range from stationary to 1000 rpm. In fact, it is quite common practice to calculate the rotating frequencies using the stationary mode shapes unchanged.

Figure 5.4.3 Changes in mode shape with rotor speed for the 1st 2nd and 3rd modes

5.4.5 Rotating modal frequencies — approximate method using Southwell’s equation

If the change in mode-shape with increasing rotor speed is ignored, then the change in natural frequency can be found by Southwell’s formula. Returning to the equation for the Lagrangian, (5.4.1), the stationary mode shapes are used in the form of the coefficient vectors, $\mathbf{a}$, which have already been calculated. Setting the Lagrangian to zero and solving for the natural frequency leads to Southwell’s formula.
\[
\frac{\omega^2}{\omega_0^2} \left( \mathbf{a}^T \mathbf{H} \mathbf{M} \mathbf{a} \right) - \left( \mathbf{a}^T \mathbf{E} \mathbf{a} \right) - \frac{\Omega^2}{\omega_0^2} \left( \mathbf{a}^T \mathbf{H} \mathbf{G} \mathbf{a} \right) = 0
\]

\[
\omega j(\Omega)^2 = \omega_n^2 + \alpha j\Omega^2 \quad (5.4.5)
\]

where \( \omega_n^2 = \omega_0^2 \frac{\mathbf{a}^T \mathbf{H} \mathbf{E} \mathbf{a}}{\mathbf{a}^T \mathbf{H} \mathbf{M} \mathbf{a}} \) and \( \alpha = \frac{\mathbf{a}^T \mathbf{H} \mathbf{G} \mathbf{a}}{\mathbf{a}^T \mathbf{H} \mathbf{M} \mathbf{a}} \)

In non-dimensional terms, this is

\[
\lambda j(\Lambda) = \sqrt{\lambda_n} \frac{\omega_n}{\omega_0} \left( \omega_n^2 + \alpha \lambda \right) \quad \text{where} \quad \lambda_n = \frac{\omega_n}{\omega_0}
\]

Figure 5.4.4 shows the changes in natural frequency with increasing rotational speed and the small differences in the result depending on whether the Southwell equation is used or the full solution (taking changes in mode shape into account). The Southwell method produces slightly higher frequencies, which is to be expected - the fuller solution should always produce lower frequencies as it is less constrained.

Dotted lines have also been plotted of multiples of the rotational speed in the same units as the modal frequencies to indicate when there is a danger of interaction. It can be seen from this graph that the 1st mode crosses the line corresponding to \( 2P \). This is twice the blade passing frequency and is one of the principal frequencies at which the tower receives excitation. The rotational speed at which this occurs can be found by solving an equation based on the Southwell equation (5.4.5).

\[
\omega_n^2 + \alpha \lambda \Omega^2 = (2\Omega)^2
\]

This is solved as a quadratic in \( \Omega \)

\[
\Omega = \frac{2P}{\sqrt{4 - \alpha \lambda}}\quad \Omega = 111.224 \cdot \text{rpm}
\]

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Figure 5.4.4 Change in modal frequency with rotor speed, comparing full solution with Southwell’s method for 1st-3rd modes
5.4.6 Model generalisation and extension

It had been hoped further to generalise the solutions to the eigenvalue equation and to the Southwell equation in order to express them algebraically in terms of blade shape parameters. This would make it possible to see the sensitivities of the blade vibration properties to these parameters. Unfortunately, this has not been achieved successfully as yet, though in principle it should be possible for simplified matrices (2x2) and deriving only 1st order dependences.

It is also worth noting that, whilst a Rayleigh-Ritz model such as the one used here is somewhat involved to set up initially, once this has been done, it should take little effort to extend it to take account of more complex taper-shapes (quadratic or cubic) or blades whose cross section changes along the blade.

With somewhat more effort it should be possible to extend the model to calculate edgewise and torsional modes of vibration and even coupled modes.

5.5 Static deflection model using Rayleigh-Ritz

In order to find the static deflection of the blade under load, an expression for the total potential energy is assembled from the energy terms already examined in terms of the, as yet unknown, coefficients of the deflected shape function, the elastic and centrifugal matrices and the work vector. In order to minimise the PE, this expression is then differentiated with respect to each of the coefficients in turn and set to zero, yielding a set of simultaneous equations. These can then be solved (in matrix form for greatest convenience) for the values of the coefficients.

Total Potential Energy:

\[
U_{TOT} = \frac{k_0}{2} \sum \nu Z \left[ a^T \left( H_E + \Lambda \alpha H_G \right) a - \frac{a^T h Q}{\varphi Z} \right]
\]  (5.5.1)

Differentiating w.r.t. \( a \) and setting to zero minimises PE and gives a set of simultaneous equations:
\[
\frac{d}{da} U_{TOT} = \frac{k}{2} v Z^2 \left[ \left( H_E + A \right)^2 G \right] \cdot a - \frac{h_Q}{\theta} = 0
\]

The solution to these gives a column vector of coefficients.

\[
\varphi Z^a = \left( H_E + A \right)^2 G \cdot h_Q \tag{5.5.2}
\]

For the sake of clarity it is helpful to have the assembled shape function normalised to take a value of unity at the tip end. This requires the calculation of a dimensionless factor, \( \varphi \), which gives the true value of the function.

\[
\text{where } \varphi Z^a = \left( \varphi Z^a \right)^T \cdot Z(1) \tag{5.5.3}
\]

This factor can then be used to calculate the normalised shape function coefficients and also to relate the stiffness of this system to the original 'basic stiffness' of the blade.

Shape coefficients:

\[
a = \begin{pmatrix} \varphi \\ Z^a \end{pmatrix} \tag{5.5.4}
\]

During the construction of the work vector \( h_Q \), a more general approach was looked at whereby the solutions could be found for unit load vectors. This can be set up in a similar manner to the simultaneous equations above but instead there are four parallel sets of equations. The solutions, once normalised, take the form of an assembled four-column coefficient matrix \( A \).

\[
\frac{d}{da} U_{TOT} = \frac{k}{2} v Z^2 \left[ \left( H_E + A \right)^2 G \right] \cdot \theta A - \frac{H_Q}{\theta} = 0
\]

\[
\theta A = \left( H_E + A \right)^2 G \cdot h_Q \tag{5.5.5}
\]

For loadings restricted to the tip-mass zone,
\[ \mathbf{\hat{A}}_{\text{tip}} = \left( \mathbf{H}_E + \mathbf{A} \cdot \mathbf{H}_G \right)^{-1} \cdot \mathbf{H}_Q \]

The sets of coefficients can readily be normalised to give a row vector of compliance factors, \( \mathbf{\hat{c}} \), and sets of shape coefficients in a four column matrix, \( \mathbf{A} \), which give shape functions having an end value of one.

The row vectors of compliance factors are given by

\[ \mathbf{\hat{c}} = \mathbf{Z}(1) \cdot \mathbf{A} \]

\[ \mathbf{\hat{c}}_{\text{tip}} = \mathbf{Z}(1) \cdot \mathbf{A}_{\text{tip}} \]

and the normalised shape coefficients are:

\[ \mathbf{A} = \frac{\mathbf{\hat{c}}}{\mathbf{\hat{c}}} \]

\[ \mathbf{A}_{\text{tip}} = \frac{\mathbf{\hat{c}}_{\text{tip}}}{\mathbf{\hat{c}}_{\text{tip}}} \]

(This is not matrix division but simply division through of each column by the corresponding single element of the compliance row vector.) Once the shape coefficients and the compliance coefficients have been calculated, it is then possible to derive the stiffness of the blade and its deflection and shape under load. Using the universal solution, and a single loading,

End deflection

\[ v \quad Z = \frac{Q}{k_0} \cdot \mathbf{\hat{c}} \cdot \mathbf{q} \]

Blade shape

\[ y \quad \varepsilon(x) = \frac{Z(x)^T \cdot \mathbf{A} \cdot \mathbf{q}}{\sum \mathbf{q}} \]

Deflected form

\[ v \quad \varepsilon(x) = \frac{Q}{k_0} \cdot \frac{Z'(x)^T \cdot \mathbf{\hat{c}} \cdot \mathbf{A} \cdot \mathbf{q}}{\sum \mathbf{q}} \]

Gradient

\[ \theta \quad \varepsilon(x) = \frac{Q}{k_0} \cdot \frac{Z''(x)^T \cdot \mathbf{\hat{c}} \cdot \mathbf{A} \cdot \mathbf{q}}{\sum \mathbf{q}} \]
In Figure 5.5.1, the end deflection calculation can be seen as applied to the example loading from the derivation of $h Q$. It can be seen that, like the calculations of modes of vibration, the deflection results appear to converge quite rapidly. For gradients and curvatures, however, the convergence is somewhat slower. This can be seen clearly in Figure 5.5.2 for the calculation of the blade tip angle.

![Figure 5.5.1 Convergence of tip deflection calculations with increasing number of terms](image)

**Figure 5.5.1 Convergence of tip deflection calculations with increasing number of terms**

![Figure 5.5.2 Convergence of tip gradient calculations with increasing number of terms](image)

**Figure 5.5.2 Convergence of tip gradient calculations with increasing number of terms**

Whilst there is little visible change in the deflected shape (see Figure 5.5.3), it is possible to show up the change more effectively by calculating the percentage error as a distribution relative to the highest order (20 terms) result.
Comparison of deflected curves

**Figure 5.5.3** Change in deflected shape with increase in number of terms in the calculation

This can be seen better by plotting the errors as a percentage of the best estimate possible with the maximum number of terms, in this case 20 terms.

\[
e_z(j, x) = 100 \frac{v_z(j, x) - v_z(n, x)}{v_z(n)}
\]
5.5.1 Calculation of bending stress and strain

The bending strain and stress can be calculated in 2 ways, from the blade curvature or from the bending moments. It should be borne in mind that no account is taken of any possible departure from a uniform tensile strain due to the centrifugal load or of any additional stress due to constraints, particularly constraint against warping at the root end.

The first method is to take the calculated blade curvature - i.e. the 2nd derivative of the deflection curve, multiplied by the maximum distance from the flexural axis, giving the maximum bending strain. The product of this with the elastic modulus gives the maximum stress. However, it has already been seen that the errors in the approximation become progressively worse in going from the deflection to the gradient and to the curvature. It should also be borne in mind that the shape functions do not fully satisfy the boundary conditions which require bending moment and shear force to be zero at the blade tip. There is thus some departure from these conditions in the solutions.

The second method, which is therefore preferable, is to calculate the residual bending moment from the difference between the bending moment due to the external load and that due to the action of the centrifugal load on the deflected shape. Multiplying by the
maximum distance of the shell from the elastic axis and dividing by the 2nd moment of area gives the maximum bending stress.

**Bending moment due to external loading:**

Being a cantilevered beam, points on the blade only experience bending moments due to loads acting further 'outboard' of them i.e. towards the tip. The bending moment at a point \( x \) due to an element of the distributed bending load at \( x_1 \) is the product of the elemental load with the distance from \( x \) to \( x_1 \) along the blade. For the loadings which run the entire blade length, the bending moment is found by integrating from \( x \) to the tip of the blade.

\[
M_Q(x) = -L^2 \int_x^1 (x_2 - x)q(x_2) \, dx_2 \quad \text{where} \quad q(x) = \frac{Q}{L}T \cdot P(x)
\]

\[
... = QL \cdot q^T \int_x^1 P(x_2)(x_2 - x) \, dx_2 = QL \cdot q^T \cdot PPP(x, x)
\]

where \( PPP(x, x) = PPP(l, x, x_l) \)

\[
... = \int_x^1 x_2 l(x_2 - x) \, dx_2
\]

\[
... = \frac{1 - x l}{l + 2} \cdot \frac{x(l - x l + 1)}{l + 1}
\]

Generalising this to the universal solution,

\[
M_Q(x) = -QL \cdot M_Q(x) \cdot q \quad \text{where} \quad M_Q(x) = PPP(l, x, x)
\]

For the illustrative figures employed before, the moment at the root is

\[
M_Q(0) = -46.74 \cdot N \cdot m
\]
Bending moment due to centrifugal loading:

The centrifugal loading only exerts a bending moment as a result of the blade’s deflection.

An element of centrifugal loading is given by:

\[
dG_{bl}(x) = \mu \cdot L^2 \cdot \Omega \cdot 2 \cdot \mathbf{P}(x) \cdot ^T \cdot \mathbf{g} \cdot dx
\]

The bending moment at \( x \) due to the loading on an element of blade at \( x_I \) is the product of the centrifugal load with the difference in bending deflection between \( x \) and \( x_I \), which is given by:

\[
dM_{Gbl}(x, x_I) = dG_{bl}(x_I) \cdot (v(x_I) - v(x))
\]

The bending moment due to the tip mass takes the same form but allowance must be made for the fact that it only runs for part of the blade length. The bending moment at \( x \) due to an element of tip mass at \( x_I \) is found from:

\[
dG_{tip}(x) = \mu_{tip} \cdot L^2 \cdot \Omega \cdot 2 \cdot \mathbf{P}(x) \cdot ^T \cdot \mathbf{g} \cdot dx
\]

so that

\[
dM_{Gtip}(x, x_I) = dG_{tip}(x_I) \cdot (v(x_I) - v(x))
\]

when \( x \geq x_{tip} \)

\[= 0 \]

when \( x < x_{tip} \)

The net bending moment at \( x \) is arrived at by integration.

\[
M_G(x) = \int_x^1 (v(x_I) - v(x)) \, dG_{bl}(x_I) + \int_x^1 (v(x_I) - v(x)) \, dG_{tip}(x_I)
\]

(5.5.6)

As with most of the other quantities, the centrifugal moment can be expressed in terms of dimensioned scalars and non-dimensional matrices and can be calculated directly from the loading vector \( \mathbf{q} \).
\[ M_G(A, x) = L \cdot \lambda^2 \cdot Q \cdot (M_G(A, x) \cdot q) \]  \hspace{1cm} (5.5.7)

where

\[
M_G(A, x) = g^T \left[ \begin{array}{c}
B(x)^T - PP(x) \cdot Z(x)^T \\
+ \frac{\mu}{\mu_{tip}} B_{if} \left( x \geq x_{tip}, x \cdot x_{tip} \right) \cdot Z(x)^T \\
+ \frac{\mu}{\mu_{tip}} PP_{if} \left( x \geq x_{tip}, x \cdot x_{tip} \right) \cdot Z(x)^T
\end{array} \right] \cdot \theta_A(A) \]  \hspace{1cm} (5.5.8)

and \( B(x)_{j,l} = B(j + \frac{1}{2}, l, x) \) (See Appendix A3)

For the numerical values used for illustration before, the moment at the root is

\[ M_G(5, 0) = 38.21 \cdot \text{N}\cdot\text{m} \]

To establish equilibrium, the sum of the moments due to external work, centrifugal load and elastic curvature must be zero at all points, so the elastic moment can be calculated from the other two.

\[ M_E(A, x) = -(M_G(A, x) + M_Q(x)) \]

\[ M_E(5, 0) = 8.52 \cdot \text{N}\cdot\text{m} \]

The distributions of the different moments turn out as in Figure 5.5.6.
This elastic moment distribution can then be used to calculate the normal stress and strain.

Maximum bending stress in the downwind surface is given by:

$$\sigma'_{bd}(x) = M_E(x) \cdot \frac{-b_{\text{max}}}{I'c} \cdot \frac{2}{(1 - c'x)^2}$$

At the root, this is $$\sigma'_{bd}(5,0) = -11.98 \cdot \text{MPa}$$ where a negative stress represents a compression.

In the upwind surface, this is

$$\sigma'_{bu}(x) = M_E(x) \cdot \frac{-b_{\text{min}}}{I'c} \cdot \frac{2}{(1 - c'x)^2}$$

giving a root stress of $$\sigma'_{bu}(5,0) = 8.77 \cdot \text{MPa}$$

The strains are found by dividing by the effective elastic modulus on the blade axis.
\[ \varepsilon_{bd}(x) = \frac{\sigma}{Ea}, \quad \varepsilon_{bu}(x) = \frac{\sigma}{Ea} \]

Downwind root strain: \( \varepsilon_{bd}(5,0) = -4.28 \cdot 10^{-4} \)

upwind: \( \varepsilon_{bu}(5,0) = 3.13 \cdot 10^{-4} \)

The total strain on the blade axis is found by adding a contribution from the centrifugal tension.

\[
G(x) = \mu \cdot \frac{L^2 \cdot \Omega^2}{2 \cdot \left( \mathbf{PP}(x)^T \mathbf{PP}(x) + \frac{\mu}{\mu} \cdot \mathbf{PP}(x)^T \mathbf{PP}(x) \right)} \cdot g
\]

\[
\sigma = \frac{G(x)}{S' \cdot c \cdot \delta t} \cdot \frac{1}{1 - c' \cdot x}, \quad \varepsilon = \frac{\sigma}{Ea}
\]

At the root, this is \( \varepsilon(0) = 4.1 \cdot 10^{-4} \)

Since the blade mostly bends downwind, it is largely the upwind surface which is in tension so the maximum total strain is found there, generally at the root.

\[
\varepsilon_{tot}(x) = \varepsilon \cdot g(x) + \varepsilon_{bu}(x)
\]

At the root this is \( \varepsilon_{tot}(5,0) = 7.22 \cdot 10^{-4} \)

Looking at Figure 5.5.7, as one would expect based on the simple examples in Section 5.1, for most of the blade length, the centrifugal tension carries the load there is little elastic bending moment or bending strain. However, near the root, it is only due to the elastic properties of the blade that it conforms to a condition of zero slope and thus much more of the bending moment is carried elastically, giving rise to significant bending strains.
5.5.2 Model generalisation and extension

As with the model for mode calculations, it had been hoped further to generalise the static deflections model. It was hoped that it would be possible to arrive at design sensitivities by expressing the matrices algebraically in terms of blade shape parameters and again, this should be possible to the first order of approximation, but it has not yet been achieved.

It was also hoped that it would be possible to arrive at an analogy to the Southwell equation in order to facilitate calculations for different rotor speeds. The deflected shapes do not change enormously with increasing centrifugal stiffness (See Figure 5.5.8), though the changes are subjectively greater than for the mode shapes and it is relatively simple to show numerically that the blade stiffness is close to being proportional to the rotor speed squared. (See Figure 5.5.9)

Attempts so far to relate the proportionality constant to known parameters have not been successful. Neither has an attempt at a Southwell-type use of the stationary shape coefficients to construct a rotor speed stiffness coefficient. More direct attempts to find an algebraic solution by expressing the inverted form of the combined
elastic and inertial matrix in terms of the separate matrices and the rotor speed coefficient were also not successful. Series expansions derived from this approach were divergent. However, I feel sure that some such generalisation is possible.

![Diagram showing change in shape of the displacement function for uniform loads as rotor speed increases.](image)

**Figure 5.5.8** Change in shape of the displacement function for uniform loads as rotor speed increases

![Diagram showing blade stiffness coefficient variation with rotor speed for different shaped loadings.](image)

**Figure 5.5.9** Blade stiffness coefficient variation with rotor speed for different shaped loadings
5.6 Deflections due to (simulated) aerodynamic loadings

The generalised model of static deflections can be applied to series of loading distributions obtained from the aerodynamic simulation of the rotor (see chapter 4).

Runs of the simulation were carried out corresponding to three conditions. The obvious one is the so-called ‘design’ condition, in which the rotor generates maximum power if it is below rated power and rated power at wind speeds over the rated value.

The ‘runaway’ condition corresponds to an unloaded rotor which generates no power and races to faster rotor speeds and higher tip-speed ratios than the design condition.

