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DYNAMIC ANALYSIS OF RAILWAY VEHICLE/TRACK INTERACTION FORCES

by

GEOFFREY A. HUNT

A Doctoral Thesis

Submitted in partial fulfilment of the requirements for the award of

Doctor of Philosophy of the Loughborough University of Technology
June 1986

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ABSTRACT

Methods of predicting the dynamic forces are developed for the cases of vehicles negotiating vertical and lateral track irregularities.

The bounds of validity of various models of the track are evaluated, from single degree of freedom, lumped parameter models to the case of a two layered beam on elastic foundation with a moving dynamic load. For the case of the lateral response of a vehicle negotiating a track switch, a finite element model of the track is also developed.

The vehicle model developed for the vertical case contains all the rigid body modes of a four axle vehicle for which primary and secondary suspension can be included with viscous or friction suspension damping. Solution of the vehicle/track interaction problem for these non-linear models is obtained by numerical integration, vehicle and track being connected by the non-linear wheel/rail contact stiffness. The most significant forces are shown to arise from the interaction of the unsprung mass and track resilience, with the vehicle modes also making a significant contribution, particularly in friction damped cases.

For the lateral case use is made of an existing model of transient vehicle behaviour containing the wheel/rail contact non-linearities, to which track resilience is added in order to predict the track forces. The model is used to predict the forces which would be anticipated at discrete lateral irregularities such as those to be found at track switches. Once again the interaction with the track introduces modes of vibration which are significant in terms of wheel/rail forces.

Comparison is made with experimental results obtained from full scale tests in the field. In one experiment the vertical track forces due to a range of vehicles negotiating a series of dipped welds in the track were measured, and in a second the lateral forces were recorded at the site of an artificially introduced lateral kink.

A particular application of the results is in the prediction of the rate of deterioration of vertical and lateral geometry due to dynamic forces. This is to offer an improved understanding of the deterioration mechanism in order to influence the future design of vehicles and track to reduce maintenance costs.
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Finally I would like to acknowledge the patience of my wife, Paula who provided much support and endured many lonely evenings without complaint.
NOTATION