The ‘gust’ condition is somewhat fictitious. In order adequately to model the behaviour of the rotor in true gust conditions, a full dynamic model is necessary. However, it was considered to be worthwhile to form an estimate of the order of magnitude of deflections and strains that would occur under such conditions. For this set of simulation runs, for each value of rotor speed, the wind speed was set to twice the design value. This corresponds to lower tip-speed ratios than the optimum.

The three conditions are compared numerically in Table 5.6.1 and graphically in Figure 5.6.1 in terms of how rotor speed varies with wind speed.

<table>
<thead>
<tr>
<th>Rotor speed (rpm)</th>
<th>wind speeds (m/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N_{rot}$</td>
<td>design $U_{a}$</td>
</tr>
<tr>
<td>400</td>
<td>7.2</td>
</tr>
<tr>
<td>600</td>
<td>12.2</td>
</tr>
<tr>
<td>1000</td>
<td>20.8</td>
</tr>
<tr>
<td>1400</td>
<td>47.0</td>
</tr>
</tbody>
</table>
5.6.1 Load distributions

The aerodynamic simulation routine gives the load distribution for a particular simulation run in the form of load density at each blade station. In this form, it cannot be applied to the blade bending model. Instead, a cubic curve must be fitted to the data which can be achieved quite simply with a least squares routine. (See Appendix A3)

All three sets of loading grow with wind speed and rotor speed, as can be seen in figures Figure 5.6.2 and Figure 5.6.3 below. However, whilst the runaway loading varies smoothly, the design condition behaves differently. Where it is plotted against rotor speed, Figure 5.6.2, there is a large level region in the slope between roughly 600 and 1000 rpm. This merely reflects the switch in the power control system from below to above rated power, which results in the ‘kink’ in the speed curve in Figure 5.6.1. A similar, though less pronounced, phenomenon is seen in the curves for the gust condition. When viewed in terms of load against wind speed, the design condition subjects the rotor to lower loads than the runaway condition at wind speeds above the rated value.
**Figure 5.6.2** Thrust load as a function of rotor speed for the 3 rotor cases

**Figure 5.6.3** Thrust load as a function of wind speed for the 3 rotor cases
5.6.2 Blade deflections

Figure 5.6.5 shows the results of calculations of the deflection of the blade tip for the three loading schemes and how they vary with rotor speed. Figure 5.6.6 shows the same comparison but plotted against wind speed. In either case it can be seen that, even at the highest speeds, the deflections at runaway are insignificant.

The gust results, however, are certainly significant. It is interesting to note that they are at their worst at 500-600 rpm, which corresponds closely to the rated speed. The turbine is running at a low tip-speed ratio, approximately half the optimum value. Here can be seen the disadvantage of designing a machine to pitch in the feathering direction. The design lift coefficient is quite low and the aerofoil has not been selected for its stalling characteristics. A stall-controlled or stall-assisted wind turbine should not suffer too severely from gusts as these would merely push the blade further into stall. In this wind turbine, a decrease in the tip-speed ratio results in an untempered increase in lift and therefore in the bending loads. Whereas in the runaway condition the centrifugal stiffening prevents even quite severe loads from
causing large deflections, this is far less effective at the low rotor speeds of the gust condition.

Although a maximum deflection of approximately 60 mm may not seem too great, it should be borne in mind that, if a real gust is treated as a step increase in wind speed, the dynamic behaviour of the blade and the aerodynamics (induction lag) are likely to result in considerable overshoot.

**Figure 5.6.5** Blade tip deflections due to different rotor conditions for a range of rotor speeds

**Figure 5.6.6** Blade tip deflections due to different rotor conditions for a range of wind speeds
Figure 5.6.7  Deflected shapes at 1000 rpm for design, runaway and gust conditions

Figure 5.6.8  Deflected shapes at 600 rpm for design, runaway and gust conditions

Figure 5.6.9  Deflected shapes for gust condition at 3 rotor speeds
5.6.3 Bending moments

As explained in section 5.4, the bending moment distribution must be calculated in order to find the strain distribution suffered by the blade.

As an example, the root bending moments due to the loadings at 1000 rpm can be broken down into the components carried by centrifugal and elastic stiffness. See Table 5.6.2.

<table>
<thead>
<tr>
<th>Table 5.6.2 Root bending moment components from external load, centrifugal stiffness and elasticity at 1000 rpm</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bending moments</td>
</tr>
<tr>
<td>-----------------</td>
</tr>
<tr>
<td>design condition</td>
</tr>
<tr>
<td>runaway condition</td>
</tr>
<tr>
<td>gust condition</td>
</tr>
</tbody>
</table>

The following Figure 5.6.10 to Figure 5.6.20, illustrate how the bending moments and their distributions vary with wind speed and rotor speed and between the different rotor conditions. In general, like the loads, the bending moments rise monotonically with wind speed and rotor speed, but the exception is that for the design condition, a peak is reached at a low rotor speed of approximately 600 rpm, corresponding to a wind speed of 12.2 m/s. (See Figure 5.6.12) This is just over the speed at which the rotor has reached the rated power, and thus, when following the design condition, it no longer tracks the optimum tip-speed ratio but runs away until only the rated power is generated. This condition of partial runaway is reflected in a similar peak in the blade deflection at this wind speed. (See Figure 5.6.6)

Regarding the bending moment distributions, it can be seen that for the runaway condition and for the gust condition, the distribution shape does not change much with speed, only the magnitude changes. However, for the design condition, the distribution is similar to that for the runaway condition at high speeds but similar to the gust condition, reflecting the switch in behaviour at rated speed.
Figure 5.6.10 Elastic component of bending moment in all three rotor conditions as a function of wind speed

Figure 5.6.11 Variation of root bending moments at runaway

Figure 5.6.12 Variation of root bending moments in the design condition
Figure 5.6.13 Variation of root bending moments in the gust condition

Figure 5.6.14 Moment distributions at runaway at 1400 rpm, 46.2 m/s

Figure 5.6.15 Moment distributions for design condition at 1400 rpm, 47 m/s
Figure 5.6.16 Moment distributions at runaway at 1000 rpm, 19.1m/s

Figure 5.6.17 Moment distributions for design condition at 1000 rpm, 20.8m/s

Figure 5.6.18 Moment distributions for gust condition at 1000 rpm, 41.7m/s
5.6.4 Blade strains

The most important figures to come out of the blending model, apart from the blade deflections, are those for the bending strains suffered by the blade. These give the clearest picture of how close the blade is to failure under the bending conditions considered. In any particular state, there are contributions to the total strain from bending and from centrifugal tension and each tends to dominate in different circumstances.
In the runaway condition (Figure 5.6.24), there is little strain due to bending, most being tensile strain due to the centrifugal load. In the gust condition (Figure 5.6.25), the reverse is the case. There is a much smaller centrifugal strain but there is also little centrifugal contribution to stiffness. The large deflections therefore give rise to large bending strains. The design condition (Figure 5.6.23) shares the strain roughly equally between tensile and bending.

It is clearly the bending strains which are the most worrying, particularly in gust conditions. They not only take the largest values, but they are the strains which vary on the most rapid time-scale, and are therefore more likely to cause problems with fatigue.

One aspect of these results is slightly more encouraging; despite the bending strain reversing sign at some point on the blade, even in the gust condition, there is just enough centrifugal tension to prevent the total strain from changing sign except by small amounts near the tip at the lower speeds. (Figure 5.6.31 and Figure 5.6.32) This is important. — when fatigue is a concern, a point on the blade which experiences a strain passing through zero in a steady state model would in reality, under dynamic conditions, experience a rapidly reversing strain. This would be the best way to cause the material to fail!

Despite this more encouraging note, the high values of bending strains strongly suggest that it would be worthwhile examining the possibility of additional load relief by incorporating some bending-twisting coupling.
Figure 5.6.21 Comparison of root bending strains for design, runaway and gust conditions

Figure 5.6.22 Comparison of total axial root strains for design, runaway and gust conditions

Figure 5.6.23 Bending and centrifugal contributions to the root strain in the design condition
Figure 5.6.24 Bending and centrifugal contributions to the root strain in the runaway condition

Figure 5.6.25 Bending and centrifugal contributions to the root strain in the gust condition
Figure 5.6.26 Strain distributions for runaway condition at 1400 rpm, 46.2 m/s

Figure 5.6.27 Strain distributions for design condition at 1400 rpm, 47.0 m/s
Figure 5.6.28 Strain distributions for design condition at 1000 rpm, 20.8m/s

Figure 5.6.29 Strain distributions for gust condition at 1000 rpm, 41.7m/s
**Figure 5.6.30** Strain distributions for design condition at 600 rpm, 24.4 m/s

**Figure 5.6.31** Variation of bending strain distribution with speed in the gust condition
Figure 5.6.32 Variation of total axial strain distribution with speed in the gust condition.

total strain distributions in gusts

--- 1000rpm
--- 600rpm
--- 400rpm
6. AEROFOIL PROFILES

6.1 Aerofoil profile selection

At the stage of designing the prototype blade assembly, most of the design criteria are based on the insights already gained from simulations with different blade shapes etc. However, the question of choosing an aerofoil profile is not so straightforward. Some insight has been gained into the characteristics desired but short of designing and testing a 'made-to-measure' profile (a lengthy and involved process) a selection must be made from existing profiles based on whatever data already exist.

It is only in recent years that much attention has been paid to the particular needs of wind turbines with regard to aerofoil characteristics. Prior to this, selections were made on the same basis as for aviation from existing aviation aerofoils. One approach has been to design aerofoils specifically for wind turbines, as their requirements are generally quite different from those of other types of wing, particularly aircraft wings.

A common theme in nearly all uses of aerofoils is that drag is generally unwanted and lift is wanted. In general, the higher the lift-to-drag ratio, the better. This is certainly true of wind turbine blades. However, in reality the perfect all purpose aerofoil section has not been and will not be developed, and in practice, one ends up with many different aerofoil sections each designed with emphasis on particular conditions and requirements.

The vast majority of work on aerofoil development has been aimed at improvements for military and commercial air flight. The requirements here are primarily concerned with drag reduction at high Reynolds' numbers, which is achieved by extending the range over which laminar flow is maintained before it trips to turbulent flow. Much work has also been put into developing aerofoils that can work at Mach numbers ever closer to one without having any regions of sonic flow. In general, aeronautical
aerofoil design pays very little attention to the behaviour of the aerofoil in stalled flow
- essentially, the aircraft falls out of the sky, which is best avoided.

Wind turbine blades, in contrast, rely on operating in stalled flow for a substantial
amount of the time. They also have to keep operating with insect and other dirt
accumulated on their blades, whereas aircraft are cleaned much more frequently.
Thus work has been carried out, with some success, on developing aerofoil profiles
specifically designed for wind turbine blades. (Tangler and Somers[ 43 ]). Indeed,
families of profiles now exist, with different properties for blade root regions, the
main run of the blade and the tip region. All of these have been designed to stall
gently. In the tip region, this happens at low lift, while the root region aerofoils do not
stall until a high lift coefficient is reached. This accords with the requirements of stall
control and stall-assist pitch control. The root region aerofoils are thick in section for
structural reasons, as in aircraft, whilst the tip sections are relatively thin to minimise
drag.

However, most of these profiles are also designed for use at relatively high Reynold’s
numbers due to the size of commercial wind turbines. For a small wind turbine which
twists towards feather as the means of speed regulation, some of the requirements are
the same as for commercial scale machines. As ever, a good lift-to-drag ratio is
important, as is insensitivity to accumulated roughness.

However the 3m diameter machine with self-twisting blades operates in a Reynold’s
number regime altogether different from larger machines and the ‘special, designed
for wind turbine’ aerofoil profile families have been optimised for the higher
Reynold’s numbers of commercial machines (with one exception). The low
maximum lift coefficient required at the tip on stalling machines is no longer a
particular benefit, though a similar condition in the negative angle-of-attack zone
would be beneficial (see Chapters 2 & 4). Lastly, the aim of maximising twist for a
given rotor speed requires the enclosed area of the blade section to be minimised
which inevitably means a small thickness-to-chord ratio. This is the opposite of the
design approach for the root-region profiles, which are thick in order to maximise
bending strength and stiffness.
Short of designing a new set of profiles, it was necessary to search for profiles which would satisfy at least some of these criteria. Conventional aviation sections were ruled out on grounds of Reynold’s number (with the exception of NACAxxxx profiles).

6.1.1 Reynold’s Number

A brief word here about Reynold’s number ‘regimes’ will help to explain the importance of this important quantity. Reynold’s number is a measure of the relative importance of viscous and inertial forces on a body immersed in a fluid flow.

In a high Reynold’s number flow, greater than, say 2 million, such as that describing the wing of a commercial ‘plane, the inertial forces tend to dominate and lead to a tendency towards early transition to a turbulent flow regime. This is relatively stable with respect to flow separation, but has relatively high shear stresses. For many years, much effort has gone into designing profiles which delay transition, prolonging laminar flow along the profile. The shear stresses are lower, and so there is a consequent reduction in drag. Most aerofoil data in the public domain cover this type of Reynold’s number regime.

In a low Reynold’s number regime, with Re less than, say, 100 000, the flow has little tendency to become turbulent as the viscous forces are relatively larger and damp out turbulent eddies. However, over the rear half of the profile on the upper surface, the flow is slowing down after its peak speed just behind the leading edge and thus increasing in pressure (Bernoulli’s law). The flow is thus proceeding against an adverse pressure gradient. The boundary layer has a tendency to form a velocity profile with a local minimum which leads to flow separation. The resulting low pressure region behind the aerofoil gives rise to very high pressure drag which far outweighs any benefits from the lower skin-friction drag associated with laminar flow.

A small amount of data are available covering this type of regime, based on work at Göttingen (See Miley [36]). It should also be said that at very low Reynold’s numbers, there is little difference between the behaviour of aerofoils except that
thickness produces nothing but adverse aerodynamic effects. It is often preferable to employ an inclined flat or curved plate to anything more sophisticated.

There is also an intermediate Reynold’s number regime which is transitional between the aforementioned scenarios. The tendency is for the flow to separate as for low Re and for the separated flow to become turbulent. The turbulent flow is more effective than the laminar flow at mixing and tends to bring energy and momentum down into the lower levels of the boundary layer. This may be enough to re-attach the flow, which continues as a turbulent boundary layer. There is thus a so-called separation bubble at some point on the profile. It is the size of this bubble which is the main determinant of drag and any departure of the lift curve from linearity. As it changes in size with changes in various flow parameters, it becomes extremely difficult to predict the variation of lift, drag and pitching moment with angle of attack. Possibly more serious is that, even when the drag data are known, they are anyway far from being smooth or reliable. The curve frequently displays a double minimum and shows a lack of consistency between different measurement conditions.

This, unfortunately, is the regime in which our 3m diameter machine will operate, with a design Reynold’s number of approximately $3.6 \times 10^5$, and just over twice that figure in extreme conditions.

To exacerbate the problems, data for this regime are not widely available and this is possibly the main factor limiting which aerofoils can be considered as candidates. The main work on aerofoils for this regime has been carried out by Selig et al [40] at Princeton University, and continues at University of Indiana at Urbana Champain [19] and is aimed at improvements in competitive model aircraft. (It should be noted that, unfortunately, ref. [19] was not available at the time the aerofoil selection was made and thus some of the aerofoils covered in it were not considered as candidates.)

Clearly, not all the requirements for improvements in model aircraft are relevant to wind turbines. However, some of the airfoils developed are worth considering as candidates and the Selig team have provided a valuable service in publishing aerofoil lift and drag data from their own wind-tunnel measurements for a wide range of
aerfoil profiles in the right Reynold's number regime, including both their own
designs and pre-existing ones.

As elsewhere, most of the work at Princeton has been aimed at reducing drag for the
appropriate operating conditions. Traditionally, it was common on such model
aircraft to add devices to the wings such as trip strips or turbulators which trigger
transition to turbulence further forward than the separation bubble, forestalling its
creation, since the turbulent boundary layer is much less likely to separate. Much of
the development work has gone towards avoiding the need for such devices, which
inevitably increase drag for those conditions when separation would not occur. This
is achieved by delaying the adverse pressure gradient as long as possible with a so-
called bubble ramp. This essentially consists of a continuation of the full aerfoil
thickness much further back than is common in most aerfoils. The last section has a
steep pressure recovery zone where separation finally and inevitably occurs but at a
higher final pressure. These profiles have been quite successful in reducing drag, in
effect by prolonging unseparated laminar flow, and also in improving the smoothness
of the aerfoil characteristics.

Other aspects of such aerfoil designs are less relevant to the operation of the wind
turbine, or may even conflict with it. For example, some of the profiles have a
moderately low drag coefficient over a wide range of angles of attack. Others
concentrate on a relatively narrow range, depending on the type of competition for
which they are intended and these are more suitable for the self-twisting blades.

Out of the limited range of profiles for which data exist at the appropriate Reynold’s
numbers, a few were selected as possible candidates according to certain criteria;

- Less than 12% thickness (to give adequate stretch-twist coupling)
- Greater than 3% camber (to shift the drag minimum to reasonable lift values)
- High lift-to-drag ratio at moderate values of lift (0.5 - 0.6)
- Large drag rise as lift falls to low values (by eye from drag polars).
Unfortunately, it was not possible to select on the basis of insensitivity to roughness, as for most aerofoils there are no data in this respect.

The main sources of data were Selig et al. [40] and Miley [36]. The candidate aerofoils were:

Aquila, DAE51, FX60-100, FX63-137, NACA6409, Spica, S4062.

For these candidates, lift and drag curves were plotted against each other and a number of parameters were calculated in order to compare them more rigorously, especially lift-to-drag ratio at a suitable value of lift, perimeter-to-area ratio, lift slope, drag-rise ratios. It is also worth considering the design angle of attack. Other more qualitative criteria were also considered, particularly the commentaries on the profiles by Selig on the basis of experience in the model aircraft world and a judgement by eye of their ease of construction based on the plotted shape.

### 6.1.2 Lift-to-drag ratio

The requirements for the wind turbine with self-twisting blades are for a good lift-to-drag ratio at a moderate value of lift, for operation at all low wind speeds. (For more detailed consideration of L to D ratio, see below.) The angle of attack at any one point on the blade will remain almost the same throughout this range as the machine should operate at approximately the maximum tip-speed ratio.

At higher wind speeds, low drag is no longer desirable as there is excess power available. Instead, the requirement is that the blade twist should be effective at regulating rotor speed. There are three aspects of an aerofoil’s characteristics which might affect its merit in this regard; lift slope, off-design drag increase and twist development.

### 6.1.3 Lift slope

If it were possible to find an aerofoil with a higher than normal value of the lift slope, then this would obviously be beneficial. Then, the lift at the tip would drop off quickly with increasing twist and even become negative lift. Unfortunately, though
there are differences between different profiles with respect to this parameter, they are very small.

### 6.1.4 Drag rise

The second aerodynamic parameter which has a major influence on speed regulation is the rate at which drag increases at low lift coefficients, in the 'negative stall' region. Clearly, the aerofoils designed to operate with low drag over a wide range are unsuitable. Those with a narrow drag 'bucket' are preferable and this fortuitously tends to occur most with thin aerofoils.

Whether or not the drag bucket is wide or narrow, it is more likely to have a large increase in drag at or around zero lift when it is centred on a high value of lift. As a rule of thumb, whilst the lift curve is shifted upwards by an increase in camber or otherwise, the curve of drag versus angle of attack does not move very much and remains approximately centred on a zero angle-of-attack, which thus corresponds to a higher lift. Thus highly cambered aerofoils or those which otherwise achieve a low angle of zero lift, tend to have a drag curve centred around a high value of lift and thus quickly develop drag as lift falls.