\(a\) = Semi-wheel spacing on bogie
\(a_0\) = Semi-wheel spacing of two axle vehicle
\(b\) = Semi-bogie spacing
\(c_b\) = Distributed lateral ballast damping/unit length
\(c_f\) = Distributed vertical foundation damping/unit length
\(C_f\) = lumped vertical track damping
\(c_w\) = Lateral damping from rail web bending/unit length
\(c_p\) = Primary vertical viscous damping rate on vehicle
\(c_s\) = Secondary vertical viscous damping rate on vehicle
\([C]_l\) = Lumped vertical track damping matrix
\(EI\) = Vertical bending rigidity of track
\(EI_h\) = Lateral bending rigidity of rail head
\(EI_l\) = Lateral bending rigidity of track
\(EI_i\) = Lateral bending rigidity of track in span \(i\)
\(F_{vr_i}\) = Lateral contact forces on left/right hand wheel of wheelset \(i\)
\(F_0\) = Amplitude of lateral harmonic track force
\(F_p\) = Primary friction damping level
\(F_s\) = Secondary friction damping level
\(g\) = Acceleration due to gravity
\(G\) = Shear modulus of rail steel
\(G(\lambda)\) = Lateral track irregularity function in wavelength domain
\(G_H\) = Hertzian contact flexibility coefficient
\(h_b\) = Height of vehicle body centre of gravity above rails
\(H_t(\omega)\) = Track receptance
\(H_u(\omega)\) = Unsprung mass on contact spring receptance
\(I\) = Rail moment of inertia about \(xx\)
\(I(t)\) = Impulse response function for track
\(I_b\) = Body yaw inertia
\(I_f\) = Bogie frame pitch inertia
\(I_w\) = Wheelset roll/yaw inertia
\(J_b\) = Body roll inertia
\(J_w\) = Wheelset polar inertia
\(k_b\) = Distributed lateral ballast stiffness/unit length
\(K_b\) = Body pitch inertia
\(k_f\) = Distributed vertical foundation stiffness/unit length
\(K_f\) = Lumped track stiffness
\([K]_l\) = Lumped vertical track stiffness matrix
\( k_w \) = Lateral stiffness due to rail web bending/unit length
\( k_p \) = Primary vertical suspension stiffness of vehicle
\( k_s \) = Secondary vertical suspension stiffness of vehicle
\( l \) = Sleeper spacing
\( l_0 \) = Semi-gauge of rails
\( m_b \) = Vehicle body mass
\( m_f \) = Bogie frame mass
\([M]_l \) = Lumped vertical track mass matrix
\( m_h \) = Rail head mass/unit length
\( m_r \) = Rail mass/unit length
\( m_s \) = Sleeper mass/unit length
\( m_t \) = Track mass/unit length
\( M_t \) = Lumped vertical track mass
\( M_w \) = Wheelset mass
\( p_i(s) \) = Fourier transform of \( y_i(x) \)
\( P \) = Vector of vertical wheel/rail contact forces
\( P_b \) = Sleeper/ballast force
\( P_c \) = Vertical wheel/rail contact force
\( P_{uti} \) = Vertical wheel/rail contact force at wheelset \( i \) of leading/trailing bogie
\( P_0 \) = Amplitude of vertical harmonic track force
\( q_i(s) \) = Fourier transform of \( z_i(x) \)
\( Q_i \) = Yaw suspension torque on wheelset \( i \)
\( Q_p \) = Primary vehicle damper force
\( Q_s \) = Secondary vehicle damper force
\( r \) = Longitudinal distance relative to moving coordinate system
\( R_i \) = Local radius of lateral track curvature at wheelset \( i \)
\( s \) = Fourier transform variable
\( S_i \) = Lateral suspension force on wheelset \( i \)
\( s_i(x) \) = Track irregularity function
\( S(\lambda) \) = Vertical track irregularity function in wavelength domain
\( S(\omega) \) = Vertical track irregularity function in frequency domain
\( t \) = Time
\( T_{1,uri} \) = Longitudinal wheel/rail force on left/right hand wheel of wheelset \( i \)
\( T_{2,uri} \) = Lateral wheel/rail force on left/right hand wheel of wheelset \( i \) (in plane of contact)
\( T_{3,uri} \) = Wheel/rail contact force on left/right hand wheel of wheelset \( i \) (normal to plane of contact)
\( U_{bi} \) = Roll suspension torque on wheelset \( i \)
\( v \) = Vehicle velocity
\( V_i \) = Vertical suspension force on wheelset \( i \)
\( x \) = Longitudinal distance
\( y_i \) = Lateral displacement of wheelset \( i \)
\( y_1 \) = Lateral displacement of rail head
\( y_2 \) = Lateral displacement of sleepers
\( z_b \) = Vehicle body vertical displacement
\( z_f \) = Bogie frame vertical displacement
\( z_l \) = Vertical track irregularity profile
\( z_w \) = Wheelset vertical displacement
\( z_0 \) = Vertical track displacement at loaded point
\( z_1 \) = Vertical displacement of rail head
\( z_2 \) = Vertical displacement of sleepers
\( Z \) = Vector of vertical vehicle displacements
\( Z_r \) = Vector of vertical vehicle displacements relative to the undisturbed track profile
\( Z_t \) = Vector of track irregularity profiles at each wheelset
\( \beta \) = Characteristic wavelength of beam an elastic foundation
\( Y_{1l/ri} \) = Longitudinal creepage on left/right hand wheel of wheelset \( i \)
\( Y_{2l/ri} \) = Lateral creepage on left/right hand wheel of wheelset \( i \)
\( \delta_{l/ri} \) = Contact angle on left/right hand wheel of wheelset \( i \)
\( \Delta r \) = Rolling radius diffence
\( \Delta t \) = Time step
\( \dot{\psi} \) = Pitch angle relative to track irregularity co-ordinates
\( \eta \) = Vertical vehicle displacements relative to track co-ordinates
\( \lambda \) = Wavelength
\( \theta_b \) = Vehicle body roll angle
\( \mu \) = Coefficient of friction
\( \rho \) = Mass density of rail steel
\( \phi_b \) = Body pitch angle
\( \phi_f \) = Bogie frame pitch angle
\( \phi_i \) = Wheelset \( i \) rotational velocity
\( \psi_b \) = Body yaw angle
\( \psi_{ci} \) = Wheelset yaw angle relative to track
\( \psi_i \) = Yaw displacement at wheelset \( i \)
\( \omega \) = Angular frequency
\( \omega_{3l/ri} \) = Spin creepage on left/right hand wheel of wheelset \( i \)
1. REVIEW OF PREVIOUS WORK

1.1 Introduction: Dynamic Forces on the Track

The primary purpose of a railway system is to transport passengers and freight. This should be achieved both comfortably in the case of passengers and without damage to the goods in the case of freight. It is also of prime importance that the railway vehicles are operated safely without risk of derailment. All of these features should be achieved economically so that the railways can compete with other modes of transport.

For these reasons much work has been carried out throughout the world on the design of railway vehicle suspensions. Many of the fundamental concepts of railway vehicle suspension design are the same as those for other land vehicles such as automobiles, there are, however, some important exceptions, these being:

i) Railway vehicles feature a solid steel wheel running on a steel rail resulting in a significant part of the vehicle which cannot be isolated by a suspension. This is the 'unsprung mass'.

ii) A large number of railway vehicles have bogies which carry typically two, but up to four axles which can result in two levels of suspension, the primary, between wheel and bogie frame, and the secondary, between bogie frame and body. From a vehicle ride point of view the bogie provides an effective filter to the vehicle track roughness.

iii) Railway vehicle axles are 'rigid', meaning that no independent rotation of the wheels can take place, and the wheels themselves are coned. This gives the wheelset steering ability on curves but also results in a potential instability on straight track known as hunting. This important feature has resulted in a very large amount of work being performed which is unique to railway vehicles.

Thus as railway vehicle speeds have increased, much work has been needed to improve vehicle suspensions in order to maintain and improve passenger ride,
at the same time trying to minimise the capital and maintenance costs of the vehicles.

The main source of input forces to the vehicle of course is the track. Ideally this would be a flat level surface having straight parallel rails but clearly this cannot be achieved in practise. Dynamic forces therefore result between wheel and rail both vertically and laterally. The forces which occur have a number of undesirable effects, namely loss of track geometry as a result of settlement, damage to track components due to overstressing and fatigue and, particularly at the higher frequencies, ground borne vibration and noise.