Other factors which are less simple to quantify also have an important influence and may be extremely subtle. In particular, the exact shape of the profile, especially the rear half of the upper surface can have a major effect on the shape of the drag curve. It is highly unlikely that any aerofoil profiles are designed to have a characteristic with a narrow drag curve, but some may have it as a by-product of the compromises that are inevitable in any design process. A wide range of low drag may be traded off to obtain a particularly low value of minimum drag at one value of lift or for a gradual stall. (For aircraft of any sort, the operating range is generally extended by the use of flaps.) In other cases, a narrow drag bucket may merely be an accident.

### 6.1.5 Design angle-of-attack

This is less important than the other factors. However, given a design value of lift, the angle of attack directly affects the set pitch angle of the blade in the final design.
When the turbine is starting up from rest, the blade is deeply stalled, so its thickness and camber become almost irrelevant. The lift under these conditions is roughly proportional to the pitch angle. Thus, a low value of design angle-of-attack is beneficial as it leads to a relatively high set-pitch angle (for the design tip-speed ratio) and therefore a better starting torque.

The aforementioned properties can act as a guide for candidate selection, but in terms of actually comparing profiles, a more quantitative approach is needed.

Lift gradient $C_L'\alpha$ is easy to quantify, the drag properties are less so. Initially, in this study, the figure used was the ratio in which the drag rose in going between two lift values, one an estimate of design lift, e.g. 0.6, the other an estimate of regulating lift, e.g. zero. This, of course could only be calculated for some of the aerofoil sections as not all had data covering lift coefficients down to zero.

Another figure used was based on the assumption that the drag curves are well behaved and that even a small upwards curvature gives some indication of the trend outside the range available. The measure is based on the change in angle-of-attack required in order to cause the drag to rise by a certain multiple. The reference value could either be the minimum drag or the drag value at a particular value of lift.

Although the ‘fairest’ aerodynamic comparison would be to measure from the minimum drag, this is beset with problems. This approach would have to be based on the assumption of designing at minimum drag, but as will be seen later, this is not necessarily the best ‘point’ for the design. It is also clear that the chord width of the blade is determined by the design lift and this in turn influences how fast twist develops. The aerofoils have therefore been compared on the basis of the rise in drag from a baseline value at a particular value of lift of 0.6, chosen to be roughly the value to be used in the design. Ideally, one would look for the angle required for a large drag rise, say a factor of 2, but for most of the profiles, such data were not available, so a factor of 1.3 was used instead. This angle change is labelled $\Delta\alpha_{1.3\times\Delta}$. We are thus measuring the ‘promise’ of the different profiles rather than their actual drag rise and we have to hope that the drag trend continues in the same manner.
6.1.6 Cross-section geometry

The other parameter that influences the ability of a blade to regulate rotor speed is its cross-section geometry. As can be seen from the twist equations in chapter 3, all other factors being equal, the twist rate is inversely proportional to the ratio $A/S$ or $A'/S'$ where $A'$ is the non-dimensional enclosed area of the aerofoil section and $S'$ is the non-dimensional perimeter length. This quantity is essentially proportional to thickness and thus does not vary greatly between the candidate profiles with the exception of FX63-137 and Spica which have higher values than the rest.

<table>
<thead>
<tr>
<th>Table 6.1.1 Comparison of aerofoil properties</th>
</tr>
</thead>
<tbody>
<tr>
<td>properties</td>
</tr>
<tr>
<td>camber</td>
</tr>
<tr>
<td>thickness</td>
</tr>
<tr>
<td>$S'$</td>
</tr>
<tr>
<td>$A'$</td>
</tr>
<tr>
<td>$A/S$</td>
</tr>
<tr>
<td>$\alpha_0$ (estimated from curve fit)</td>
</tr>
<tr>
<td>$\alpha_{0.6}$</td>
</tr>
<tr>
<td>$C_L^{-\alpha}$</td>
</tr>
<tr>
<td>$C_{D_{00}}$</td>
</tr>
<tr>
<td>$C_{D_{0.6}}$</td>
</tr>
<tr>
<td>$C_{D_{00}}/C_{D_{0.6}}$</td>
</tr>
<tr>
<td>$\Delta \alpha_{1,3\delta}$</td>
</tr>
</tbody>
</table>

| properties | AQUILA | DAE51 | FX60-100 | FX63-137 | NACA6409 | S4062 | SPICA |
| $L/D_{0.6}$ | 50.21 | 60.61 | 59.29 | 41.30 | 49.38 | 53.98 | 48.66 |
| $(A/S)/C_{L^{-\alpha}}$ | 0.307 | 0.314 | 0.280 | 0.388 | 0.309 | 0.319 | 0.376 |
| $(A/S)-\Delta \alpha$ | 0.096 | 0.090 | 0.153 | 0.188 | #N/A | 0.113 | 0.120 |

180
Figure 6.1.1 Comparison of candidate aerofoil profiles

Figure 6.1.2 Comparison of lift curves
Initial selection was based on a subjective weighing of the various factors. Out of the aerofoils for which the full range of data was available, Spica had the best drag rise over the lift range but had a poor A/S ratio, FX60-100 was far better on this score but poorer in terms of drag rise, whilst S4062 was a good compromise. The S4062 was designed by Prof. Michael Selig as an advance on the successful S4061 with the intention of further improving his ideas about bubble ramps, but it failed to deliver
and proved to have a narrower range of low drag - hence its apparent suitability for this project.

Later, the measure of angle of attack change required for a particular drag rise was used and this allowed a more quantitative comparison as well as allowing the examination of all the aerofoils. By multiplying the angle change required by the A/S ratio one gets a figure which is related to how much centrifugal load is needed to induce a given rise in drag. Of course this is not the full picture with regard to regulating the turbine, as the lowering of lift has a direct effect in reducing torque and thrust as well as affecting the axial induction factor and therefore the flow angle. However, the figure should give some picture of the fitness of an aerofoil for the task. On this figure, DAE51 clearly has the best (lowest) score. It also had the best lift-to-drag ratio at the appropriate value of lift.

Taking into account the slope of the lift curve, the reciprocal of this figure can also be multiplied by the A/S ratio to give a measure of how fast a blade loses its lift as the centrifugal load increases. On this figure, FX60-100 has the best (lowest) score with a clear margin of nearly 10% from the others. DAE51 is the fourth best, though the difference in this figure from AQUILA (2nd) and NACA6409 (3rd) is only about 2%.

It is a difficult matter to decide which measure of fitness should be given greater weight in choosing an aerofoil, as this requires a detailed picture of the relative contributions of drag and lift in a retarding torque developed by the outer portion of the wind turbine. This question was looked at briefly in an earlier chapter but realistically, all that was shown was that in different circumstances either one or the other could dominate. There would appear to be no sensible way of combining the two figures to give a single quantity for comparing aerofoils and instead a choice has to be made in a more subjective way.

In the end, DAE51 was selected. This was based on the guess that the differences in the drag curves would make more difference than the smaller differences in the lift curves. From the construction point of view, DAE51 has a simpler shape, having a lower/pressure surface which is almost flat, compared with the more markedly recumbent lower surface of the FX60-100. It also achieves the same lift at a much
lower angle-of-attack, which should give much better starting torque. However the FX60-100 may have other benefits: a recumbent pressure surface close to the trailing edge is known to give an aerofoil’s characteristics a measure of insensitivity to roughness. At the stage of second and third prototypes, it may be necessary to reconsider aerofoil choice.

The DAE51 was designed by Prof. Mark Drela [12] for the propeller of the Daedalus project human powered flight from Crete to Santorini. This profile was intended for an airscrew rather than a wing and most importantly has been proven successful in real flight.

Having selected DAE51, a few problems have to be addressed. The most crucial one is that, as mentioned before, the wind tunnel data available do not extend over an adequate range of lift. The second is that the aerofoil, like almost all others, has been designed to have a sharp trailing edge. In practice for the sake of manufacture and robustness, the trailing edge has to have a finite thickness, but this will inevitably have an effect on the aerodynamic characteristics. Thirdly, there is the question of choosing a design point for the aerofoil.

6.2 Aerofoil profile 2D aerodynamics simulation

6.2.1 Extending data range

Lacking the funding, facilities or expertise to conduct a series of wind tunnel tests to obtain experimental data over the necessary range, an attempt was made to cover the gap using a software aerofoil aerodynamics simulation. The software used was Xfoil.

Owing to a lack of appropriate aerofoil data in the right range of angles-of-attack, some means had to be found of estimating that data in order to carry out meaningful simulations of the regulating behaviour of the self-twist mechanism. It was decided to employ an aerofoil aerodynamics simulation package called Xfoil, developed by Prof.
Mark Drela at MIT. This package has been shown to be particularly good at simulating low to medium Reynold’s number flows (Drela and Giles [11]).

The method of simulation is not a full Navier Stokes approach. Classical potential flow panel methods are used for the flow outside the boundary layer which determines the pressure distribution. Classical integral methods are used for calculating the development of the boundary layer, which determines the flow displacement. The overall calculation iterates between the two until they converge. There are special methods used to make allowance for separation bubbles and it is also possible to change the panel density over some parts of the foil to improve resolution in critical zones.

It is relatively easy to run Xfoil over the normal range of angles of attack. With some care, it is also not hard to cover the positive stall region. It should be born in mind that there are inherent instabilities in flow which involves separation, particularly the early stall region. In the real flow, there is always a measure of hysteresis in that at a particular angle of attack, the flow may separate from different stations on the blade depending on the angle of attack history. It is inevitably going to be difficult to achieve a unique solution of the flow equations when this is the case, and it is thus harder to make the iterative solution converge. Some success can be achieved by increasing the angle of attack very gradually.

For each new calculation, the program uses boundary layer thickness and pressure distributions from the previous calculation as the starting point for the new calculation. Small changes in angle of attack are much more likely to lead to a converged solution than large step changes. Despite this, where the flow is more unstable, this is to some extent reflected in the stability of the equations and it is much harder to reach a convergent solution. As most aerofoils have been designed specifically not to stall too harshly with increasing angle of attack, simulating partially stalled conditions is not too hard.

However the negative stall region, which is of primary interest in this study, is much more difficult. In effect, one is using an aerofoil with a concave suction surface. Unlike the more normal arrangement, this does not have the progressive increase in
negative slope of the normal suction surface which eventually becomes too steep for 
flow to remain attached. Instead, the zone immediately after the leading edge zone 
dips away sharply, causing a strongly adverse pressure gradient. This tends to lead to 
separation, though the flow may reattach further back towards the tail where the slope 
is more gradual.

From the simulation point of view, the change from an attached flow to a separated 
flow may happen over a very small change in angle of attack or even instantaneously. 
Because the solver uses each calculated boundary layer and pressure distribution as 
the starting configuration for the next calculation, this can cause serious problems. 
There is too great a discrepancy between one solution and the next and the solver may 
fail to converge. This seems to occur at around the same angle of attack every time, 
between -4° and -5°, and may genuinely reflect a sharp change in the aerofoil flow, 
perhaps from attached to detached flow. For this reason, it has proved very hard to 
obtain exactly those data points which are required, though some picture of the trend 
in the drag curve can tentatively be guessed at.

A further problem is that the parameters of the simulation strongly affect both the 
ability of the solver to find a convergent solution and the solution it eventually finds. 
Thus, whilst it is always difficult to find a convergent solution when the separation 
bubble is growing, it may paradoxically be slightly easier when there is only poor 
resolution of the bubble due to coarse panelling. A finer panel concentration in the 
region of the bubble may resolve the boundary layer parameters more accurately, but 
make the solution less stable; with a small bubble, the flow parameters tend to vary 
very sharply over a short distance, making convergence impossible. Thus, by varying 
the operating parameters for the solver, for instance the panel distribution, one can 
obtain different drag curves for the same aerofoil under the same flow conditions. 
One of these may extend over a wider range of angles-of-attack than the other and yet 
could be a poorer reflection of the true aerofoil behaviour.

In practice, the drag data obtained appear to fall onto two curves. (Figure 6.2.2) (The 
curves marked ‘xfoil’ etc. are different runs of the simulation software, that marked 
‘selig’ is wind tunnel data from Selig et al. [40].)
The lift curves, Figure 6.2.1, by contrast, are remarkably consistent. However, they differ noticeably from the wind tunnel data in that the linear part of the simulated curve shows a greater slope and continues further before stalling, giving a much higher maximum lift. Over the negative angle of attack range, the lift curve remains linear, with no tendency to level off, despite the drag curve looking very much like an incipient stall. As described above, the solver always seems to lose the ability to find a convergent solution between -4 and -5 degrees, so obviously the lift curve also remains unknown beyond this point.

![Lift curve comparison](image)

**Figure 6.2.1 Comparison of simulation lift data with wind tunnel data**
Figure 6.2.2 Comparison of simulation drag data with wind tunnel data

Having used the simulation software, it is, unfortunately, somewhat difficult to discern the underlying relationship between the simulated data and the wind tunnel data. To help with this, wind tunnel data and simulated data were plotted on one chart with equivalent data for the S4062 profile.
Figure 6.2.3  Comparison of lift data for DAE51 and S4062 aerofoils

As can be seen from the shape comparison in Figure 6.1.2, DAE51 and S4062 are very similar in shape. The wind tunnel data show them to have a significant difference in zero-angle lift (and zero-lift angle) but this difference does not appear in the simulated data for lift vs. angle of attack.
Figure 6.2.4  Comparison of drag data for DAE51 and S4062 aerofoils

The chart of drag vs. lift (Figure 6.2.4) shows that, over the known range, the drag behaviour of the two profiles are indeed very similar. As in the lift curve, the simulations appear to show stall to occur at much higher positive values of lift than in the wind tunnel data and again shows the two aerofoils to behave similarly. In the negative lift region, however, the simulations suggest that they behave very differently, with DAE51 showing a much earlier drag rise than S4062. DAE51 reaches a drag coefficient of 0.02 at a lift of -0.025 whilst S4062 only reaches the same drag at a lift of -0.115. Whether or not this reflects a real difference is hard to say. The DAE51 curve seems to be very close to the wind tunnel data curves of both aerofoils but it is hard to rely on this when the simulated curves differ so greatly from the wind tunnel curves over the one region where good data are known to exist, namely for positive lift.

All that can be said is that it is probably safe and conservative to utilise the existing S4062 data over the negative region for the DAE51. If they differ at all, it is probably in the direction of DAE51 having the earlier drag rise and thus regulating better.
6.2.2 Simulating the effect of a thick trailing edge

For the sake of the manufacturing process and also for the sake of the final blade being reasonably robust, the aerofoil is not in practice constructed with a sharp trailing edge. Instead, the outer layer of carbon fibre is extended into the trailing edge, giving it roughly the thickness of two layers of cloth. This comes to a thickness of approximately 0.66 mm, which, with a chord width at the tip of 73 mm translates to a proportional thickness of 0.009.

There are several conceivable ways of adding this thickness to the trailing edge and the question that arises is of the effect of these different methods on the fitness of the aerofoil to the tasks of generating power and of speed regulation. Initially, the comparisons were made on the S4062 profile.

The reasoning behind the different possible approaches to the profile modifications is that there are expected to be two main effects of the increase in trailing edge thickness. One is on the aerodynamics, particularly the drag characteristic. The other is on the rate of twist development.

Thin aerofoil theory is based on the premise of a sharp trailing edge being the site of flow separation whilst flow is attached along the entire width of the aerofoil itself. When that trailing edge is sharp, the trailing separated flow is itself very thin. When the trailing edge has a definite thickness, whether flush or chamfered, there is most unlikely to be attached flow behind the two edges. There is likely to be a region behind the aerofoil of separated flow at low pressure which is likely to give rise to pressure drag. Less certain is how much additional drag there would be as the exact point of separation and the air pressure in the separated region cannot be predicted without either measurement or simulation. Even more difficult to predict is the influence of the thick trailing edge on flows partially separated from the aerofoil body, such as a separation bubble. Does the thick trailing edge promote or delay separation?

As has been explained before, the rate of twist development is dependent on the ratio between the perimeter length and the enclosed area of the profile. Thickening the trailing edge does not affect the perimeter length significantly, but the enclosed area is
inevitably increased, thereby reducing the twist developed. How can this effect be minimised without unduly affecting the desirable aerodynamic characteristics of the aerofoil?

On the assumption that it is the slope of the surfaces which is ‘seen’ by the boundary layer, it might be possible to attempt to change the gradients over the aerofoil surfaces as little as possible. Thus, for instance, keeping the shape the same but scaling it up by the appropriate amount and truncating it close to the trailing edge would keep the shape largely the same and might maintain an identical pressure distribution to the original aerofoil across most of the blade. This might also be achieved by adding a constant thickness throughout. On the other hand, both these approaches would add greatly to the enclosed area.

Adding a ‘wedge’ of the requisite thickness at the trailing edge, tapering to nothing at the leading edge, would halve the added area, but would change the slope of the ‘pressure recovery’ zone on the upper surface, behind the thickest section, (possibly beneficially). Reducing the slope of this zone would reduce the adverse pressure gradient against which the boundary layer must battle and thus might delay separation and thus might reduce drag. Other approaches might confine the modification to a small zone at the rear of the profile but might also change the aerodynamic behaviour in ways unknown.

The different methods considered are explained below and shown in Figure 6.2.5 to Figure 6.2.7. (Vertical scale is exaggerated).

The unchanged profile is designated NOR.

**Truncate and scale.** The profile is truncated by an appropriate length at the tale and scaled back up to the correct chord. (On graphs, designated as TR.)

**Pad.** The appropriate trailing edge thickness is added across the full width of the blade (equally to the top and bottom surfaces). (Designated PAD.)
Figure 6.2.5 Profiles of NOR, PAD and TR sections

**Wedge.** A wedge is added to the profile starting at zero thickness at the leading edge to the full thickness of 0.009 times the chord at the trailing edge. (Designated WJ, though unfortunately the data for this version are missing.)

**Half-wedge.** As above, but only the ‘rear’ (diminishing) part of the profile is wedged. This starts at the location where the upper (suction) surface is farthest from the chord-line. (Designated HWJ.)

**Half-wedge.** As above, but only the quarter (approximately - actually 22%) of the profile is wedged. (Designated QWJ.)
Shim. As above, but only the last ~15% of the rear part of the profile is wedged. (Designated SHL.)

In addition, versions of ‘shim’ were created in which all the thickness was added to respectively the upper and lower surfaces. (Designated SHM and SHN respectively.)
The outcome was that most of the modifications produced very similar effects on the aerodynamics, particularly the lift - so much so, that if they are all plotted on the same set of axes, most of them are difficult to distinguish from each other. They all raised the slope of the lift curve from 0.107 to 0.110 approximately. The exception is the truncated profile, TR, which raised the slope by less and lowered the lift at zero incidence, such that at 'working' angles of attack, the lift is very close to that produced by the unmodified profile.

![Figure 6.2.8 Comparison of lift characteristics for different trailing edge modification schemes](image)

**Figure 6.2.8 Comparison of lift characteristics for different trailing edge modification schemes**
Figure 6.2.9 Comparison of drag characteristics for different trailing edge modification schemes

Figure 6.2.10 Enlargement of drag characteristics for different trailing edge modification schemes
When it comes to the effect on drag, there were also great similarities in the low drag bucket between the various modified profiles, though they were easier to distinguish than on the lift curve. Again, TR caused the smallest increase in drag, from 0.0088 to 0.0092 at a lift of 0.6, whilst the other profiles raised it to between 0.00935 and 0.0096. It would seem that (with the exception of PAD), the largest increases in drag are produced by the modifications which are confined to the smallest part of the profile. This is not altogether surprising, as they entail much more sudden changes in the slope of the surface. These differences are, however, still small and are only really visible on a magnified scale. The differences in lift-to-drag ratio are still very small.

The unmagnified drag curve, extending over a wider range of lift shows that there is some effect of the modifications on the stall behaviour, making it marginally more gentle but again this is a small effect.