To combat these problems regular track maintenance is required to maintain track quality and safety margins. Geometry is maintained by tamping and lining machinery which levels and lines the track and packs extra ballast underneath the sleepers. Damaged track components are usually replaced on a spot basis if the numbers are not too great. Some of the typical failures which result from, or are worsened due to dynamic loads are listed below:

i) Rails: Failure of welds due to fatigue (usually site made welds) and failure of the rails themselves as a result of growing fatigue cracks. The former are usually the result of some form of defect in the rail weld which precipitates a fatigue crack growth, whereas the latter usually start at the rail surface, being caused by unfavourable contact stresses. These flaws are generally detected before failure, however, by ultrasonic or by eddy current methods.

ii) Sleepers: Failure of sleepers due to cracking of the concrete or loosening of the rail fastening shoulders. The sleeper can also become badly worn due to abrasion by the ballast.

iii) Ballast: Some direct damage due to loading can occur where the dynamic forces are large, but damage is also caused by mechanised maintenance as a result of deterioration of geometry and by general weathering. The main mode of 'failure' is due to clogging of the voids by fine particles resulting in a drainage problem.

At a certain point in the life of the track any one of the above problems or a combination of such problems becomes sufficiently bad to precipitate complete
renewal of a length of track, at which point all the components are renewed and some of the serviceable ones are used on minor lines.

It is clear therefore that when considering vehicle/track interaction problems there is another important area to be considered apart from the vehicle ride and that is the track forces and the track damage that they cause. Furthermore a significant proportion of the track forces appear as a result of dynamic vehicle/track interaction, particularly due to irregularities in the running surface, and these are found in practice to be of comparable magnitude to the static forces. An improved knowledge of the forces on the track would, therefore, have many benefits including an improved knowledge of the level of forces to be used for component design and an ability to define required track and vehicle standards. Furthermore if the track forces could be related to the amount of track damage this would enable the costs of changes in the traffic to be more accurately calculated and would also enable the benefits of advanced vehicle suspensions such as active suspensions to be evaluated.

There are significant differences between the mechanisms of vehicle/track interaction vertically and laterally. For the vertical case forces arise purely due to irregularities in either the vehicle or the track. The system is essentially linear apart from the mechanism of wheel/rail contact, where local deformations of the wheel and rail result in a degree of resilience having the characteristics of a spring which is stiffening with load. There is a negligible effect of the lateral vibration upon the vertical.

In the case of lateral vibration there is no rigid contact between wheel and rail. Small lateral deviations from the equilibrium rolling position on the track cause a change in the rolling radii on the two wheels due to the coning angle. The effect of this is to cause a steering action of the wheelset back towards the equilibrium position. Furthermore longitudinal and lateral forces on the individual wheels of the wheelset cause small deformations to occur within the wheel/rail contact patch and 'creepages' occur. The rolling wheelset therefore does not experience a sudden change from pure rolling to pure sliding but a gradual change can occur, resulting in a mismatch between the wheel rotational velocity and the vehicle forward velocity. Creepage is defined as the difference in velocity divided by the forward velocity and can be defined in the longitudinal, lateral and spin situations as described later. For small oscillations the creep force/creepage relationship is normally considered to be linear. For larger amplitude oscillations, however, this is not so and also flange
contact can be reached which results in a rapid change in the angle of plane of contact meaning that a component of the vertical force also acts laterally.

The irregularities in the track which cause the dynamic forces come from a variety of sources and also cover a wide spectrum of wavelengths. All frequencies of the dynamic system are therefore excited to varying degrees. The main zone of interest to the vehicle designer is at frequencies less than 10 Hz. It is in this frequency range that the vehicle rigid body modes are found and vibration of the vehicle body occurs thus affecting passenger comfort. At frequencies higher than this and up to 100 Hz resonance of the wheel/track system can occur and as a result much of the spectral component of the track force appears in this range. Many of the vehicle component bending modes also occur in this region, however they have negligible effect on the track forces as a result of being well damped and isolated from the wheel/rail contact zone. The first bending mode of the body is often a consideration for passenger comfort though, being typically around 10 Hz.

At frequencies greater than 100 Hz most of the force is due to motion of the wheelset and track with some of the higher modes of track vibration becoming important as well as the non-linear contact stiffness between wheel and rail. Short wavelength irregularities such as corrugations (which are described below) can result in frequencies approaching 1000 Hz which can also create a severe noise problem. It is only recently that the effect of some of these higher frequency forces on the track have been studied.

The sources of track irregularities can be summarised as follows:

i) Joints and welds: The fishplated rail joint has long been considered one of the main geometric faults in the rail surface. As a result of repeated loading and unloading the joint begins to wear, this begins a cycle of increasing dynamic load, differential ballast settlement at the joint and more joint wear. A large dip therefore results in the track with an angular discontinuity at the joint.

The introduction of continuously welded rail has resulted in an improvement of the geometry where rails are joined together, however welds still represent a significant source of geometric error, particularly where a poor weld has been made or the top surface has been badly ground when finishing off the weld. Welds
and joints also present discrete irregularities laterally if poorly aligned.

ii) Random roughness: This in itself can come from a variety of sources, at the shorter wavelengths irregularities in the shape of the rail are important. These obviously come from the manufacturing process and from damage in handling. At the longer wavelengths the track vertical profile is dominated more by the underlying ballast profile. This is difficult to modify permanently by mechanical means because the ballast tends to settle at a greater rate the more it is lifted when tamping. For this reason the periodic shape of jointed track can reappear in track which is re-railed with cwr (continuously welded rail) even if the welds do not coincide with old joint positions.

iii) Variations in track parameters: Large dynamic forces can occur at changes in track support conditions, for instance where a single sleeper is poorly supported or where a change occurs in the underlying foundation conditions. These generally apply more to the vertical case, however a situation where this applies laterally is given below.

iv) Track Structures: Bridges represent a significant transition when negotiated by rail vehicles. However this is a large subject in itself and is not covered within the scope of this report. Another important structure, however, is the track switch or turnout. This provides both an unavoidable discrete irregularity where the switch rail joins the through straight rail and also significant changes in lateral stiffness through the switch due to the varying construction. This change of stiffness also applies to a certain extent vertically.

The point where the two rails of the turnout cross obliquely is called the crossing. A gap is obviously necessary at this point to allow the wheel flanges to pass through, which can result in large vertical and lateral forces.

All of the above cases represent inputs to the track which cause dynamic forces. It is known that dynamic forces cause progressive deterioration of the
geometry of the track and therefore it is anticipated that the irregularities described could be made to grow by the forces they produce. This is certainly true for the dipped rail joint case.

Aims of the Work: There are clearly a significant number of benefits to be obtained from a thorough knowledge of the forces which result due to railway vehicles travelling along the track. Indeed a certain amount of work has been done by various administrations as is described in the following section. The majority of these pieces of work, however, split into two main areas, on the one hand the analysis of vehicle based forces derived by making simple assumptions about the response of the track (often that it is rigid) and on the other hand the analysis of wheel/rail forces with more sophisticated track models and simple vehicle models (generally an unsprung mass).

The purpose of this work is to establish under what conditions such simplifications are valid and if there are additional benefits to be obtained from refining the total vehicle/track models. The particular area of interest for the work being the forces causing track damage.

The first area of consideration is therefore the track, where it is essential to establish which are the important components to be represented in models, and also to investigate the importance of the moving load effects on the structure.

Vehicle models are then considered which are incorporated into models of the total vehicle/track system and the importance of the vehicle modes of vibration upon the wheel/rail forces evaluated, both vertically and laterally. Certain non-linearities of the vehicle suspension are also considered such as friction damping and local wheel/rail contact effects vertically, and non-linear wheel/rail contact geometry conditions laterally. The results of the models are then analysed and compared where possible with the results of experimental validation work.

As a final example the results from the track force modelling are used to consider a very important area of track damage, namely deterioration of track geometry. Predicted dynamic forces, along with empirical relationships describing the effect of repeated loading upon permanent movements of track both vertically and laterally, are used to make estimates of the deterioration in the track geometric quality with traffic. Such work has potential benefits in the
design of future vehicles and track to minimise track maintenance costs, and allow economic evaluations of traffic policy decisions to be made, for example when considering the cost benefits of introducing higher speeds, or freight vehicles with higher axle loads.

1.2 Vertical Vehicle/Track Interaction

Much of the available work concerning railway vehicle and track dynamics has been performed in Japan, the USA and Europe, notably Britain, France and Germany. All of these countries have modern railways but the conditions under which they operate are in some cases quite different. For example there is a strong difference between the long distance, high tonnage freight routes of North America and the high speed routes of Japan and France. Many of the engineering problems faced by the various railways however have been similar and a great deal of interchange of theories and method has contributed to the evolution of current railway practises.

One of the initial problems of note to be approached was that of the moving vertical load on the railway track. In 1972 Fryba (1) studied this problem in some detail which was not related specifically to railway track problems but considered the effect of moving loads on a variety of structures. The particular case of relevance to the railway track situation was the moving load on a beam on Winkler elastic foundation which had also been studied previously by Kenney (2) and Mathews (3,4).

The main feature of this problem was that for the case of a quasi-static moving load, a critical speed was discovered at which (in the absence of damping) the displacement beneath the load approached infinity. The critical speed was defined by:-

\[ v_c = \left( \frac{4EIk}{m^2} \right)^{0.25} \]

where \( EI \) is the bending rigidity of the beam, \( k \) the distributed support stiffness and \( m \) the mass per unit length of the beam. This critical speed is shown to be comfortably in excess of present railway operating speeds although Fortin (5) has reported very large deformations beneath the wheels of a train suspected to have reached a speed close to the critical speed at a site on soft peat.
Sauvage (6) also considered this problem but once again with a moving quasi-static load. Mathews, Fryba and Fortin all presented solutions to the case of a moving harmonic load in the absence of damping and Fryba also considered moving mass effects. Due to the complexity of the equations, however, the general problem of the moving harmonic load with damping was not solved, although Stadler and Shreeves (7) did present a transient solution based upon obtaining the impulse response for the beam on Winkler foundation.

For the vehicle dynamicist, particularly if the main criterion is passenger ride, the effect of track resilience is not considered to be too important and the vehicle can be considered to be negotiating the unloaded or in some cases the statically loaded profile. This is probably quite reasonable as the frequency range of interest is generally lower than the frequency at which the track stiffness has an effect. When considering wheel/rail forces, however, it is most important to include the track resilience.