As a matter of curiosity, for the three different ‘shimmed’ profiles, where the modification is confined to the last 15% of the chord, the effects of making the addition to the upper or lower surface or half to each are just as might be expected. The different positions of the added material act in a very similar manner to an adjustable flap or changes to the amount of camber. They increase or decrease the lift.
over the entire range, and leave the drag almost entirely unaffected except that stall starts to occur at slightly change values of lift and angles of attack. This might even be considered to be a beneficial change.

![Graph of lift characteristics](image1)

**Figure 6.2.12** Comparison of lift characteristics for different trailing edge modification schemes

![Graph of drag characteristics](image2)

**Figure 6.2.13** Comparison of drag characteristics for different trailing edge modification schemes

If the aerodynamic characteristic were the only consideration, then the obvious choice would be the ‘truncate and scale-up’ modification in order to cause the least loss in
efficiency. However, as has been said so many times in the course of this thesis, the ability of the machine to regulate speed is also crucial. The truncated profile, because of the need to scale it up to the correct chord, has a larger increase in enclosed area than any of the other modifications, whilst the 'shimmed' profile has the least. Whilst the latter has the largest increase in drag, as has been said, this is small and the twist development must therefore have priority. The modification should be confined to only a small part of the profile.

Table 6.2.1 Comparison of geometrical and aerodynamic properties for different trailing edge modification schemes

<table>
<thead>
<tr>
<th>properties</th>
<th>$S'$</th>
<th>$A'$</th>
<th>$A/S$</th>
<th>$C_{L}^{-}\alpha$</th>
<th>$C_{D0-6}$</th>
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<tbody>
<tr>
<td>NOR</td>
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<td>0.108</td>
<td>0.0088</td>
</tr>
</tbody>
</table>

Only two of the modifications were investigated on the DAE51 profile and these were found to have similar effects to those on the S4062. The lift slope was increased from 0.105 to 0.108 by both the half wedge and the quarter wedge. The quarter wedge raised the drag from 0.0082 to 0.0088 at a lift of 0.6, whilst the half wedge raised it to 0.0087. This increase in drag appears to remain roughly constant down to zero lift.

The final blade design was based on the quarter wedge profile, with the thickening wedge covering the last 22% of the profile. (This was, in fact an oversight. The 'shl' type thickening should have been used.)
Figure 6.2.14 Comparison of aerodynamic characteristics for different trailing edge modification schemes

Figure 6.2.15 Enlargement of lift characteristics for different trailing edge modification schemes on DAE51
Figure 6.2.16 Comparison of drag characteristics for different trailing edge modification schemes on DAE51

Figure 6.2.17 Enlargement of drag characteristics for different trailing edge modification schemes on DAE51
6.3 Aerofoil design lift selection

Most descriptions of the process of wind turbine blade design (e.g. Jansen and Smulders [27]) involve simply finding the angle-of-attack at which the chosen aerofoil attains its maximum ratio of lift-to-drag. This is seen as particularly important in designing for high tip-speed ratios. At the time of the design of the prototype self-twisting blades, this was broadly followed. At Reynold's numbers of around 2-300 000, the DAE51 aerofoil reaches a maximum lift-to-drag ratio at lift coefficients of around 0.9 and angles of attack of around 4°. For the actual design, slightly different values of lift and angle-of-attack were used in the end, as these were found not to drop the lift-to-drag ratio by very much.
Since the design process was completed work has been published which points out the shortcomings of the conventional approach. It may be appropriate at high Reynold’s numbers, in which the drag curves do not vary very much with Reynold’s number. However, as explained by Giguère and Selig, [19] at the low Reynold’s numbers encountered in small wind turbines, it is inadequate. Choosing the Reynold’s number first and then selecting the best lift-to-drag ratio from the data at that Reynold’s number does not make sense. The chord-width of the blade at the design point is directly affected by the design value of lift, but the Reynold’s number is itself directly affected by the chord. Thus it is the values of lift-to-drag ratio from a whole range of curves which should be compared whilst taking into account the variation of Reynold’s number with lift. (Strictly speaking, the influence of axial and tangential induction factors on the velocity component of Reynold’s number should also be taken into account but these are small and can be safely neglected.)

The result of this modified approach is bound to yield a different choice of angle-of-attack from the conventional approach. Almost invariably, for a given lift coefficient, the drag coefficient increases as Reynold’s number is reduced. This is bound to push the choice towards lower values of lift which then give designs with wider blades.

Figure 6.3.1 Standard lift-to-drag ratio curves for DAE51 aerofoil
This relationship is expressed quantitatively by Giguère and Selig in the form of the reduced Reynolds’s number \( R \) which is equal to the product of lift coefficient with Reynolds’s number. This is based on the notion that it is the product of lift coefficient and the chord that is set for each station by the design equations. Thus, \( R \) is calculated for each blade station as a preliminary to calculating the chord. The lift and drag data are then searched for the maximum value of lift-to-drag ratio at that value of reduced Reynolds’s number.

Figure 6.3.2 Lift-to-drag ratio curves for DAE51 aerofoil at constant reduced Reynolds’s number
The flaw in this approach is that it requires a very full set of lift and drag data over a range of Reynolds’s numbers and this is precisely what is very rarely available. As an example, for a small wind turbine to generate approximately 1 kW at a wind speed of 10 m/s, at a rotational speed of 500 rpm, at 80% radius the reduced Reynolds’s number should be approximately 350 000.

Going through the lift and drag data for, say, the DAE51 aerofoil profile and generating columns for lift-to-drag ratio and reduced Reynolds’s number shows that only the highest Reynolds’s number curve reaches such a value of reduced Reynolds’s number at all. For some other aerofoils in Selig’s own collection of aerofoil data, this value isn’t reached at all. Thus, finding the best lift-to-drag ratio becomes a fruitless search.

Such comparisons can, however, be made to a limited extent at lower values of $\tilde{R}$. What is found (and this is only a tentative conclusion) is that the optimum lift coefficient does not vary enormously between different values of $\tilde{R}$. 

---

**Figure 6.3.3** Lift-to-drag ratio curves for DAE51 aerofoil at constant reduced Reynolds’s number
Table 6.3.1  DAE51 characteristics at set values of reduced Reynold’s number

<table>
<thead>
<tr>
<th>$\mathcal{R}$</th>
<th>max. $L/D$</th>
<th>$Re_{\text{max}}$</th>
<th>$C_{L_{\text{max}}}$</th>
<th>$\alpha_{\text{max}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>75</td>
<td>42.13</td>
<td>143</td>
<td>0.525</td>
<td>0.354</td>
</tr>
<tr>
<td>100</td>
<td>51.58</td>
<td>153</td>
<td>0.656</td>
<td>1.501</td>
</tr>
<tr>
<td>150</td>
<td>68.00</td>
<td>218</td>
<td>0.689</td>
<td>1.715</td>
</tr>
</tbody>
</table>

Figure 6.3.4  Lift coefficients at maximum lift-to-drag ratio against reduced Reynold’s number for FX60-100, DAE51, & S4062 aerofoils

Thus it is possible to make an educated guess at the correct value of lift as being in the range 0.6 to 0.7 for the higher value of $\mathcal{R}$. The corresponding angle-of-attack is in the range 0.7° to 1.7° (by linear interpolation). This contrasts markedly with the 4° angle-of-attack (lift approximately 0.9) chosen on the basis of $L/D$ at constant Reynold’s number in the conventional way and, unfortunately, used in the blade design.
7. **ROTOR DESIGN AND CONSTRUCTION**

Having looked at the predicted effect on rotor behaviour of each of the design parameters, it is necessary to incorporate the lessons learnt into a prototype blade design which can be tested empirically. As well as the need to maximise the development of induced-twist, there are other considerations in the design.

The broad headings for these are:

- Power generation
- Starting torque
- Speed regulation
- Blade survival
- Ease of manufacture and assembly

### 7.1.1 Power generation

For maximum power generation;

- The aerofoil should be selected for maximum lift-to-drag ratio.
- Design angle of attack should correspond to maximum L-D ratio.
- The blade chord distribution should be as close as possible to the design for maximum power, with twist and taper.
7.1.2 Starting torque

The wind-turbine should develop adequate torque for self starting at a reasonable cut-in speed, such as 2.5 - 3 m/s. Thus;

- The blade chord should be as large as possible.
- The set-pitch angle should be as large as possible.
- The blade should preferably be twisted and be well tapered (giving a wide root region).

7.1.3 Speed regulation

The requirement for speed regulation requires both the amount of induced twist in the blade and the aerodynamic effect of the induced twist blade to be maximised.

To achieve this;

- The rotor should be designed for a high tip-speed ratio (high centrifugal loads and narrow tip region)
- The blade should have a thin section, (See Chapter 4 Section 5)
- should have little or no taper (narrow root region).
- There should be additional tip-mass.
- The aerofoil section should be selected for maximum lift-slope (although there is little variation between profiles on this characteristic) (See Chapter 6 Section 1)
- The aerofoil should also have a narrow region of minimum drag, with the drag coefficient rising as sharply as possible outside the design lift range.
- The aerofoil should be highly cambered for the minimum drag to occur at a positive value of lift.
- The composite lay-up should be optimised for maximum shear per unit load, and thus have a principal fibre angle of approximately 25° in carbon or aramid fibre. (See Chapter 3 Section 2)
7.1.4 Blade survival

The blades should be able to survive its expected extreme loads and fatigue loads without damage. The blades should also be robust in that they should resist wear and impact damage.

To these ends;

- The composite lay-up should be optimised for maximum shear per unit normal strain, i.e. a principal fibre angle of approximately 15° in carbon or aramid fibre. There should be glass incorporated, preferably in the outer layer when carbon is the main fibre, in order to protect against impact damage. This should be perpendicular to the principal fibre angle.

- Holes and sharp changes in section should be not be incorporated in the design to avoid stress concentrations. Enclosed elements such as the core and the tip-mass should not terminate suddenly but be ‘scarfed’.

- There should be adequate chord at the blade root to withstand bending moments.

- Avoid loads being carried through the hub as far as possible - construction to be two blades in one piece.

- The hub region of the blade should be made rigid with a wood or metal insert.

- The fibre should continue into the trailing edge of the blade, which therefore can not be truly sharp.

7.1.5 Ease of manufacture

The blades should not be inordinately difficult to manufacture or assemble. Thus;

- The trailing edge cannot be perfectly sharp but must be of finite thickness. This will have aerodynamic consequences which need to be investigated. (See Chapter 6 Section 2)

- The material should not be aramid fibre (e.g. Kevlar) as it is hard to cut, particularly ‘on the bias’.
• The blade should not be twisted

• The core should not be made of foam as originally intended

• The blade must have some system for mounting on the hub - locating grooves or bushed holes.

It should be noted that some of these ‘requirements’ conflict and a compromise needs to be found between them. For instance, wide chord and substantial twist are good for power generation and starting torque but conflict directly with induction of twist and ease of manufacture. More subtly, the requirement of speed regulation would favour the choice of fibre angle as the optimum value for twist per unit load, whereas the requirement of blade survival would favour the optimum angle for twist per unit strain. Each aspect of blade design must be examined in turn and a suitable design choice made.

7.2 Blade Construction

7.2.1 Taper

In particular, the requirement for zero or low taper to give maximum induced-twist conflicts with the requirement for high taper for maximum resistance to bending moment. In practice, due to the high rotational speeds, the bending loads are largely carried by centrifugal stiffening and so it is reasonable to specify the planform to give a high degree of induced-twist. The prototype design tapers from a chord of 111 mm at the root to 73 mm at the tip.

7.2.2 Twist

Twist in the initial blade shape, here designated pre-twist, gives some benefit in terms of power coefficient, but is particularly useful in terms of starting torque. When a wind turbine is at rest, the blades are deeply stalled and regardless of their actual profile, they act in the same manner as flat plates. To a large degree, the lift in such a state is proportional to the sine of twice the angle of attack. Hence, having at least
part of the blade at a substantial pitch angle, albeit the root region with a short moment-arm, helps considerably to give start-up torque. In normal circumstances, it should not be too difficult to manufacture a fibre-composite blade with twist. However, the blades being built for this project have the fibre laid-up at an angle ("on the bias"). This tends to make the job of handling the fabric and laying it up in the mould much more difficult and this would be compounded considerably for a twisted blade. For this reason, an untwisted blade design was chosen. However, later developments of the machine may incorporate some twist if the prototype is found not to develop enough starting torque.

7.2.3 Material

For reasons of ease of handling during manufacture, aramid fibre composite were ruled out for the material. It is even difficult to cut on the material axes. It would be very hard to cut correctly at a fibre angle of 22°. T300 Carbon fibre was chosen for the principal fibre direction, with additional glass fibre (E glass) in the orthogonal direction. The matrix material is a cold-curing epoxy resin. The design fibre volume fraction is 55%. The skin thickness is 1.33 mm.

7.2.4 Fibre angle

Fibre angle can be optimised either for maximum twist per unit load or for maximum twist per unit strain. Since it is the longitudinal tensile strain which appears to be the most severe, the optimisation should be per unit strain for maximum strength. However, practical considerations dictate more of a compromise. The per strain optimum angle requires a considerably larger tip-mass in order to develop the required twist. Unfortunately, with a thin sectioned blade, there is not a lot of space in which to fit such a tip mass and it would therefore extend a long way down the blade. There is a diminishing return as this occurs as the centrifugal field is lower at smaller radii requiring progressively more added mass for a given effect. As a compromise, a fibre angle was chosen roughly half way between the two optima. The angle chosen was 22° (relative to the long axis of the blade).
7.2.5 Blade number

At an early stage a decision was taken to opt for a two-bladed design. The high tip-speed ratio of the design requires quite a low solidity. Achieving this with three blades would result in very narrow blades. There are two problems with such narrow blades. Firstly, from the practical point of view, construction could become awkward, with a blade thickness at the tip of approximately 4 mm. It would be very hard to fit much tip-mass into the shell of the blade. Secondly, three narrow blades would have a lower Reynolds' number than two wider blades, and would thus lead to higher drag.

Another reason for opting for a two-bladed design concerns the hub design. See the next section.

7.2.6 Hub and blade construction

The turbine is, in comparison to most small wind turbines, a very high speed machine. It has to withstand rotor speeds of up to 1300 rpm and thus severe centrifugal tensile loads. In such situations, the most common site for failure is at the root attachment, partly because the accumulated load is at its greatest but also because attachments generally act as stress concentrators. To avoid some of these problems, a design was chosen with the two blades constructed in one piece. The tensile load in each blade thus does not have to be transferred to the hub at all but is carried through and reacts against the similar load in the opposite blade.

The original intention was to have the blade assembly clamped in position on the hub with a ‘U’ bolt type of arrangement or a bolted retaining bar. The correct positioning would have been ensured by suitable geometry, such as shoulders or notches. Although such a form of attachment would also give rise to stress concentration, as long as large radii are used, the stress would not be as severe as for holes. However, this was ruled out as impractical by Professor V. Middleton of Nottingham University who was responsible for the manufacturing process and much of the detailed design around the hub region of the blade. Four bushed holes were incorporated in the hub region of the blade to allow for bolting to the hub.
To prevent the blade from twisting away from the hub and to provide some protection against crushing, a shaped sheet metal insert was also incorporated in the hub region. This runs the full width of the blade and the full diameter of the hub. It projects a short distance into the 'free' part of the blade, but here it has to twist in order to allow for the set-pitch angle of the blade. It also tapers quite sharply. The aim of scarfing this piece in to the general blade construction is to avoid stress concentration problems here.

7.2.7 Blade cores

The blade cores were machined from balsa wood. Once manufactured, the core is only about 4 mm thick at the tip and 8 mm thick at the root. It would thus be too flexible for convenient handling if made from polyurethane foam. This was in fact found to be the case when the glass fibre test piece was constructed (See Chapter 3).

7.2.8 Tip masses

As was mentioned in chapter 3, it was not initially recognised that the tip-masses would need to be so great. Once this was recognised, it was felt that using solid lumps of lead might be potentially dangerous. If there were a failure of the retaining material, a solid lump of lead, thrown at the blade tip speed could act as a formidable and dangerous projectile. Various schemes were considered, ranging from a bag of shot (as used in the original project - see Karaolis [31]), to lead sheet interleaved between the composite skin and the core material. The method chosen was to set lead powder in a polymer matrix, in this case unfoamed polyurethane.

The masses were 509g each.

7.3 Manufacture

As explained in an earlier chapter, the test piece (used for direct validation of the simple twisting model in a tensile testing machine) was constructed of glass fibre by hand lay-up. Even for that item, elliptical in section with no sharp edges, the process...
was extremely difficult and control of fibre angle and volume fraction was, quite frankly, poor. It was felt that for the prototype blade assembly it was essential to use a process which would give better quality control.

One possibility would be the ‘filament-winding’ process, in which filament or tape (in this case probably ‘pre-preg’) is wound onto a correctly shaped spandrel. This would then be vacuum-bagged and cured. The advantage of this process is the very precise control it affords of the fibre angles as long as the winding process is correctly designed. However, there are several reasons why this process would not be practical. The desire for a once-through construction would require a spandrel that could be left in place as part of the finished blade assembly. The spandrel would need to be stiff enough to resist bending due to the tension in the filament/tape as it was being wound and yet would be only 6 mm deep at mid-span and would have to be soft enough not to interfere with the centrifugal loading of the composite skin. It would either need supporting at the tip end, which does not fit easily with the requirements of the design or need to have such stiffness as a cantilever. The slightly reflex lower surface and sharp trailing edge of the aerofoil section are not very compatible with filament winding. Curing would require a kiln able to accommodate a 3m object.

The alternative, and the manufacturing method chosen, is ‘Resin transfer moulding’ (RTM). In this process, the fibre reinforcement is cut to shape, generally from a woven or stitched fabric, and is laid-up dry in a closed mould. (Middleton et al. [35]). This facilitates handling of the material and allows close control of the fibre angles. In general, the mould is double-sided in order to control the fibre volume fraction closely. For a tubular object such as a wind-turbine blade, it is more common to employ some kind of a moulded or machined core placed inside the fabric which acts as the other half of the mould and must clearly be made to a high degree of precision. The outer mould would still consist of two separable halves.