Beer et al (8) included the track resilience in a simple form when modelling the forces at a dipped rail joint, the results being published in 1974. The track was treated as a parallel spring/damper system isolated at each wheelset of a bogie vehicle, the mass of the track was neglected as was the contact flexibility between wheel and rail. The body mass of the vehicle was also considered to be 'ground', connected to the bogie by a spring and viscous damper. The primary suspension, however, was modelled as a spring and friction damper combination.

The shape of the track at the dipped rail joint was taken to be a symmetrical quarter cosine shape having a depth of 10 mm and an angular discontinuity of 10 mrad. Solution of the system equations was by an analogue computer approach. The results showed an oscillatory wheel/rail force history from the impact at the joint, the maximum of which was shown to rise linearly with speed. A reduction in the assumed track stiffness also resulted in a reduction in impact force. It was also concluded that a reduction in the wheel radial stiffness would also have a beneficial effect on track forces.

A second paper covering a similar subject which appeared around the same time was by Jenkins et al (9) in which the methods due to Lyon (10) were used to predict the wheel/rail vertical forces due to a variety of track and vehicle defects, of which one was the dipped rail joint. In this case the track was modelled as an
infinite beam on a damped Winkler elastic foundation, the load being considered to be stationary. The vehicle model was considered to consist simply of a rigid wheelset with the non-linear Hertzian (11) spring contact model between the wheel and rail. In this model the deflection (and contact stress distribution) which occurs between two elastic bodies in forced contact can be calculated. For the railway wheel on rail situation this results in a relationship of the form:-

\[ y = G_H P \]

where \( y \) is the relative displacement of wheel and rail
\( P \) is the contact force
\( G_H \) is the Hertzian flexibility coefficient, which can be calculated from the formulae for elastic contact when the elastic properties and the principle radii of curvature in two planes at an angle are known.

The wheel/rail contact spring, therefore, has the characteristics of a stiffening spring, having a tangent stiffness at typical axle loads an order of magnitude higher than the effective track stiffness.

Lyon used a Laplace transformation method to solve the equations of the beam on Winkler foundation from which the impulse response function was derived. This was then used with a time stepping integration solution of the lumped parameter vehicle equations and the convolution integral to incorporate the track and non-linear contact effects. The vehicle model used in the analyses presented was basically an unsprung mass with bogie and body motions neglected. Refinements were made to this, however, to incorporate traction motor connections and wheel radial flexibility (for resilient wheels) when studying locomotives.

The dipped rail joint was studied as the major track irregularity. The wheel rail force histories predicted were shown to be oscillatory in nature as in the work of Beer et al (8) with the major difference that two characteristic frequencies dominated the response as shown in Fig 1.1 due to the inclusion of an additional degree of freedom. This resulted in two peaks to the decaying response of each frequency which were termed \( P_1 \) for the higher frequency peak and \( P_2 \) for the lower. The \( P_1 \) force was found to be associated with vibration of predominantly the track on the Hertzian contact spring and caused a peak force some 0.25 - 0.5 msec after crossing the joint. The \( P_2 \) force was found to be the
response of the wheelset on the effective track resilience, occurring much later at typically 6 - 8 msec. These forces and their positions on the track were associated with unfavourable alternating stress conditions around the circumference of bolt holes in the rails at the joint, which were prone in practice to failure by fatigue.

**TOTAL WHEEL/RAIL FORCE**
**STATIC RAIL FORCE = 86200.0 N.**

![Graph showing wheel/rail contact force at dipped rail joint](image)

**Fig. 1.1** Wheel/Rail Contact Force at Dipped Rail Joint

A study of the various vehicle and track parameters on the forces at dipped joints was made. The conclusions from this were that for constant track and vehicle parameters the peak forces (both $P_1$ and $P_2$) were approximately proportional to the product of the dip angle and speed and that the total dip depth (for the simple vehicle model) was unimportant. Approximate formulae for the $P_1$ and $P_2$ forces were also presented which would enable simple
comparisons of vehicle types to be made. These formulae were derived by finding equivalent lumped mass systems to define the response and resulted in the following equations:

\[ P_1 = P_0 + 2av \left( \frac{k_H m_e}{1 + m_e/m_u} \right)^{1/2} \]

\[ P_2 = P_0 + 2av \left( \frac{m_u}{m_u + m_t} \right)^{1/2} \left( 1 - \frac{c_t n}{4(k_t (m_u + m_t))^{1/2}} \right) (k_t m_u)^{1/2} \]

where

- $2\alpha$ is the total dip angle
- $v$ is the vehicle speed
- $k_H$ is a chord stiffness to the Hertzian contact stiffness
- $m_u$ is the vehicle unsprung mass
- $k_t, c_t, m_t$ are the effective lumped track stiffness, damping and mass.

The latter can be calculated from the distributed values as follows:

\[ k_t = \frac{2k}{\beta} \]

\[ m_t = \frac{3m}{2\beta} \]

\[ c_t = \frac{3c}{2\beta} \]

Where \( \beta = \left( \frac{k}{4EI} \right)^{0.25} \)

$m_e$ is the effective track mass for the $P_1$ force calculation which is calculated from:

\[ m_e = m \left( \Gamma \left( \frac{3}{4} \right) \Gamma \left( \frac{5}{4} \right) \sqrt{\frac{2}{4}} \right)^{4/3} \left( \frac{4EI}{k_h} \right)^{1/3} \]

These formulae are now fairly widely used within BR for quick assessment of track forces, $m_e$ being typically equivalent to the mass of a 0.4m length of track.
The paper also considered the track forces due to random track surface irregularities and also the likely forces due to flat spots on wheels which result due to heavy braking. Extensive experimental work based upon dipped rail joints was also reported and compared with the predictions made by the theory, for which good agreement was obtained. The method was also used to study the effect of fitting resilient wheels to locomotives having a high unsprung mass, for which the theoretical and experimental results both showed useful benefits. This latter subject was also covered in some detail by Bjork (12).