With the fabric and core inside the mould, the latter is clamped and resin is injected under pressure. Heating elements in the mould are employed to improve the flow of the resin and for curing the blade in place, though the materials used do not require high temperature curing.
Table 7.3.1 Prototype blade design parameters

<table>
<thead>
<tr>
<th>BLADE DIMENSIONS</th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>hub radius</td>
<td>$r_h$</td>
<td>0.148 m</td>
<td></td>
</tr>
<tr>
<td>reference position</td>
<td>$r_{ref}$</td>
<td>1.015 m</td>
<td></td>
</tr>
<tr>
<td>ref. chord</td>
<td>$c_{ref}$</td>
<td>0.0811 m</td>
<td></td>
</tr>
<tr>
<td>tip chord</td>
<td>$c_1$</td>
<td>0.0731 m</td>
<td></td>
</tr>
<tr>
<td>root chord</td>
<td>$c_0$</td>
<td>0.1095 m</td>
<td></td>
</tr>
<tr>
<td>Blade length</td>
<td>$L$</td>
<td>1.302 m</td>
<td></td>
</tr>
<tr>
<td>skin thickness</td>
<td>$t$</td>
<td>0.00133 m</td>
<td></td>
</tr>
<tr>
<td>length of tip mass</td>
<td>$L_{tip}$</td>
<td>0.497 m</td>
<td></td>
</tr>
<tr>
<td>initial set angle</td>
<td>$\beta_{set}$</td>
<td>4°</td>
<td></td>
</tr>
<tr>
<td>measured twist</td>
<td>$\Delta \beta$</td>
<td>6° @ rotor speed $N_{rot}$ = 577 rpm</td>
<td></td>
</tr>
<tr>
<td>skin mass</td>
<td>$m_{skin}$</td>
<td>0.543 kg</td>
<td></td>
</tr>
<tr>
<td>tip mass</td>
<td>$m_{tip}$</td>
<td>0.508 kg</td>
<td></td>
</tr>
<tr>
<td>total blade mass</td>
<td>$m_{skin}+m_{tip}$</td>
<td>1.051 kg</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>MATERIAL PROPERTIES</th>
<th>carbon/epoxy</th>
<th>glass/epoxy</th>
</tr>
</thead>
<tbody>
<tr>
<td>Elastic Modulus $^\parallel$</td>
<td>$E_{a1}$</td>
<td>125 GPa</td>
</tr>
<tr>
<td></td>
<td>$E_{b1}$</td>
<td>9.4 GPa</td>
</tr>
<tr>
<td>Shear Modulus $^\perp$</td>
<td>$G_{ab1}$</td>
<td>5.7 GPa</td>
</tr>
<tr>
<td>Poisson’s Ratio $^\parallel$</td>
<td>$\nu_{ab1}$</td>
<td>0.33</td>
</tr>
<tr>
<td>Density $^\parallel$</td>
<td>$\rho_{l}$</td>
<td>1467 kg m$^{-3}$</td>
</tr>
<tr>
<td>fibre angle</td>
<td>$\theta_{l}$</td>
<td>22°</td>
</tr>
<tr>
<td>thickness</td>
<td>$t_{1}$</td>
<td>0.66 mm</td>
</tr>
</tbody>
</table>

Based on: manufacturer’s values

Text-book values

Relative to the blade’s long axis

Transformed properties of Laminate

| Elastic Modulus (axial) | $E_x$ | 28.0 GPa | |
| Elastic Modulus (trans.) | $E_y$ | 18.5 GPa | |
| Shear Modulus | $G_{xy}$ | 7.5 GPa | |
| Poisson’s Ratio | $\nu_{xy}$ | 0.53 | |
| Lekhnitski coeff. | $\eta_x$ | 1.25 | |
| Density | $\rho_{xy}$ | 1687 kg m$^{-3}$ | |
| Tip-mass density | $\rho_{tip}$ | 5650 kg m$^{-3}$ | |

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7.4 Performance prediction

A series of simulations were carried out for the blade design set out above in order to predict the wind turbine's speed regulation and power performance and in order to obtain blade loading data for predictions of blade deflections and strains. (See Chapter 5).

The power curve in Figure 7.4.1 is not particularly informative as it merely reflects the intended means of operation whereby the power is limited through control of the excitation of the generator. Below rated power it is meant to track the maximum power point. The torque curve is rather more interesting. Below rated power, it has to increase almost quadratically in order to deliver maximum power. At rated power, there is a sharp downturn as progressively less torque is required to deliver the rated power as the speed increases. The sharpness of the cusp in the curve would in practice be almost impossible to achieve and would almost certainly lead to unstable operation. It would be preferable to smooth off the curve.

![Figure 7.4.1 Torque and power curves against wind speed for design operation](image)

Figure 7.4.2 has the same quantities as before but this time plotted against rotor speed. This represents a control schedule for the wind turbine. The controller cannot sense
wind speed directly, but can sense rotor speed and can set a demand current (which has a simple relationship with torque).

Finally, Figure 7.4.3 shows the resulting rotor speed behaviour of the turbine. The 'kink' in the speed curve for the generating condition reflects the downturn in torque once the rotor hits the rated power. After this point it gradually approaches the runaway curve. The corresponding curves for an unregulated rotor have been plotted in the same figure for comparison. It can be seen that the unregulated rotor reaches a speed of 1300 rpm at a wind speed of 12 m/s, whereas the twisting rotor can keep going till 64 m/s before it reaches the same speed - more than 5 times the wind speed.
Figure 7.4.3 Rotor speed curves for design operation and runaway, comparing twisting blade rotor with unregulated rotor.
MODEL VALIDATION AND BLADE TESTING

8.1 Driven-spin testing of the rotor

In order to validate further the simple model of blade twisting, a test was designed to rotate the blades under controlled conditions, at fixed speeds, rather than to rely on natural wind conditions and logging of collected data. It was felt necessary to check that the experiment could be performed satisfactorily in terms of the power needed to drive the rotor and in terms of the expected loads and deflections of the blades. For this purpose, a simplified aerodynamic model was developed, treating the rotor as a propeller, the derivation of which is set out in Appendix A4. It should be noted that this model is quite crude, completely neglecting stall, and is only designed to give a conservative estimate of the power requirements.

Table 8.1.1 Rotor characteristics for spin test calculations

<table>
<thead>
<tr>
<th>blade length</th>
<th>$L$</th>
<th>1.302 m</th>
<th>zero lift angle</th>
<th>$\alpha_0$</th>
<th>-5°</th>
</tr>
</thead>
<tbody>
<tr>
<td>hub radius</td>
<td>$hL$</td>
<td>0.148 m</td>
<td>lift slope</td>
<td>$C_{L'\alpha}$</td>
<td>0.1 deg$^{-1}$</td>
</tr>
<tr>
<td>initial twist</td>
<td>$\Delta \beta$</td>
<td>0°</td>
<td>min. drag</td>
<td>$C_{D_{min}}$</td>
<td>0.004</td>
</tr>
<tr>
<td>set pitch</td>
<td>$\beta_0$</td>
<td>4°</td>
<td>drag curvature</td>
<td>$k_d$</td>
<td>0.001 deg$^{-2}$</td>
</tr>
<tr>
<td>running twist</td>
<td>$\Delta \theta(\Omega)$</td>
<td>25.4°</td>
<td>@ rotor speed</td>
<td>$N_{rot}$</td>
<td>1500 rpm</td>
</tr>
</tbody>
</table>

The results of the calculation show that, once the increasing blade twist is taken into account, the power rises along a curve which is close to being a quartic or a quintic and reaches very high values at the higher rotor speeds. These are, however, unrealistic. At anything greater than 8-900 rpm and 7-9° of twist, we would expect the crucial tip region to be stalled and the linear lift and quadratic drag characteristics
no longer to reflect the real situation. The figure of 3.6 kW at 800 rpm is probably not unrealistic.

Table 8.1.2 Spin test aerodynamic calculations

<table>
<thead>
<tr>
<th>rotor spd. (krpm)</th>
<th>twist (°)</th>
<th>thrust (kN)</th>
<th>torque (kNm)</th>
<th>power (kW)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.3</td>
<td>1.0</td>
<td>0.000</td>
<td>0.006</td>
<td>0.2</td>
</tr>
<tr>
<td>0.4</td>
<td>1.8</td>
<td>0.002</td>
<td>0.005</td>
<td>0.2</td>
</tr>
<tr>
<td>0.5</td>
<td>2.8</td>
<td>0.012</td>
<td>0.010</td>
<td>0.5</td>
</tr>
<tr>
<td>0.6</td>
<td>4.1</td>
<td>0.039</td>
<td>0.017</td>
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<tr>
<td>0.7</td>
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<td>0.8</td>
<td>7.2</td>
<td>0.183</td>
<td>0.043</td>
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<tr>
<td>0.9</td>
<td>9.2</td>
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<td>0.067</td>
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<td>1.0</td>
<td>11.3</td>
<td>0.543</td>
<td>0.103</td>
<td>10.8</td>
</tr>
<tr>
<td>1.1</td>
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<td>0.847</td>
<td>0.158</td>
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<tr>
<td>1.2</td>
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<td>30.2</td>
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<td>22.2</td>
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<td>0.532</td>
<td>78.0</td>
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<td>1.5</td>
<td>25.4</td>
<td>3.367</td>
<td>0.775</td>
<td>121.7</td>
</tr>
</tbody>
</table>

Figure 8.1.1 Blade twist and spin test power demand

Figure 8.1.2 Spin test thrust load and torque demand
8.1.1 Spin test experimental details and results

The completed blades were lab tested to measure the degree to which twist developed as a function of rotor speed. A subsidiary purpose for this test was to see whether the blade bending under load conformed to predictions.

The experimental set-up was as follows:

**General Set-up:** The blade was bolted to a hub-piece machined to take it, which in turn was fixed to the keyed shaft of a 2 kW dc motor. The motor was mounted on a steel frame with its shaft on a vertical axis. The blade thus rotated in a horizontal plane, approximately 2/3m off the floor. The motor was connected to a variable dc supply consisting of a ‘Variac’ and rectifier.

**Rotational speed measurement:** The shaft was marked with 6 white index marks using typing correction fluid. A reflection based opto-switch sensor was used to pick up the marks. A frequency-to-current converter with a PC operating as an ammeter were used to measure the frequency. This was calibrated against a factory-calibrated hand-held tachometer.

The rotor was run at speeds up to 577 rpm. Above this speed, the vibration and noise became quite alarming, and the test was stopped, though the problem may have been due to inadequate anchoring of the motor and frame.

Two methods were used to measure the twist at the blade tip.

**Method 1: Stroboscopic photography**

**Stroboscope:** The stroboscope used was a GenRad GR 1538-A Strobotac. It was mounted on the same tripod as the camera.

A second optical pickup was used higher up the motor shaft, with only one white mark, driving a Schmitt-trigger and giving a 5v +ve going pulse in order to synchronise the stroboscope to the blade in accordance with its requirements.

**Camera:** The camera used was a Minolta X500, with standard 50 mm f1.7 lens mounted on a tripod level with and 0.36m from the blade tip. The shutter speed was set to cover up the duration of 1 blade revolution. Film used was Ilford FP4 plus with
a speed of 125 ASA. A 150 mm steel ruler was mounted vertically in frame just beyond reach of the blade tip to act as a reference.

At the film speed used (125 ASA), the strobe has an exposure guide number of approx. 18 ft. It was mounted at a distance of 14" (41 cm) from the blade tip, making an effective distance of 32" according to the guide book. (Due to the parabolic reflector, the effective source is 18" behind the flash-tube). The ideal aperture setting was thus calculated as f6.8, with additional exposures at 1 and 2 stops above and below this value. Settings were thus f3.4, f4.8, f6.8, f9.5, and f13.

Two of the prints obtained can be seen in Figure 8.1.3. Both the twist of approximately 6° and the deflection of approximately 4 cm can be seen, (Note that it is the stationary blade which droops due to gravity, and that this decreases as the blade rotates at moderate speeds.)

In order to measure the flapwise and twist deflections, the images from the negatives were projected using a photographic enlarger onto it's baseboard. It was desired to have the projected blade tip positions from all the tests superimposed on one sheet for measurement purposes. Because negative film had been used, the images could not be multiply printed to achieve this. Instead, the positions of the leading and trailing edge for each test were marked in pencil on a sheet of paper, the images of the index rule having been moved to coincide with an index line marked on the paper.

Once this was complete, the marks on the sheet were measured using a standard technical drawing table. The deviations of leading edge and trailing edge were measured, from which both flapping and twisting deflections could be calculated. The twisting deflection was also measured directly with the angle Vernier scale on the drawing table.

The speed measurements for this test were taken from the shaft encoder alone. Unfortunately, the next day the calibration on this was found to be out by approximately 10%. It was unclear whether or not it had been correct on the first day.
Results:

Figure 8.1.3 Stroboscopic photographs showing blade-tip twist at standstill and at 577 rpm

Table 8.1.3 Dimensions for scaling of measurements made on enlarger - projected image

<table>
<thead>
<tr>
<th></th>
<th>10</th>
<th>9 cm</th>
</tr>
</thead>
<tbody>
<tr>
<td>scale length on enlargement</td>
<td>le</td>
<td>16.8 cm</td>
</tr>
<tr>
<td>chord length on enlargement</td>
<td>ce</td>
<td>13 cm</td>
</tr>
<tr>
<td>actual chord length on blade</td>
<td>co</td>
<td>6.96 cm</td>
</tr>
</tbody>
</table>

Table 8.1.4 Blade deflection and twist measurements from strobe photography

<table>
<thead>
<tr>
<th>rotor speed Nrot (rpm)</th>
<th>speed² (krpm²)</th>
<th>flapwise position z (cm)</th>
<th>twist δβ (°)</th>
<th>leading edge zLE (cm)</th>
<th>trailing edge zTE (cm)</th>
<th>bending deflection Δ(cm)</th>
<th>strobe twist δβ₁ (°)</th>
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<tbody>
<tr>
<td>0</td>
<td>0.000</td>
<td>7.3</td>
<td>0.5</td>
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<td>2.4</td>
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<td>0.00</td>
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<td>106</td>
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<td>9.4</td>
<td>0.5</td>
<td>6.1</td>
<td>6.2</td>
<td>3.96</td>
<td>0.00</td>
</tr>
<tr>
<td>202</td>
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<td>1.33</td>
<td>9.3</td>
<td>9.6</td>
<td>2.20</td>
<td>0.88</td>
</tr>
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<td>301</td>
<td>0.091</td>
<td>11.3</td>
<td>2.5</td>
<td>9.8</td>
<td>10.4</td>
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<td>351</td>
<td>0.123</td>
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<td>10.1</td>
<td>10.8</td>
<td>1.66</td>
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<tr>
<td>452</td>
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<td>11.3</td>
<td>4.83</td>
<td>9.5</td>
<td>10.5</td>
<td>1.90</td>
<td>3.96</td>
</tr>
<tr>
<td>503</td>
<td>0.253</td>
<td>11</td>
<td>5.67</td>
<td>9</td>
<td>10.3</td>
<td>2.09</td>
<td>5.27</td>
</tr>
<tr>
<td>577</td>
<td>0.333</td>
<td>10.5</td>
<td>7</td>
<td>7.8</td>
<td>9.4</td>
<td>2.65</td>
<td>6.58</td>
</tr>
<tr>
<td>577</td>
<td>0.333</td>
<td>10.5</td>
<td>7</td>
<td>7.8</td>
<td>9.4</td>
<td>2.65</td>
<td>6.58</td>
</tr>
</tbody>
</table>

on index scale direct angle measurement from projection calculated from projected linear measurements
Method 2  Laser spot deflection method

Laser and mirror: An acrylic mirror, 3.5 mm square was fixed to the underside of one blade, 1 cm from the tip using a self-adhesive pad. (Hot-melt glue was found to distort the mirror slightly). For balance, a second square was attached to the other tip with the reverse side showing.

The laser used was a standard battery operated laser pointer of the type used in lectures. This was attached to the motor frame, using a laboratory clamp and boss, pointing almost vertically upwards, inboard of the blade tip. It was set to point slightly outwards radially in order to improve safety for the experimenter recording the spot deflection. The reflected spot positions were recorded by pen manually on a sheet of graph paper attached to a board on the floor.

The speed measurements for this test were taken using a factory-calibrated hand-held tachometer, pointed directly at a white spot painted on one blade tip.

The blade flap-wise deflection and twist were calculated trigonometrically from the measured geometry of the set-up. Strictly speaking, in order to disentangle the 3 unknowns - twist, bending deflection, and slope - from the 2 sets of measurements - radial and tangential spot deflection, more measurements would be needed, e.g. laser deflections from a mirror on the top surface of the blade. However, such an arrangement was not found to be practical, particularly in the limited time available. Instead, a linear relationship was assumed between slope and deflection, based on the bending model, and this allowed calculations of the blade twist to be made.

Figure 8.1.4  Schematic of set-up for laser deflection method
### Table 8.1.5 Laser deflection set-up geometry

<table>
<thead>
<tr>
<th>mirror to motor shaft</th>
<th>Rm</th>
<th>144 cm</th>
<th>initial spot position relative to mirror:</th>
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<tbody>
<tr>
<td>hub height</td>
<td>( h_b )</td>
<td>67.5 cm</td>
<td>radial ( r_0 )</td>
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<tr>
<td>stationary blade tip height</td>
<td>( z_0 )</td>
<td>61.5 cm</td>
<td>tangential ( t_0 )</td>
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<tr>
<td>laser height</td>
<td></td>
<td>36 cm</td>
<td>initial reflected laser beam angle:</td>
</tr>
<tr>
<td>laser to motor shaft</td>
<td></td>
<td>135 cm</td>
<td>radial ( \phi_0 )</td>
</tr>
<tr>
<td>laser spot to motor centre, initial value</td>
<td></td>
<td>152 cm</td>
<td>tangential ( \varphi_0 )</td>
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<tr>
<td>laser angle (rad.)</td>
<td>( \phi_l )</td>
<td>19.4 °</td>
<td>stationary tip deflection</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>stationary blade-tip angle</td>
</tr>
</tbody>
</table>

### Table 8.1.6 Laser deflection measurements and calculated results

<table>
<thead>
<tr>
<th>rotor speed ((\text{rpm}))</th>
<th>rot. spd. sq ((\text{krpm}^2))</th>
<th>measured radial ((\text{cm}))</th>
<th>measured tangl. ((\text{cm}))</th>
<th>tip flap angle ((^\circ))</th>
<th>bending at tip ((\text{cm}))</th>
<th>twist at tip ((^\circ))</th>
</tr>
</thead>
<tbody>
<tr>
<td>( N_{\text{rot}} )</td>
<td>( rr )</td>
<td>( tt )</td>
<td>( \theta_t )</td>
<td>( \Delta_r )</td>
<td>( \delta \beta )</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>0.000</td>
<td>0.0</td>
<td>0.0</td>
<td>6.01</td>
<td>6.00</td>
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<td>3.93</td>
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<td>14.4</td>
<td>2.93</td>
<td>2.92</td>
<td>6.15</td>
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</table>

*assumed* prop. to angle
Figure 8.1.5 Predictions of bending deflection and twist and measured values by strobe photography and laser deflection

Figure 8.1.6 Predicted and measured trajectory of laser spot
8.1.2 Discussion of results

The predictions of blade twist seem to be very good – better than 5% – based on the laser deflection method. However, the spot became quite hard to detect at high rotor speeds and would be even harder to detect at higher speeds still.

It is clear that, despite its simplicity, the model used is quite adequate for predicting the induced twist, at least over the range of speeds tested and for the single celled tube construction used.

Slightly worse results were obtained by stroboscopic photography. These are understandable in the light of the erroneous calibration of the tachometer used in that test. The stroboscope was perfectly adequate for ‘freezing’ the blade motion and synchronised well. The main problem with the method appears to lie with reading the small twist angles off the photographs.

The predictions of blade bending deflections are clearly not as good as those for blade twist. This is to be expected and is not worrying, as the loadings are based on the crude propeller model of the rotor and would not be expected to be accurate. For a
validation of the bending model, it would be necessary to apply known loadings to the blades under rotating conditions.

The vibration problems encountered at the highest speeds could possibly have been ameliorated if the motor frame had been better anchored. This would have allowed the tests to continue to higher rotor speeds.

8.2 Field testing of runaway behaviour

The complete prototype wind turbine is now installed and under test at Marlec's factory site in the grounds of Marlec's factory in Corby, Northants. (See Figure 8.2.1). Wind speed, rotor speed, voltage and current are logged whenever the machine is in operation.

Under supervision, it was allowed to run unloaded, with no excitation. Wind speed and rotor speed were logged at 1 second intervals. The results for two such tests are presented in Figure 8.2.2 as well as the behaviour predicted by the simple rotor model. The data were subsequently averaged at 30s intervals, and sorted into 2m/s wide wind speed bins, and the mean wind speed and rotor speed for each bin calculated. The binned performance curve is presented in the same figure.
Figure 8.2.1 Photograph of prototype wind-turbine in operation at the site of Marlec Engineering Co. Ltd.
The prototype turbine would seem to regulate runaway speed well at the wind speeds examined and the binned curves and the means do not depart drastically from the predicted behaviour. However, in reality little inference can be drawn from such a short test run. The wide scatter in the raw data is likely to be due to the short averaging time. Brief gusts and lulls of wind do not give the wind turbine enough time to reach a steady speed.