In 1975 the problem of a rail vehicle negotiating a rail joint was also studied by Ahlbeck et al (13). The track was modelled as a lumped spring, damper and mass at each wheelset. The stiffness of the spring being derived from a static calculation of the deflection of a beam on Winkler foundation model into which the stiffness of the rail pad, wooden sleeper, ballast and subgrade were included. A formula was also offered to calculate the effective ballast and subgrade stiffness from the moduli of the two materials viz:-

\[ k_{\text{ballast}} = \frac{C(l-w)E_b}{\log \left[ \frac{1}{w(l+C.h)} \right]} \]

where \( E_b \) = Young's modulus for ballast
\( l \) = length of loading area
\( w \) = width of loading area
\( h \) = ballast depth
\( C = 2 \tan \alpha \)
\( \alpha \) = angle of internal friction (20° assumed for ballast)

For the subgrade:

\[ k_{\text{subgrade}} = k_0(l+C.h)(w+C.h) \]

where \( k_0 \) = soil modulus

The ballast subgrade stiffness can then be simply calculated:

\[ \frac{1}{k_{\text{ballast subgrade}}} = \frac{1}{k_{\text{ballast}}} + \frac{1}{k_{\text{subgrade}}} \]
This calculation was based upon the assumption that each sleeper was supported on a pyramid of material. Because of the overlap due to adjacent pyramids one half of this value was used in the beam on Winkler foundation calculation. The relationship between displacement and distance from the load for the track model was also used to include the effect of interaction of a nearby wheel of the same bogie of a vehicle. The effective rail mass to be included in the model was calculated by considering the rail and pad system to be acting as a beam on Winkler foundation on its own and calculating the parameters to give an equivalent lumped system when vibrating in the first mode.

The vehicle model used for comparison with experimental data was a half bogie vehicle having a lumped half body mass over a bogie with two wheelsets (no roll degrees of freedom were included). This was solved on an analogue computer enabling certain non-linearities to be included such as friction damping in the vehicles and non-linear stiffness of the ballast. The conclusions made from the comparisons were that a non-linear ballast stiffness was needed in order to predict both the quasi-static and dynamic responses and that the damping of the track was high, between 50 and 100 percent of critical damping based upon the natural track frequency. The authors also concluded that the lumped parameter modelling was inadequate for considering the higher frequencies of the response of the vehicle/track system.

A very simple model of an unsprung mass on a lumped parallel spring/damper system was used by Sato and Satoh (14) to investigate the effects on vertical track forces of running trains at higher speeds, this work being also published in 1975. The model was used to predict the likely increase in wheel/rail forces which would result from an increase in train speed on the Japanese SHINKANSEN railway network. The paper concluded that an increase in speed from 210 km/hr to 260 km/hr would require that a decrease in track stiffness of 20-30% or a reduction in the track irregularities of 10% be made in order to maintain the forces at the previous levels. Improvement in these two parameters had been achieved in the field by grinding the rail surface on particularly poor sites and by the introduction of resilient ballast mats beneath the sleepers. Test with this arrangement showed that a significant reduction in track forces was obtained in line with the theoretical predictions.

A model said to be capable of predicting the track response at higher frequencies was also presented later by Sato (15). This was offered primarily as a means of predicting the sources of noise on the Japanese National Railroads
but consideration was also given to ballast forces. The track model used for this case was a double layer beam on elastic foundation in which the two layers represented the pad and ballast layers. For the case of slab track the lower layer was also capable of having bending rigidity. For the prediction of the high frequency response, only the wheelset of the vehicle was considered. The paper once again considered the beneficial effects of using ballast mats, which were shown to result in a reduction in ballast forces at all frequencies of loading on slab track. Good agreement with experimental results was also obtained at frequencies of up to 2000 Hz. For this the rail pad stiffness was taken to have increasing stiffness and reducing damping characteristics with frequency.

Newton and Clark (16) studied two possible track models for the prediction of forces due to wheelflats on railway vehicles in 1979. The first model considered was an infinite single layer beam on Winkler foundation having no provision for pad flexibility effects. This was compared against a two layer model including the pad layer and also including the symmetric bending modes of the sleeper. The latter model, although including the discrete supports at sleepers was also restricted to a finite length which was chosen such that reflections from the ends were damped out before returning. The vehicle model used was once again a single wheelset but with two additional spring/mass systems in series, representing the bogie mass, body mass and primary and secondary stiffness and damping.

The beam on Winkler foundation model was solved by the method due to Lyon (10) while the discrete support model was solved by modal analysis methods using the techniques of Wittrick and Williams (17) to assemble the normal modes of the complete structure from those of the component parts. The results of both models were compared against experimentally measured results of sleeper loads and rail strains due to wheelflat impacts. The authors found that good agreement was obtained from both models for lower speed impact, however at higher speeds, higher frequencies were apparent in the response and the beam on Winkler foundation model tended to over predict the force levels. The reason for this was found to be due to the lack of rail pad flexibility in the model and the authors suggested that a two layer beam on Winkler foundation model would probably give better results.