In order to be more certain about the behaviour of the turbine, much longer test runs would need to be performed, long enough to allow averaging periods which correspond to the time it takes for the rotor speed to adjust.
9. CONCLUSIONS AND RECOMMENDATIONS

9.1 Conclusions

Prior to the work on this project, it was already well established that orthotropic materials, such as fibre-reinforced composites, develop shear strain when loaded off the principal axes and that this can be used to induce twist in tubular structures such as wind turbine blades if they are laid-up appropriately.

9.1.1 Validity of models

It has clearly been demonstrated that the method of modelling the development of twist in the blade under the action of both mechanical and inertial (i.e. centrifugal) loads is capable of predicting blade twist to within 5%.

The only (limited) test of the rotor aerodynamics suggests that there may be some shortcomings in the model, though it is hard to identify precisely where.

The Rayleigh-Ritz blade bending model has not yet been subjected to any kind of validation for static bending loads. Applied to analysis of the (stationary) natural frequencies of the blade, the model was within 10% of the measured fundamental frequency and within 25% of higher modes.

9.1.2 Design Guidelines

Predictions have been made of the effect of design parameter changes on rotor behaviour but due to the expense of blade manufacture, only one blade set has been made. The sensitivity of rotor behaviour to design parameters has not therefore been validated. However the design lessons from the paper study have informed the design
of the prototype which has been seen to regulate rotor speed satisfactorily, and the design guidelines can therefore be said to be successful. They can also be said to be successful in that the degree of speed regulation achieved is considerably better than that achieved when a conventional blade shape was used with an asymmetric fibre lay-up.

9.1.3 Feasibility of concept

It has been shown experimentally that, with a lay-up that runs the principal fibres in a helical pattern, blades can be made to develop twist under the action of the centrifugal loads acting on their own mass and on added mass as they rotate.

The model of speed regulation has to some extent been validated experimentally and the predictions show the rotor to be fast but not unreasonable. The theoretical predictions of rotor runaway behaviour and the stretching and bending loads it would suffer show them to be within material failure strain limits, at least in the static sense and the static blade deflections are predicted to remain within safe limits. The concept of speed regulation with self twisting blades can thus reasonably be said to be feasible.

9.2 Recommendations for further work

9.2.1 Field testing of rotor performance

Although at the time of writing the wind-turbine is currently operating as a wind turbine at the premises of Marlec Engineering Ltd., it is essential that a full programme of performance assessment should be carried out in the field. This could then be compared with the computer-based performance predictions. The minimum would be a programme of monitoring wind speed and direction, rotor speed and yaw orientation and power generated in order to derive torque and power curves and some measure of cut-in speed. Additionally, it would be valuable to monitor the loads on the blades and the tower.
9.2.2 Manufacturing methods

In order to take the self-twisting machine closer to being a commercially viable product, it is important that consideration should be given to the most appropriate manufacturing method. Which method is chosen should be seen to depend not only on technical issues but also, possibly more importantly, on a clearer picture of the future market for such a machine. The method used to manufacture the prototype, in which it was laid-up by hand, followed by injection of resin into the mould, is quite labour intensive. On the other hand, a method in which the reinforcing fibre fabric were pre-formed into the appropriate shape in an automated process might save labour but would be very capital intensive. The relative economics of such options would depend to a great extent on the likely volume of production.

9.2.3 Validation of existing blade bending models using current prototype rotor

Tests should be carried out on the blades to determine the blade flapping, edgewise and torsional modes of vibration under rotating conditions for at least one rotational speed in order to validate existing computer models and in order to facilitate their further development. If possible, a test should be carried out under rotating conditions to validate the bending response to static loading.

9.2.4 Further development of the blade model

The Rayleigh-Ritz based blade model should be developed further in a number of ways. The model should be extended to cover a wider range of blade shapes. Edgewise bending and torsional modes of vibration should be analysed and also the coupling between all three degrees of freedom. Algebraic derivation of the matrices for both the static and vibration models should be carried out to establish sensitivities of the blade stiffness and frequencies to all the blade design parameters.
9.2.5 Aerodynamic model refinement
In the light of the limited field testing carried out so-far, if the shortcomings in the
aerodynamic model of the rotor are confirmed by further testing, then it will be
necessary to look at how it can be improved. This may only require refinement of the
blade-element model or it may require a fuller inclusion of wake structure in the
model.

9.2.6 Wind-tunnel testing - power performance and
yawing moments
It would be valuable to perform a full program of tests on the rotor in a wind tunnel
with full load control in order to derive a series of performance curves in terms of tip-
speed ratio and power and torque coefficients. Some measurements should also be
carried out in static cross-flows to determine the yawing moments on the rotor as well
as the power performance in yawed situations. If possible, loads should also be
measured in constant-rate yawing situations.

9.2.7 Modelling of rotor dynamic and yawing behaviour
The self-twisting wind-turbine has highly flexible blades, and there is thus a greater
danger of problems regarding its dynamic behaviour than in more conventional
designs. This is most likely to occur when there is significant coupling between
modes. There is also the danger of problematic yawing behaviour.

The results of the stationary and rotating modal tests (see above) and the yawing
moment tests should be used to develop a dynamic and yawing model of the rotor in
order to identify potentially dangerous situations which, in the field, might only
appear in high winds. Particular concerns are the possibilities of a classical flutter
instability and of chaotic yawing behaviour. Ideally the dynamic model should allow
the identification of the sensitivities of the dynamic problems to the various design
parameters.
9.2.8 Design improvements

In the light of the work by Giguère & Selig [11], it would be worth looking at whether or not aerofoil sections even thinner than the one used in the prototype would improve performance or whether they would merely worsen the danger of instabilities.

Following from the same work, it would be valuable to look at redesigning the blades to operate at higher pitch angles and wider chords in order to raise the Reynold's number and possibly to improve starting-torque.

Other design improvements should be made in the light of work on dynamic behaviour, 'tuning out' instabilities by making changes to the blade mass distribution and if necessary providing some means of damping yaw motion.

9.2.9 Other forms of coupling

It would be valuable to examine the feasibility of using other forms of material coupling for passive regulation of wind turbines. In particular, the possibility should be examined of using bending twisting coupling (as set out by Karaolis [31]) as well as stretch-twist coupling. For example, stretch-twist coupling could be used as overall speed regulation and bending-twisting coupling could be used for gust-load alleviation.
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A1. APPENDIX TO CHAPTER 3

A1.1 Propeller effect – centrifugal untwisting

A wind turbine blade which is rotating and is set at a pitch relative to the rotor plane, experiences a couple, due to the centrifugal field, which tends to flatten it out into the rotor plane.

Consider a blade which is rotating at $\Omega \text{ rad/s}$. A mass element $dm$ is situated in a cross section at a distance from the axis of rotation of $xL$ measured along the $x$ axis i.e. along the length of the blade. If bending displacements are neglected, the position of the mass element within the blade section is a distance $y.c(x)$ from the shear centre in the chordwise direction, where $c(x)$ is the chord at that section, and a distance $z.c(x)$ in the thickness direction. It experiences a centrifugal force $dF$ which can be resolved into two components. $dF_x$ acts parallel to the $x$ axis, whilst $dF_t$ acts perpendicularly to the axis of rotation, and perpendicularly to the axis of the blade, i.e. tangential and in the rotor plane.

\[ (x+h).L \]

\[ dF \]

\[ \beta \]

\[ \cos(\beta) \]

\[ dF = dm \cdot \Omega^2 (y.c(x) \cos \beta - z.c(x) \sin \beta) \]

Figure A1.1.1 Origin of propeller-effect couple

If the blade section at that point is at a pitch angle $\beta$, then the tangential force component, $dF_t$ is given by
This gives rise to a couple which tends to align the blade element with the plane of rotation.

\[ ddQ = dF_i (y \cdot c(x) \sin \beta + z \cdot c(x) \cos \beta) \]
\[ = \Omega^2 dm \cdot c(x)^2 \left( (y^2 - z^2) \cos \beta \sin \beta + yz \cdot \left( \cos^2 \beta - \sin^2 \beta \right) \right) \]
\[ = \Omega^2 dm \cdot c(x)^2 \left( \frac{1}{2} (y^2 - z^2) \sin 2\beta + yz \cos 2\beta \right) \]

The effect of this for the whole cross section is given by integrating over that section.

\[ dQ(r) = c(x)^2 \cdot \Omega^2 \left[ \frac{1}{2} \sin(2\beta) \right] \left( \int y^2 - z^2 \, dm \right) + \cos(2\beta) \left( \int yz \, dm \right) \]

\[ \int y^2 \, dm \quad \text{and} \quad \int z^2 \, dm \quad \text{and} \quad \int yz \, dm \text{ are the section moments of inertia } I_{zz}, I_{yy} \text{ and } I_{yz}. \]

For the skin and the tip mass separately, which have uniform density, these correspond to the product of the density and the respective second moments of area and product moment.

Furthermore, these can be expressed as products of dimensionless forms of the quantities, relating only to section shape, with section dimensions.

For the skin,

\[ \int y^2 \, dm = \rho \cdot t \cdot c(x)^3 \cdot I'_{sz} \]
\[ \int z^2 \, dm = \rho \cdot t \cdot c(x)^3 \cdot I'_{sy} \]
\[ \int yz \, dm = \rho \cdot t \cdot c(x)^3 \cdot I'_{sz} \]

Giving the couple element as

\[ dQ(x) = \Omega^2 \rho \cdot L \cdot t \cdot c(x)^3 \cdot dx \left[ \sin(2\beta) \cdot \left( I'_{sz} - I'_{sy} \right) + \cos(2\beta) \cdot I'_{syz} \right] \]
and for the tip-mass:

\[
\int z^2 \, dm = \rho \ \text{tip} \left( c(x)^4 \cdot I'_{ay} - t \cdot c(x)^3 \cdot I'_{sy} \right)
\]

\[
\int y^2 \, dm = \rho \ \text{tip} \left( c(x)^4 \cdot I'_{az} - t \cdot c(x)^3 \cdot I'_{sz} \right)
\]

\[
\int yz \, dm = \rho \ \text{tip} \left( c(x)^4 \cdot I'_{ayz} - t \cdot c(x)^3 \cdot I'_{syz} \right)
\]

(where \( I'_{az} \) and \( I'_{sz} \) are the 2nd moments of area of the enclosed area and of the perimeter of the blade section respectively)

so that in this case the couple element is

\[
dQ_{\text{tip}}(x) = \Omega \ \text{tip} \cdot L \cdot dx \left[ c(x)^4 \left[ \sin(2\beta) \cdot (I'_{az} - I'_{ay}) + \cos(2\beta) \cdot I'_{ayz} \right] \right.
\]

\[
\left. + t \cdot c(x)^3 \left[ \sin(2\beta) \cdot (I'_{sz} - I'_{sy}) + \cos(2\beta) \cdot I'_{syz} \right] \right]
\]

The total restoring moment experienced at \( x \) also consists of the two components;

Due to the skin;

\[
Q_{\text{skin}} = \int_x^1 \ dQ = \Omega \ \text{tip} \cdot L \cdot t \left[ (I'_{sz} - I'_{sy}) \int_x^1 \ sin(2\beta) \cdot c(x)^3 \ dx \right.
\]

\[
\left. + I'_{syz} \int_x^1 \ cos(2\beta) \cdot c(x)^3 \ dx \right]
\]

Due to the tip-mass;

\[
Q_{\text{tip}} = \int_x^1 \ dQ_{\text{tip}}
\]
This can be simplified considerably. All the second moments apart from $I'_{az}$ and $I'_{sz}$ can be neglected as they are orders of magnitude smaller. As with the original twist calculations, the moment of inertia can be approximated by a cubic distribution, rather than the quartic, by generating another fictitious skin thickness:

$$t''_{tip} = \left( \frac{I'_{az}}{I'_{sz}} \cdot c \left( \frac{x_{tip} + 1}{2} \right) - t \right)$$

Giving a moment distribution of:

$$\int y^2 dh = \rho \cdot t''_{tip} \cdot I'_{sz} \cdot c(x)^3 \cdot t''_{tip}$$

The element of couple is then expressed as

$$dQ_{tip}(x) = \Omega^2 \cdot \rho \cdot L \cdot t''_{tip} \cdot I'_{sz} \cdot c(x)^3 \cdot \sin(2 \cdot \beta) \cdot dx$$

The total restoring moment experienced at $x$ also consists of the two components;

Due to the skin;

$$Q_{bt} = \int_x^1 \cdot dQ = \Omega^2 \cdot \rho \cdot L \cdot t' \cdot I'_{sz} \cdot c(x)^3 \cdot \sin(2 \cdot \beta) \cdot dx$$

A4
Due to the tip-mass

\[
Q_{\text{tip}}(x) = \int_{x}^{1} \ dQ = \Omega + \rho \ t_{\text{tip}} \ t''_{\text{tip}} \ t'_{sz} \int_{x}^{1} \sin(2\beta(x)) \cdot c(x)^3 \ dx
\]

These integrals are unfortunately not amenable to closed form solution as the angle is itself a function of \(x\).

Given the couple \(Q(x)\) acting at a particular point, the twist can be calculated, again using Batho-Bredt.

\[
\frac{d}{dl} \theta = -Q \int \frac{1}{4 \cdot A^2} ds
\]

Applying the chord distribution which is known and the fact that the shear modulus and skin thickness are constant around the section,

\[
\frac{d}{dx} \beta = \frac{L \cdot S'}{4 \cdot A^2 \cdot G} \cdot \frac{Q(x)}{c(x)^3}
\]

The 'untwist due to the blade's own mass can then be expressed as:

\[
\delta \beta = \Omega \cdot K \cdot W(x)
\]

where \(K \cdot W := \frac{\rho \cdot L^2 \cdot S'}{G \cdot A^2}
\]

\[
W(x) := I'_{sz} \int_{0}^{x} \frac{1}{x} \frac{1}{(1 - c'x)^3} \ dx
\]
As regards the tip mass, things are inevitably more complex. For values of $x < x_{tip}$ the couple due to all the tip mass material can be calculated and applied as a constant.

$$\frac{d}{dx} \beta = \frac{L \cdot S' \cdot \cdot \cdot Q(x)}{4 \cdot A^2 \cdot t \cdot G \cdot c(x)^3} = \Omega 2 \cdot \frac{\rho \cdot t \cdot t'' \cdot t_{tip}}{\rho \cdot t} \cdot k_{wtip} \cdot K \cdot W \cdot \frac{k_{wtip}}{(1 - c \cdot x)^3}$$

where $k_{wtip} = \int_{x_{tip}}^{1} sin(2 \cdot \beta (x)) \cdot (1 - c \cdot x_{tip})^3 dx$

Integrating this to find the cumulative effect,

$$\delta \beta_{tip} = \Omega 2 \cdot \frac{\rho \cdot t \cdot t'' \cdot t_{tip}}{\rho \cdot t} \cdot k_{wtip} \cdot K \cdot W \left[ \frac{1}{(1 - c \cdot x_{tip})^3} \right]_{0}^{x}$$

$$\delta \beta_{tip} = \Omega 2 \cdot \frac{\rho \cdot t \cdot t'' \cdot t_{tip}}{\rho \cdot t} \cdot k_{wtip} \cdot K \cdot W \left[ - \frac{1}{2} \cdot x \cdot \frac{2 - c \cdot x}{(1 - c \cdot x)^2} \right]$$

The value at the start of the tip-mass material is obtained by substituting $x_{tip}$.

$$\delta \beta_{x_{tip}} = \Omega 2 \cdot \frac{\rho \cdot t \cdot t'' \cdot t_{tip}}{\rho \cdot t} \cdot k_{wtip} \cdot K \cdot W \left[ \frac{1}{2} \cdot x_{tip} \cdot \frac{2 - c \cdot x_{tip}}{(1 - c \cdot x_{tip})^2} \right]$$

For $x >$ the same integral applies as for the blade skin, but the limit should run from to $x$. The twist as far as $x_{tip}$ is simply taken from $\delta \beta_{x_{tip}}$.

$$\delta \beta_{tip(x)} = \Omega 2 \cdot \frac{\rho \cdot t \cdot t'' \cdot t_{tip}}{\rho \cdot t} \cdot k_{wtip} \cdot K \cdot W (W(x) - W(x_{tip})) + \delta \beta_{x_{tip}}$$

At a rotor speed of 1200 rpm, when the blade has twisted through 28.1° (from a set pitch of 4°), the untwist is 5.8° to the first order. (Strictly, the whole process should be iterated). Of this figure, approximately 3.9° comes from the tip-mass and 1.9° from the blade mass.
It can be seen that these figures are not large but are quite significant at the higher rotor speeds. The propeller effect has therefore been included as an iterative procedure in the blade twisting model for the purpose of the aerodynamic simulations in the Chapter 4.

A2. APPENDIX TO CHAPTER 4

A2.1 Fitting a linear chord distribution to the 'Glauert ideal' chord distribution

Let the ideal chord distribution be designated \( c_a(x) \)

The fitted linear distribution is given by \( c(x) := c_0(1 - c'x) \) in the usual way.

However, it is desirable to constrain the distribution to a fixed value of chord at the reference station such that \( c(x_{\text{ref}}) = c_{\text{ref}} \cdot c_0(1 - c'x_{\text{ref}}) \). It is also helpful to make a substitution \( c = c_0c'd \).

The distribution can now be expressed as \( c(x) = c_0 - c'dx \)

The error with respect to the ideal is squared and weighted according to distance from the centre of rotation. The total weighted squared error for the whole blade is given by

\[
\int_0^1 (x + h) \cdot e(x)^2 \, dx = \int_0^1 (x + h) \cdot [\left( c_0 - c'dx \right) - c_a(x)]^2 \, dx
\]

Expanding this,
\[(\pi + h) \left[ (c_0 - c_1 x) - c_a(x) \right]^2 \]

\[= (\pi + h) \cdot c_a(x)^2 + \left[ 2 \cdot c_1 d^2 + (2 \cdot h_1 c_0 - 2 \cdot c_0^2) \cdot x - 2 \cdot h_1 c_0 \right] \cdot c_a(x) \ldots
\]

\[= \left[ c_1^2 \cdot x^3 + (h_1 c_0^2 - 2 \cdot c_1 c_0 d) \cdot x^2 + (c_0^2 - 2 \cdot h_1 c_0 c_d) \cdot x + h_1 c_0^2 \right] dx
\]

In order to minimise the total weighted squared error, the integration is carried out from 0 to 1, and the result is differentiated with respect to the blade taper coefficient, \(c'\), and set to zero.

\[\frac{d}{dc'} \int_0^1 (\pi + h) \cdot e(x)^2 \, dx = 0 \]

\[= \frac{d}{dc'} \int_0^1 (\pi + h) \cdot c_a(x)^2 \, dx \ldots
\]

\[= \left[ c_1^2 \cdot x^3 + (h_1 c_0^2 - 2 \cdot c_1 c_0 d) \cdot x^2 + (c_0^2 - 2 \cdot h_1 c_0 c_d) \cdot x + h_1 c_0^2 \right] dx
\]

The integral products involving the ideal chord distribution are best calculated numerically as no closed form expression is available. These are designated as \(I_0, I_1\), and \(I_2\).

\[I_0 = \int_0^1 c_a(x) \, dx \quad I_1 = \int_0^1 x \cdot c_a(x) \, dx \quad I_2 = \int_0^1 x^2 \cdot c_a(x) \, dx
\]

(In practice the integrals are carried out by the trapezium rule,)

\[I_0 := \frac{1}{2} \cdot \sum_{j=0}^{j_{\text{max}}-1} \left( c_{a_j} + c_{a_{j+1}} \right) \cdot (x_{j+1} - x_j)
\]
\[ I_1 = \frac{1}{2} \sum_{j=0}^{j_{\text{max}}-1} \left( c a_j x_j + c a_{j+1} x_{j+1} \right) \left( x_{j+1} - x_j \right) \]

\[ I_2 = \frac{1}{2} \sum_{j=0}^{j_{\text{max}}-1} \left[ c a_j (x_j)^2 + c a_{j+1} (x_{j+1})^2 \right] \left( x_{j+1} - x_j \right) \]

Thus,

\[ 0 = 2 \frac{d}{dc} \left[ c d I_2 + \left( h c d - c c_0 \right) I_1 - h c 0 I_0 \right] \]

\[ + \frac{d}{dc} \left[ \left( h + \frac{1}{2} \right) c 0^2 - \left( h + \frac{2}{3} \right) c d c 0 + \left( \frac{1}{4} + \frac{1}{3} h \right) c d^2 \right] \]

Each of these differentials is best calculated separately.

\[
\begin{align*}
\frac{c}{c_0} &= \frac{c_{\text{ref}}}{1 - c_{\text{x ref}}} & \frac{d}{dc} c_0 &= \frac{c_{\text{ref}}}{(1 - c_{\text{x ref}})^2} \\
\frac{c}{c^2} &= \frac{c_{\text{x ref}}}{1 - c_{\text{x ref}}} & \frac{d}{dc} c^2 &= \frac{c_{\text{ref}}}{(1 - c_{\text{x ref}})^2} \\
\frac{d}{dc} c_0 &= 2 \cdot c_0 \frac{d}{dc} c_0 = 2 \cdot \frac{c_{\text{ref}}}{1 - c_{\text{x ref}}} \frac{c_{\text{ref}}}{(1 - c_{\text{x ref}})^2} = 2 \cdot x_{\text{ref}} \frac{c_{\text{ref}}^2}{(1 - c_{\text{x ref}})^3} \\
\frac{d}{dc} c^2 &= 2 \cdot c \frac{d}{dc} c^2 = 2 \cdot \frac{c_{\text{ref}}}{1 - c_{\text{x ref}}} \frac{c_{\text{ref}}}{(1 - c_{\text{x ref}})^2} = 2 \cdot \frac{c_{\text{ref}}^2}{(1 - c_{\text{x ref}})^3} \\
\frac{d}{dc} c_0 &= c_0 \frac{d}{dc} c_0 + c \frac{d}{dc} c^2 = c_{\text{ref}} \frac{d}{dc} c_0^+ \vdots
\end{align*}
\]
\[
\frac{...}{1-c'x_{ref}} = \frac{c_{ref}}{1-c'x_{ref}} \left(1-c'x_{ref}\right)^2 + \frac{c_{ref} \cdot c_{ref} x_{ref}}{1-c'x_{ref}} \left(1-c'x_{ref}\right)^2 = \left(1+c'x_{ref}\right) \frac{c_{ref}^2}{(1-c'x_{ref})^3}
\]

Substituting back in for \(c_0\) and \(c_d\), to return the equations to being expressed in terms of the one variable, \(c'\) and the two fixed parameters, \(c_{ref}\) and \(x_{ref}\),

\[
2 \frac{d}{dc'} \left[ c_d I_2 + \left(h \cdot c_d - c_0\right) I_1 - h \cdot c_0 I_0 \right]
\]

\[
...= 2 \cdot \frac{c_{ref}}{(1-c'x_{ref})^2} \left[ I_2 + I_1 \cdot (h - x_{ref}) - h \cdot x_{ref} I_0 \right]
\]

Similarly,

\[
\frac{d}{dc'} \left[ \left(h + \frac{2}{3}\right) c_0^2 - \left(h + \frac{2}{3}\right) c_d c_0 + \left(\frac{1}{4} + \frac{1}{3} h\right) c_d^2 \right]
\]

\[
...= \frac{c_{ref}^2}{(1-c'x_{ref})^3} \left[ \left(h + \frac{1}{2}\right) 2 \cdot x_{ref} - \left(h + \frac{2}{3}\right) \left(1+c'x_{ref}\right) + \left(\frac{1}{4} + \frac{1}{3} h\right) 2 \cdot c' \right]
\]

\[
...= \frac{1}{6} \frac{c_{ref}^2}{(1-c'x_{ref})^3} \left[ \left(12 h + 6\right) x_{ref} - \left(6 h + 4\right) \right] + \left(6 h + 4\right) x_{ref} + 4 h + 3 \right] c'
\]

The two terms in the equation are equated

\[
0 = 2 \cdot \frac{c_{ref}}{(1-c'x_{ref})^2} \left[ I_2 + I_1 h - I_1 x_{ref} - h x_{ref} I_0 \right]...
\]

\[
+ \frac{1}{6} \frac{c_{ref}^2}{(1-c'x_{ref})^3} \left[ \left(12 h + 6\right) x_{ref} - \left(6 h + 4\right) \right] + \left(6 h + 4\right) x_{ref} + 4 h + 3 \right] c'
\]

and rearranged to solve for \(c'\), the blade taper, giving
\[
\frac{c'}{c_r} = 2 \frac{\left(6h + 3\right) x_{ref} - \left(3h + 2\right)}{\left(6h + 4\right) x_{ref} - \left(4h + 3\right)} - 6 \left[h x_{ref} I_0 + \left(x_{ref} - h\right) I_1 - I_2\right]
\]

\[
= \frac{\left(6h + 3\right) x_{ref} - \left(3h + 2\right)}{12 x_{ref} \left[h x_{ref} I_0 + \left(x_{ref} - h\right) I_1 - I_2\right]}
\]

---

**ideal profile**

---

**linear fit**

---

spanwise position (m)

---

chord (m)
A3. APPENDIX TO CHAPTER 5

A3.1 Elastic curves for uniform blade

Concentrated tip-mass, concentrated end load

The differential equation is given by:

\[ EI_0 \frac{d^2v}{dr^2} + G_0 (v_{EG} - v(r)) - Q_0 (R - r) = 0 \]

\[ EI_0 \frac{v_{EG}}{L^2} \frac{d^2y}{dx^2} + m_{tip} (1 + h) \Omega^2 v_{EG} (1 - y) - Q_0 L (1 - x) = 0 \]

\[ \frac{d^2y}{dx^2} + \Lambda^2 (1 - y) - \frac{(1 - x)}{\varphi_{EG}} = 0 \]

where \( \Lambda^2 = m_{tip} (1 + h) \Omega^2 \frac{L^3}{EI_0} \) and \( \varphi_{EG} = \frac{Q_0 L^3}{v_{EG} EI_0} \)

The DE has solutions of the following form

\[ y = A + B (1 - x) + C \exp(\Lambda x) + D \exp(-\Lambda x) \]  \( (A3.1) \)

\[ y' = -B + \Lambda C \exp(\Lambda x) - \Lambda D \exp(-\Lambda x) \]  \( (A3.2) \)

\[ y'' = \Lambda^2 C \exp(\Lambda x) + \Lambda^2 D \exp(-\Lambda x) \]  \( (A3.3) \)

Applying appropriate boundary conditions for a cantilever beam,

From Equ. (A2.1.1) with zero deflection at the root:

\[ y(0) = 0, \quad \Rightarrow \quad A + B + C + D = 0 \]  \( (A3.4) \)

From Equ. (A2.1.2) with zero slope at the root:
\[ y'(0) = 0, \quad \Rightarrow \quad -B + \Lambda.C - \Lambda.D = 0 \quad (A3.5) \]

From Equ. (A2.1.3) with zero curvature (bending moment) at the free end

\[ y''(1) = 0, \quad \Rightarrow \quad \frac{C}{\gamma} + \gamma.D = 0 \quad \text{where} \quad \gamma = \exp(-\Lambda) \quad (A3.1) \]

Normalising the displacement function \( y \) to unity at the free end:

\[ y(1) = 1, \quad \Rightarrow \quad A + \frac{C}{\gamma} + \gamma.D = 1 \quad (A3.2) \]

giving:

\[ A = 1 = \frac{\Lambda.(1+\gamma^2)-(1-\gamma^2)}{\Lambda.(1+\gamma^2)-(1-\gamma^2)} \]

\[ B = \frac{-\Lambda.(1+\gamma^2)}{\Lambda.(1+\gamma^2)-(1-\gamma^2)} \]

\[ C = \frac{1}{\Lambda.(1+\gamma^2)-(1-\gamma^2)} \]

\[ D = \frac{-\gamma^2}{\Lambda.(1+\gamma^2)-(1-\gamma^2)} \]

and

\[ y_{eg} = \frac{\Lambda.(1+\gamma^2).x-(1-\gamma^2)+\exp(-\Lambda.x)-\gamma^2.\exp(\Lambda.x)}{\Lambda.(1+\gamma^2)-(1-\gamma^2)} \quad (A3.6) \]

Substituting back into the original differential equation gives an expression for \( \phi \) and therefore for \( v_{EG} \):

\[ v_{EG} = \frac{Q}{k_{EG}} = \frac{\phi_{EG} \cdot \varphi}{k_0} \quad (A3.7) \]

where

\[ \varphi_{EG} = \frac{k_0}{k_{EG}} = \frac{1}{\Lambda^2} \left( 1 - \frac{\xi}{\Lambda} \right) \quad \text{and} \quad \xi = \frac{1-\gamma^2}{1+\gamma^2} \quad (A3.8) \]

A3.1.2 Concentrated tip-mass, distributed load

The differential equation is now given by:

\[ EI_0 \cdot \frac{d^2 v}{dr^2} + G_e (v_{EGd} - v(r)) - \frac{q}{2} . (R-r)^2 = 0 \]

\[ EI_0 \cdot \frac{v_{EG}}{L^2} \cdot \frac{d^2 y}{dx^2} + m_{tip} \cdot L(1+h) \cdot \Omega^2 \cdot v_{EGd} \cdot (1-\gamma) - \frac{q}{2} \cdot L^2 \cdot (1-x)^2 = 0 \]
\[
\frac{d^2 y}{dx^2} + \Lambda^2 (1-y) - \frac{(1-x)^2}{2\varphi_E} = 0
\]  
(A3.9)

Try solutions of the form

\[
y = A + B(1-x)^2 + C \exp(\Lambda x) + D \exp(-\Lambda x)
\]  
(A3.10)

\[
y' = -2B(1-x) + \Lambda C \exp(\Lambda x) - \Lambda D \exp(-\Lambda x)
\]  
(A3.11)

\[
y'' = 2B + \Lambda^2 C \exp(\Lambda x) + \Lambda^2 D \exp(-\Lambda x)
\]  
(A3.12)

If the differential equation itself (A2.1.9) is applied to the trial solution, expressions for some of the unknowns are obtained.

\[
B = \frac{1}{2 \Lambda^2 \varphi} \\
A = 1 - \frac{1}{\Lambda^4 \varphi} = 1 + \frac{2B}{\Lambda^2}
\]

Applying appropriate boundary conditions of zero deflection and slope at the root to Eqn. (A2.1.10) and its derivatives:

\[
y(0) = 0, \quad \Rightarrow \quad 1 + B \left(1 + \frac{2}{\Lambda^2}\right) + C + D = 0
\]  
(A3.3)

\[
y'(0) = 0, \quad \Rightarrow \quad \frac{2B}{\Lambda} + C - D = 0
\]  
(A3.4)

Zero curvature (bending moment) at the free end:

\[
y''(1) = 0, \quad \Rightarrow \quad \frac{2B}{\Lambda} + C \exp(\Lambda) + D \exp(-\Lambda) = 0
\]

From these:

\[
-\frac{1}{B} = 2\varphi \Lambda^2 = \frac{\Lambda^2 - 2\Lambda + 2 - 4\zeta}{\Lambda^2} \\
\text{where} \quad \zeta = \gamma \cdot \frac{1 - \gamma \Lambda}{1 + \gamma^2}
\]

\[
C = -\frac{1}{2} \frac{\Lambda^2 - 2\Lambda + 2}{\Lambda^2} \cdot B - \frac{1}{2} \\
D = -\frac{1}{2} \frac{\Lambda^2 + 2\Lambda + 2}{\Lambda^2} \cdot B - \frac{1}{2}
\]

Giving the shape as:
\[
\gamma_{ego} = \frac{\Lambda^2 (2x - x^2) - 2(\Lambda + \zeta) (1 - \exp(-\Lambda x)) - 2 \cdot \zeta (1 - \exp(\Lambda x))}{\Lambda^2 - 2\Lambda + 2 - 4\zeta} \quad (A3.13)
\]

and the end deflection as:

\[
\gamma_{ego} = \Phi_{ego} \cdot \frac{aL}{k_0} \quad \text{where} \quad \Phi_{ego} = \frac{\Lambda^2 - 2\Lambda + 2 - 4\zeta}{4\Lambda^4} \quad (A3.14)
\]

A3.2 Calculation of blade mass and tip mass

The mass of a blade element is expressed in terms of blade parameters.

\[
dm = \mu \cdot \left(1 - e^{-x}\right) \cdot L \cdot dx = \mu \cdot 0^L \cdot P(x)^T \cdot m \cdot dx
\]

The integral of this expression over the whole blade gives the blade mass.

\[
m_{bl} = \int_0^1 \mu \cdot 0^L \cdot P(x)^T \cdot m \cdot dx
\]

Only the polynomial needs to be integrated. Expressed in terms of the whole column vector,

\[
PP(x) = \int_x^1 P(x) \; dx
\]

where \(PP(x) = PP(l, x)\)

and \(PP(l, x) = \frac{1}{l+1} \cdot (1 - x^{l+1})\)

The blade mass is then given by

\[
m_{bl} = \mu \cdot 0^L \cdot PP(0) \cdot m \quad m_{bl} = 0.5385 \cdot \text{kg}
\]

For the tip-mass, a mass element is given by
The elemental mass is integrated over the length of the tip-mass to give its mass

\[ m_{\text{tip}} = \int_{x_{\text{tip}}}^{1} \rho_{\text{tip}} \cdot L \cdot \left[ \frac{c' \cdot \left(1 - c' \cdot x\right) \cdot \left(1 - c' \cdot x\right)}{S' \cdot t \cdot \left(1 - c' \cdot x\right)} \right] \, dx \]

The elemental mass is integrated over the length of the tip-mass to give its mass

\[ m_{\text{tip}} = \int_{x_{\text{tip}}}^{1} \rho_{\text{tip}} \cdot L \cdot \left[ \frac{c' \cdot \left(1 - c' \cdot x\right) \cdot \left(1 - c' \cdot x\right)}{S' \cdot t \cdot \left(1 - c' \cdot x\right)} \right] \, dx \]

\[ m_{\text{tip}} = 0.5122 \cdot \text{kg} \]

For the sake of simplicity this mass distribution in the tip-mass can be approximated adequately by representing it as a constant skin of different thickness from the blade skin and thus having the same distribution as it. This simplifies the calculation of the mass and centrifugal matrices. The mass of the tip mass is calculated in a similar manner with the integration only being carried out over the appropriate part of the blade length.

\[ dm_{\text{tip}}(x) = \rho_{\text{tip}} \cdot L \cdot \left[ \frac{c' \cdot \left(1 - c' \cdot x\right) \cdot \left(1 - c' \cdot x\right)}{S' \cdot t \cdot \left(1 - c' \cdot x\right)} \right] \, dx \]

\[ m_{\text{tip}} = 0.5122 \cdot \text{kg} \]

Where \( \mu_{\text{tip}} := \rho_{\text{tip}} \cdot t_{\text{tip}} \cdot S' \cdot c' \cdot 0 \)

And \( t_{\text{tip}} := \frac{c' \cdot A'}{S'} \cdot \left[ 1 - \left(\frac{x_{\text{tip}}}{2}\right) \right] - t \)
\[ m'' \text{_{tip}} = \mu \cdot L \cdot \int_{x \text{_{tip}}}^{1} (1 - c \cdot x) \, dx = ... \]

\[ m'' \text{_{tip}} = \mu \cdot L \cdot \mathbf{PP}(x \text{_{tip}}) \cdot m \]  \hspace{1cm} (A3.2)

\[ m'' \text{_{tip}} = 0.5095 \cdot \text{kg} \]

**A3.3 Integral functions** \( B(j, l, x) \) and \( C(j, k, l, x) \).

Define the two functions as follows:

\[ B(j, l, x) = \int_{x}^{1} x \cdot \cos(j \cdot \pi \cdot x) \, dx \]

\[ C(j, k, l, x) = \int_{x}^{1} x \cdot Z(j, x) \cdot Z(k, x) \, dx \]

where \( Z(j, x) = \cos(j \cdot \pi \cdot x) - \cos \left( \frac{j + 1}{2} \cdot \pi \cdot x \right) \)

\( B(j, l, x) \) can be integrated by parts to yield a recurrence relation which can be used to calculate its value numerically at any value of \( x \)

\[
\int_{x}^{1} x \cdot \cos(j \cdot \pi \cdot x) \, dx = \left( \frac{x}{j \cdot \pi} \cdot \sin(j \cdot \pi \cdot x) + \frac{l^2}{j^2 \cdot 2} \cdot \cos(j \cdot \pi \cdot x) \right)_{1}^{x} \\
+ \frac{l \cdot (l - 1)}{j^2 \cdot \pi^2} \int_{x}^{1} x^{l - 2} \cdot \sin(j \cdot \pi \cdot x) \, dx
\]
\[
\begin{align*}
... &= \frac{1}{j \pi} \left( \sin(j \cdot \pi) - x^l \sin(j \cdot \pi \cdot x) \right) ... \\
&+ \frac{l}{j^2 \pi^2} \left( \cos(j \cdot \pi) - x^{l-1} \cos(j \cdot \pi \cdot x) \right) ... \\
&+ \frac{l(l-1)}{j^3 \pi^3} \left( \sin(j \cdot \pi) - x^{l-2} \sin(j \cdot \pi \cdot x) \right) ... \\
&+ \frac{l(l-1)(l-2)}{j^4 \pi^4} \left( \cos(j \cdot \pi) - x^{l-3} \cos(j \cdot \pi \cdot x) \right)
\end{align*}
\]

and
\[
... = \frac{1 - x^{l+1}}{l + 1}
\]

when \( j = 0 \)

\[
B(j, l, x) := \begin{cases} 
0, & j = 0, \frac{1 - x^{l+1}}{l + 1}, \frac{1}{j \pi} \left( \sin(j \cdot \pi) - x^l \sin(j \cdot \pi \cdot x) \right) ... \\
+ (l > 0) \left[ \frac{l}{(j \pi)^2} \left( \cos(j \cdot \pi) - x^{l-1} \cos(j \cdot \pi \cdot x) \right) \right] ... \\
+ \left[ (l > 1) \frac{l(l-1)}{(j \pi)^2} B(j, l-2, x) \right]
\end{cases}
\]

(A3.1)

Regarding the function \( C(j, k, l, x) \), a trigonometric identity can be applied allowing it to be expressed in terms of simpler integrals which are of the same form as \( B(j, l, x) \):

\[
\cos(j \cdot \pi \cdot x) \cdot \cos(k \cdot \pi \cdot x) = \frac{1}{2} \left( \cos((j - k) \cdot \pi \cdot x) + \cos((j + k) \cdot \pi \cdot x) \right)
\]

First, the terms in \( C(j, k, l, x) \) are multiplied out.

\[
C(j, k, l, x) = \int_0^1 x^l Z(j, x) \cdot Z(k, x) \, dx
\]
\[ C(j, k, l, x) = \frac{1}{2} \int_{x}^{1} x^l \cos(j \cdot \pi \cdot x) \cos(k \cdot \pi \cdot x) \, dx \]

\[ = \frac{1}{2} \int_{x}^{1} x^l \cos(j \cdot \pi \cdot x) \cos\left(k + \frac{1}{2} \right) \pi \cdot x \, dx \]

\[ = \frac{1}{2} \int_{x}^{1} x^l \cos\left(j + \frac{1}{2} \right) \pi \cdot x \cos(k \cdot \pi \cdot x) \, dx \]

\[ = \frac{1}{2} \int_{x}^{1} x^l \cos\left(j + \frac{1}{2} \right) \pi \cdot x \cos\left(k + \frac{1}{2} \right) \pi \cdot x \, dx \]