Grassie and Cox (18) also considered a two layer beam on Winkler foundation model more recently in which the lower layer had bending rigidity in the vertical direction thus including the modes of sleeper vibration. The discrete
support at each sleeper was not included on the basis that this was only of importance in the vicinity of the 'pinned-pinned' resonance of the rail at around 750 Hz. A uniformly supported infinite model could therefore be handled, and solved for a stationary harmonic force by means of a Laplace transform solution for both the rails and sleepers. The vehicle model was once again a single wheelset.

The model was used to compare results obtained from experimental measurements of sleeper displacement and sleeper bending strains obtained by Dean (19) for a rail with artificially introduced 'corrugations' in the rail surface, which was negotiated at a variety of speeds. The wavelength of the corrugations of 60 mm resulted in a variety of forcing frequencies up to 750 Hz. The agreement obtained with the experimental work was reasonable at the lower frequencies of excitation but less so at the higher frequencies. This was put down to the loss of contact between wheel and rail which was known to be occurring at the higher speeds but which was not accommodated in the model.

The effect of the sleeper bending modes was shown to be important above the first natural frequency of the sleeper at around 200 Hz when compared to a model in which the sleepers were assumed to be rigid. The effects of altering pad stiffness and sleeper depth were also investigated as a means of reducing sleeper bending strains on corrugated track. It was found that the most beneficial results were obtained by reducing the pad stiffness, and the conclusion was drawn that the pad was the critical element in the isolation of the sleeper from high frequency dynamic loads. It was suggested that further benefits would also be obtained if damping of sleeper bending modes could be increased.

Much of the work which has been carried out on the vertical vehicle track interaction problem, therefore, seems to fall into two main categories:-

i) The study of complex models of the vehicle with simple models of the track (typically lumped spring and damper and sometimes including lumped track mass) which are thought to be valid at the lower frequencies of interest. At what limit of frequency they are no longer valid, however is not clearly defined.

ii) The study of simple models of the vehicle (typically rigid wheelsets with a wheel/rail contact stiffness) with complex models of the track which are thought to be valid at the higher frequencies.
1.3 Lateral Vehicle/Track Interaction

When considering the lateral vehicle/track interaction problem the mechanism of contact between wheel and rail is quite different as described earlier. The main features of this mechanism can be summarised once more by considering what takes place when a rolling wheelset is displaced laterally:

i) Changes in the angle of contact occur, particularly on the outer wheel when flange contact is approached. This causes a change in the lateral component of vertical contact force.

ii) The rolling radii of the two wheels change, resulting in a 'steering' effect and the production of longitudinal and lateral creepage forces. These increase gradually with creepage up to the gross sliding condition. Due to the inclination of the contact patch a spin creepage also occurs which results in a component of lateral force.

iii) As the wheel moves laterally across the rail head, changes in the contact patch size and shape occur which alter the way in which the creep forces are generated. Under particular conditions of wheel or rail wear contact can occur at two separate locations.

The non-linear effects which result from these particular features are summarised in Fig 1.2.

The curving ability of railway vehicles has been studied as early as 1835 (20) when the concept of rolling radius difference and the advantages of radial alignment in curves were first addressed. It was not however until 1935 that Porter, (21) and closer to the present day, Newland (22) and Boocock (23) began to develop more practical theories. In these, yaw flexibility of a wheelset was allowed but restrained by a linear spring, thus the wheelset was able to partially adopt a radial position in curves without necessarily involving flange contact. The creep force/creepage relationships and the wheel/rail contact geometry conditions were linearised, however, thus restricting the predictions to small displacements and in practise limiting the results to large radius curves only. Porter's work was restricted to the gross sliding case only and therefore only valid for very sharp curves.
Fig. 1.2 Lateral Wheel/Rail Contact Non-Linearities

- Stationary (dry friction)
- Moving with Longitudinal Creepage
- Moving without Longitudinal Creepage
Work by Kalker, (24) initially in 1967, concerning the rolling contact of elastic bodies enabled the creep force/creepage relationships to be calculated for a large range of values of lateral, longitudinal and spin creepage right up to the gross sliding situation. The complex nature of these calculations meant that these were computationally intensive to implement in their exact form and were therefore most easily used in simplified or in tabular form from which values could be interpolated.

Elkins and Gostling (25) used the tabulated creepage data along with tabulated data from measured wheel/rail contact conditions to produce a more general steady state curving theory. The non-linear equations for the vehicle/track system were solved iteratively and no flexibility of the track was included as this was considered to be unnecessary for the steady state situation. A version of the solution, however, was prepared which included the velocity dependent terms in the creepage force expressions. The reason for this was to prepare a model which would be valid for the transient curving situation.