This derivation can be repeated in a similar manner for \( C'(j, k, l) \) and \( C''(j, k, l) \), the first and second derivatives of \( C(j, k, l, x) \) respectively.

\[ C'(j, k, l, x) = \int_{x}^{1} x^l \cdot \cos(j \cdot \pi \cdot x) \cdot \cos(k \cdot \pi \cdot x) \, dx \]

\[ C''(j, k, l, x) = \int_{x}^{1} x^l \cdot \cos(j \cdot \pi \cdot x) \cdot \cos(k \cdot \pi \cdot x) \, dx \]
where \( Z'(j,x) := \left( j + \frac{1}{2} \right) \pi \cdot \sin \left( \left( j + \frac{1}{2} \right) \pi \cdot x \right) - j \cdot \pi \cdot \sin(j \cdot \pi \cdot x) \)

Using the trigonometric identity for products of sines and the definition of \( B(j,l,x) \),

\[
\sin(j \cdot \pi \cdot x) \cdot \sin(k \cdot \pi \cdot x) = \frac{1}{2} \left( \cos((j - k) \cdot \pi \cdot x) - \cos((j + k) \cdot \pi \cdot x) \right)
\]

\[
C'(j,k,l,x) := \frac{\pi^2}{2} \left[ j \cdot k + \left( j + \frac{1}{2} \right) \left( k + \frac{1}{2} \right) \cdot B(j - k, l, x) \ldots \right]
\]

\[
+ j \cdot \left( k + \frac{1}{2} \right) \cdot B(j - k - \frac{1}{2}, l, x) \ldots
\]

\[
+ j \cdot k \cdot B(j + k, l, x) - \left( j + \frac{1}{2} \right) \cdot k \cdot B(j - k + \frac{1}{2}, l, x) \ldots
\]

\[
+ \left[ j \cdot \left( k + \frac{1}{2} \right) + \left( j + \frac{1}{2} \right) \cdot k \right] \cdot B(j + k + \frac{1}{2}, l, x) \ldots
\]

\[
+ \left( j + \frac{1}{2} \right) \cdot \left( k + \frac{1}{2} \right) \cdot B(j + k + 1, l, x)
\]

(A3.3)

Finally,

\[
C''(j,k,l,x) = \int_x^1 x^2 \cdot Z''(j,x) \cdot Z''(k,x) \, dx
\]

where \( Z''(j,x) := \left( j + \frac{1}{2} \right) \cdot \pi^2 \cdot \cos \left( \left( j + \frac{1}{2} \right) \pi \cdot x \right) - j^2 \cdot \pi^2 \cdot \cos(j \cdot \pi \cdot x) \)

Again using the identity for products of cosines and the definition of \( B(j,l,x) \),

\[
\cos(j \cdot \pi \cdot x) \cdot \cos(k \cdot \pi \cdot x) = \frac{1}{2} \left( \cos((j - k) \cdot \pi \cdot x) + \cos((j + k) \cdot \pi \cdot x) \right)
\]
\[ C''(j,k,l,x) := \frac{\pi^4}{2} \left[ j^2k^2 + \left( j + \frac{1}{2} \right)^2 \left( k + \frac{1}{2} \right)^2 \cdot B(j-k,l,x) \right. \]
\[ \left. + \left( j + \frac{1}{2} \right)^2 \left( k + \frac{1}{2} \right)^2 \cdot B(j+k+1,l,x) \right) \]
\[ + \left( j + \frac{1}{2} \right)^2 \cdot \left( k + \frac{1}{2} \right) \cdot B(j+k,l,x) \]
\[ \left. + \left( j + \frac{1}{2} \right)^2 \cdot \left( k + \frac{1}{2} \right)^2 \cdot B(j-k+1,l,x) \right] \]

(A3.4)

**A3.4 Derivation of centrifugal bending moment operator**

The centrifugal bending moment is found from the following integral.

\[ M_G(x) = \int_x^1 (v(x_1) - v(x)) \, dG(x_1) + \int_x^1 (v(x_1) - v(x)) \, dG_{\text{tip}}(x_1) \]

(A3.1)

Splitting this into its component parts and expanding each separately,

Taking each part of the integrals one at a time, first consider the blade-shell contribution.

\[ \int_x^1 v(x_1) \, dG(x_1) = \mu \cdot L^2 \cdot \Omega \cdot 2 \cdot \nu \cdot Z^T \cdot a \cdot P(x_1)^T \, dx_1 \cdot g \]

\[ \cdots = \mu \cdot L^2 \cdot \Omega \cdot 2 \cdot \nu \cdot Z^T \cdot B(x) \cdot g \]
where \( B(x) = \int_x^1 x^2 \cdot Z(x) \, dx \cdot B(j, l, x) - B(j + \frac{1}{2}, l, x) \)

and \( B(j, l, x) = \int_x^1 x \cdot \cos(j \cdot \pi \cdot x) \, dx \)

\[
\int_x^1 v(x) \, dG(x) = \mu \cdot L^2 \cdot \Omega^2 \cdot v \cdot Z^T \cdot a^T \cdot B(x) \cdot g \\
... = \mu \cdot L^2 \cdot \Omega^2 \cdot v \cdot Z^T \cdot a^T \cdot Z(x) \cdot PP(x)^T \cdot g
\]

Thus the moment at \( x \) due to the centrifugal load acting on the blade-shell mass is given by:

\[
M_{Gbl}(x) = \mu \cdot L^2 \cdot \Omega^2 \cdot v \cdot Z^T \cdot (B(x) - Z(x) \cdot PP(x)^T) \cdot g \quad (A3.2)
\]

Consider now the different parts of the integral for the tip-mass contribution.

When \( x \geq x_{tip} \):

\[
\int_x^1 v(x) \, dG_{tip}(x) = \mu_{tip} \cdot L^2 \cdot \Omega^2 \cdot v \cdot Z^T \cdot B(x) \cdot g
\]

\[
\int_x^1 v(x) \, dG_{tip}(x) = \mu_{tip} \cdot L^2 \cdot \Omega^2 \cdot v \cdot Z^T \cdot Z(x) \cdot PP(x)^T \cdot g
\]

and when \( x < x_{tip} \):

\[
\int_x^1 v(x) \, dG_{tip}(x)
\]

\[
... = \int_{x_{tip}}^1 v(x) \, dG_{tip}(x) = \mu_{tip} \cdot L^2 \cdot \Omega^2 \cdot v \cdot Z^T \cdot B(x_{tip}) \cdot g
\]
Thus the moment at $x$ due to the centrifugal load acting on the blade-tip mass is given by:

$$
M_{G_{tip}}(x) = \mu_{_{tip}} L^2 \cdot \Omega^2 \cdot v \cdot Z \cdot a^T \cdot (B(x) - Z(x) \cdot PP(x)^T) \cdot g
$$

when $x \geq x_{tip}$

and

$$
M_{G_{tip}}(x) = \mu_{_{tip}} L^2 \cdot \Omega^2 \cdot v \cdot Z \cdot a^T \cdot (B(x_{tip}) - Z(x) \cdot PP(x_{tip})^T) \cdot g
$$

when $x < x_{tip}$

It is now possible to separate the dimensioned quantities and to roll up the entire shape-dependent part of the equation into a column vector operator $mG$.

$$
m G(x) := (B(x) - Z(x) \cdot PP(x)^T) \cdot g
$$

$$
m G_{tip}(x) := \frac{\mu_{_{tip}}}{\mu} \cdot \left( B \left( x_{tip}, x, x_{tip} \right) \right) \cdot \left( Z \cdot PP \left( x_{tip}, x, x_{tip} \right) \right)^T \cdot g
$$

Combined with the universal shape coefficients, it is possible to generate a four-column matrix operator $M\ G$:

$$
M \ G(A, x) := \left( m G(x)^T + m G_{tip}(x)^T \right) \cdot \partial A(A, x)
$$
The centrifugal moment can then be found directly from the loading vector.

\[ M_G(\Lambda, x) = L \cdot \Lambda^2 \cdot Q \cdot (M_G(\Lambda, x) \cdot q) \]  

(A3.5)

### A3.5 Cubic curve fitting to loading data

When fitting a curve to the aerodynamic loading data, as it is the bending moment which dominates the bending equilibrium, it makes sense to give weightings to the data which favour a close fit at the tip at the expense of the fit at the root.

In this case, the load curves were first normalised by dividing through by the net root load (found by numerical integration) and thus expressed as the data series \( q_i \). The position on the blade was expressed in non-dimensional form as \( x_i \).

The error is expressed in terms of the unknown cubic coefficients and the data points, weighted by \( xx \). The expression is then differentiated and set to zero for each coefficient in turn, to yield a set of 4 simultaneous equations.

Assuming a cubic curve is to be fitted, a 4x4 matrix and a 4 element column matrix are formed. The equation can be solved by matrix inversion to give the necessary coefficients in the form of a column vector \( q \).

\[ \Sigma XX \cdot q - \Sigma XY = 0 \]  

(A3.1)

where \[ \Sigma XX_{k,l} = \sum_{i=1}^{st} (xx_i)^{k+l} \]

\[ \Sigma XY_i = \sum_{i=1}^{st} (xx_i)^k \cdot q_i \]
$st$ is the number of blade stations, and $k$ and $l$ run from 0 to 3.

so $q = \sum XX^1 \cdot \sum XY$ \hspace{1cm} (A3.2)

The fit is not spectacularly good for the load distribution itself but is extremely good when the shear force or bending moment distributions are compared with those derived directly from the data.

Figure 3.5.1 Example aerodynamic loading data and cubic curve fit

Figure 3.5.2 Fit for shear force by numerical and algebraic integration
Figure 3.5.3 Fit for bending moment by numerical and algebraic integration
A4. APPENDIX TO CHAPTER 8

A4.1 Spin Test Aerodynamics

For the purposes of the spin-test, we must redefine the inflow as, like a helicopter in hover, there is only induced flow, no undisturbed flow through the rotor, so the former cannot be expressed as a proportion of the latter. The most obvious approach is to express it in terms of the tip-speed $R\Omega$ or the local blade-speed $r\Omega = xR\Omega$. Thus if the induced velocity is $u$, then the induced speed ratio is given by;

$$\lambda_i = \frac{r\Omega}{u}$$

It is also necessary to define the tangential induced flow, but here we can stick with the same convention as for the windmill. If the tangential induced velocity is $v$, then the induction factor $a'$ is given by;

$$a' = \frac{v}{r\Omega}$$

The in-plane flow seen by the blade is then given by;

$$V_1 = r\Omega + v = (1+a')xR \Omega$$

The flow angle is then given by; (assuming small angle approximations are valid)

$$\phi = \tan \phi = \frac{u}{v + xR \Omega} = \frac{1}{\lambda_i(1+a')}$$

and the wind speed seen by the blade is given by;

$$W_1^2 = U_1^2 + V_1^2$$

$$= (xR \Omega)^2 (1/\lambda_i^2 + (1+a')^2)$$

$$= (xR \Omega)^2 (1+a')^2 (1+\phi^2)$$

Starting from the usual momentum theory approach for an annular element of thrust force,
\[ dT = -m' \Delta U \]
\[ = -\rho U_1 dA \Delta U \]

It is unfortunately not possible to make the usual transition to dimensionless force coefficients as division by the usual factor of \( \frac{1}{2} \rho U_\infty^2 \) is meaningless when \( U_\infty = 0 \).

Making the substitutions \( U_1 = \frac{1}{2} \Delta U = u = xR\Omega/\lambda_i \) and \( dA = 2\pi R^2 x \, dx \) gives

\[ dT = -2 \rho \frac{2\pi R^2 x \, dx (xR\Omega)^2}{\lambda_i^2} = -4\pi \rho R^4 \Omega^2 x^3 \, dx / \lambda_i^2 \]

Similarly, for an annular element of torque,

\[ dQ = -m' r \Delta V \]
\[ = -\rho U_1 dA 2rV_I \]
\[ = -\rho xR\Omega/\lambda_i \times 2\pi R^2 x \, dx 2 \times xR a' xR\Omega \]
\[ = -4\pi \rho R^5 \Omega^2 x^4 \, dx / \lambda_i \]

Looking now at the aerodynamic forces on the blade elements themselves, let us now make a number of simplifications.

Let us assume that the lift coefficient is 2D and linear with angle of attack i.e.

\[ C_L = C_{L'} (\alpha-\alpha_0) \]

Substituting for \( \alpha \) with \( \alpha = \phi - \beta \),

\[ C_L = C_{L'} (\phi-(\alpha_0+\beta)) = C_{L'} (\phi - \phi_0) \text{ where } \phi_0 = (\alpha_0 + \beta) \]

The elemental lift force is then given by

\[ dL = \frac{1}{2} \rho W_1^2 c \, dr \, C_i \]
\[ = \frac{1}{2} \rho c R^3 \Omega^2 x^2 \, dx \, (1 + a')^2 (1 + \phi^2) \cdot C_{L'} (\phi - \phi_0) \]

Let us assume that drag coefficient is quadratic, centred on \( \alpha = 0 \). Thus
\[ C_D = C_{D_{\text{min}}} + k_d \alpha^2 = (C_{D_{\text{min}}} + k_d \beta^2) - 2k_d \beta \phi + k_d \phi^2 = k_0 - k_1 \phi + k_d \phi^2 \]

The elemental drag force is given by

\[
d D = \frac{1}{2} \rho W^2 c d r C_d = \frac{1}{2} \rho c R^3 \Omega^2 x^2 d x (1+\alpha)^2 \left(1+\phi^2\right) \left(k_0 - k_1 \phi + k_d \phi^2\right) \]

Resolving these forces onto axes parallel to and perpendicular to the plane of rotation gives thrust and torque;

\[
d T = d L \cos \phi + d D \sin \phi = \left(1 - \frac{1}{2} \phi^2\right) d L + \phi d D
\]

\[
= \frac{1}{2} \rho c R^3 \Omega^2 x^2 d x (1+\alpha)^2 \left(1+\phi^2\right) \left[ C_L' \left(1 - \frac{1}{2} \phi^2\right) \left(\phi - \phi_0\right) + \phi \left(k_0 - k_1 \phi + k_d \phi^2\right) \right]
\]

\[
d Q = r \left(d L \sin \phi - d D \cos \phi\right) = x R \left(\phi d L - \left(1 - \frac{1}{2} \phi^2\right) d D\right)
\]

\[
= \frac{1}{2} \rho c R^4 \Omega^2 x^3 d x (1+\alpha)^2 \left(1+\phi^2\right) \left[ C_L' \alpha \phi \left(\phi - \phi_0\right) + \phi \left(k_0 - k_1 \phi + k_d \phi^2\right) \right]
\]

Equating the two expressions for thrust (remembering to allow for the number of blades, \(n\)),

\[
d T = \frac{1}{2} \rho n c R^3 \Omega^2 x^2 d x (1+\alpha)^2 \left(1+\phi^2\right) \left[ C_L' \left(1 - \frac{1}{2} \phi^2\right) \left(\phi - \phi_0\right) + \phi \left(k_0 - k_1 \phi + k_d \phi^2\right) \right]
\]

\[
= -4 \pi \rho R^4 \Omega^2 x^3 d x / \lambda_i^2
\]

Dividing through by the common terms, and making the substitution \(\phi = 1/\lambda_i (1+\alpha)\),

\[
\left(1 + \phi^2\right) \left[ C_L' \left(1 - \frac{1}{2} \phi^2\right) \left(\phi - \phi_0\right) + \phi \left(k_0 - k_1 \phi + k_d \phi^2\right) \right]
\]

\[
= -C_L' \alpha \phi_0 + \left(C_L' \alpha + k_0\right) \phi - \left(\frac{1}{2} C_L' \alpha \phi_0 + k_1\right) \phi^2 \ldots
\]

\[
+ \left(\frac{1}{2} C_L' \alpha + k_0 + k_d\right) \phi^3 + \left(\frac{1}{2} C_L' \alpha \phi_0 - k_1\right) \phi^4 - \left(\frac{1}{2} C_L' \alpha \phi_0 - k_d\right) \phi^5
\]
\[
\frac{-8\pi xR}{nc\lambda_i^2(1 + a')^2} = -\frac{4\phi^2}{\sigma} \quad \left(= \frac{1}{\frac{1}{2}\rho(1 + a')^2 n c^2 \Omega^2 x^2} \frac{dT}{dx}\right)
\]

where local solidity, \(\sigma = \frac{nc}{2\pi xR}\)

Dropping any terms of higher than 2nd order, we are left with a quadratic which can be solved for \(\phi\).

\[
C_L' a\phi_o - \left(C_L' a + k_o\right)\phi + \left(\frac{1}{2} C_L' a\phi_o + k_i - 4/\sigma\right)\phi^2 = 0
\]

\[
\phi = \frac{\left(C_L' a + k_0\right) \pm \sqrt{\left(C_L' a + k_0\right)^2 - 4C_L' a\phi_o \left(\frac{1}{2} C_L' a\phi_o + k_i - 4/\sigma\right)}}{2\left(\frac{1}{2} C_L' a\phi_o + k_i - 4/\sigma\right)}
\]

To recap,

\[
C_L' a = \frac{dC_L}{d\alpha} \quad \phi_0 = \alpha_0 + \beta \quad \sigma = \frac{nc}{2\pi xR}
\]

\[
k_d = \frac{1}{2} \frac{d^2 C_p}{d\alpha^2} \quad k_o = C_{d_{\text{min}}} + k_d \beta^2 \quad k_i = 2k_d \beta
\]

If we now equate the momentum and blade element expressions for torque, and similarly divide through by the common terms, we can obtain an expressions for \(a'\) and \(\lambda_i\) in terms of \(\phi\).

\[
\begin{aligned}
(1 + \phi^2).\left(C_L' a\phi(\phi - \phi_o) + (1 - \frac{1}{2} \phi^2)(k_o - k_i \phi + k_d \phi^2)\right) \\
= k_o \left(C_L' a\phi_0 + k_i\right)\phi + \left(C_L' a + k_d - \frac{1}{2} k_o\right)\phi^2 \\
\ldots - \left(C_L' a\phi_0 - \frac{1}{2} k_i\right)\phi^3 + \left(C_L' a - \frac{1}{2} k_d\right)\phi^4
\end{aligned}
\]

\[
\frac{-8\pi xR a'}{nc(1 + a')^2 \lambda_i} = -\frac{a'}{(1 + a')} \frac{4\phi}{\sigma} \quad \left(= \frac{1}{\frac{1}{2}(1 + a')^2 \rho n c R^4 \Omega^2 x^2} \frac{dQ}{dx}\right)
\]

\[
\frac{a'}{(1 + a')} = d = -\sigma/4 \left[k_o/\phi - \left(C_L' a\phi_o + k_i\right) + \left(C_L' a + k_d - \frac{1}{2} k_o\right)\phi + \ldots\right]
\]

\[
\lambda_i = \frac{x R \Omega}{\phi(1 + a')} \quad \quad a' = \frac{d}{1 - d}
\]
The three values for $\phi a'$ and $\lambda_i$ can then be substituted back into the momentum equations to obtain values for $dT$ and $dQ$. Crude estimates of total thrust $T$ torque $Q$ and power $P$ could be obtained by applying just one set of values of the flow constants to the whole disc but it is better to obtain $T$ and $Q$ by integrating $dT$ and $dQ$ respectively over the whole disc. $P$ is obtained by multiplying the torque by the rotational speed.

$$T = \int_h^1 \frac{dT}{dx} \, dx, \quad Q = \int_h^1 \frac{dQ}{dx} \, dx, \quad P = Q \Omega.$$