The results of the theoretical predictions were compared with a range of experimental results for which the agreement was found to be very good. Comparison with the results of Porter (21) and Boocock (23) also gave good agreement and was able to show the bounds of validity of the simpler models for which neither small creepage in the case of Porter's work nor large creepage in the case of Boocock's work was applicable. The main conclusions made by the authors were as follows:-

i) The leading outer wheel tends to make flange contact in curves, the forces on the track tend to be gauge spreading on both rails.

ii) The trailing axle rarely makes flange contact except on the inner rail of very sharp curves.

iii) The wheelsets attempt to align themselves radially.

iv) The curving forces are somewhat different in wet and dry weather due to changes in the coefficient of friction between wheel and rail. Lubrication reduces the forces and the wear rate.

v) A nett longitudinal force exists on a wheelset in a curve, thus increasing drag.
It is apparent therefore that when considering lateral wheel/rail forces due to vehicle/track interaction it is important to give due consideration to the creepage forces and to the non-linear contact conditions. Various options present themselves when considering a theoretical approach to the non-linear problem. The most simple, and obviously the most tempting method is to use linearised equations which much simplifies the equations and solution. It is widely accepted, however, that to do this is only valid at small amplitudes of deviation from the track centre line and the worst lateral track forces are likely to occur under flange contact conditions. A second approach used by workers is to use simplified versions of the theory, for example those due to Kalker (26) or Vermuellen and Johnson (27) in which the contact conditions are much simplified. Once again these methods work best in conditions of simple contact. The third option is to use the more exact theory which is complex to implement and time consuming to obtain the solution, or as an alternative to the latter, to produce the solution beforehand in tabular form and use interpolation for a continuous solution.

The majority of the research work which has been carried out on the dynamic response of railway vehicles has been on considerations of lateral stability on straight track. For example the work of Wickens, (28) published in 1965, which used the approximate formulae for creep forces due to Vermuellen and Johnson (27) and the dynamic equations for both a single wheelset and a four wheeled vehicle to predict the stability limits for hunting. The damping available to lateral oscillations is shown to first increase with speed and then fall to eventually reach zero, and subsequently to become negative. The point at which the damping becomes zero is defined as the critical speed of the vehicle and is a function of the lateral and torsional wheelset stiffnesses, the inertia, and the conicity of the wheels.

The above calculations for stability do not require any definition of the track geometry and indeed much of the work has been carried out on the assumption of straight track. Dokainish et al (29) considered the response of a railway vehicle to a sinusoidal irregularity. Simplified non-linear creep forces expressions were used restricting the excitation to a single wavelength of excitation and the compliance of the track was neglected. Cooperrider (30) on the other hand used linear creep force expressions thus enabling the prediction of vehicle response due to measured track geometry for which the power spectral density had been obtained. Due to the restriction to small amplitudes, lateral
and longitudinal creep forces only were obtained and not flange contact forces. The work was, however, able to show the reduction in available tractive effort due to the creep forces caused by lateral oscillation.

A paper in which the track resilience was included was presented by Helms and Strothmann (31) in which the two rails were allowed to move laterally individually, restrained by a spring/viscous damper combination in each case. Once again the creep force functions were linearised for small displacements and a solution obtained in the frequency domain for a prescribed track PSD. The results were compared with experimentally obtained values for lateral acceleration of wheelsets and vehicle body. Some agreement between the theoretical and experimental results was obtained and both results showed that excitation of the wheelset laterally on the track was apparent at a frequency of approximately 16Hz and that this had some effect on the body response.

Solution of the lateral dynamic response of railway vehicles is therefore shown to be quite complex. However it is quite clear that for a consideration of the forces acting on the track it is necessary to consider the situation of flange contact of wheel on rail which is a non-linear problem. The few authors who have approached this particular problem have tended therefore to use numerical simulation techniques, to handle these non-lineararities.

Young and AppaRao (32) described such a model in which the response of a four wheeled vehicle having seventeen degrees of freedom was solved by numerical time stepping integration techniques for the case of a vehicle negotiating a spiral transition curve. A simple spring was used to represent track flexibility under flange contact conditions, and approximate creep force expressions were used. Experimental data for wheelset displacement was available and was compared with the predictions which showed reasonable agreement. Errors in the predictions were attributed to errors in the alignment of the curve which had not been measured but which had been assumed to match the design profile. The response obtained for this situation, however, was essentially quasi-static due, presumably to the relatively gentle nature of the curve negotiated.

The problem of the dynamics of curve entry was addressed also by Cooperrider and Law (33). The approach taken by these authors was to use tabulated data describing the wheel/rail contact non-linearities and simplified expressions for the non-linear creep force/creepage relationships. The rail
flexibility was also included in a simplified manner by modifying the wheel/rail contact data to reflect the relative displacements which would occur between wheel and rail if the rail also moved laterally. Functions were used to describe the track geometry which was that of a transition into a constant radius curve, and the solution of the model which was a half model of a bogie passenger coach, was once again by numerical time stepping integration.

The results showed that for vehicles with stiff yaw suspensions, which generally showed poor curving ability, significant flange impacts could occur on curve entry which resulted in large lateral impact forces. These were oscillatory in nature and typically 1-5 times the value which would have been predicted by steady state curving. Generally these forces decayed rapidly but under certain conditions they could persist throughout the curve without flange contact being maintained. No explanation was given for this phenomenon but the need for experimental validation was emphasised.

The problem of general track irregularities and in particular the geometry at track switches was presented by Clark et al (34). Both the wheel/rail contact non-linearities and the creep force/creepage non-linearities were handled by means of interpolation from tabulated data, the former being from measured wheels and rails. The vehicle model was of a two axled freight vehicle having eleven degrees of freedom. Rail flexibility was also allowed for by incorporating lumped spring and viscous damper elements at each wheel position, and the solution was by numerical time stepping integration to obtain time histories of the required variables. The measured lateral geometry of the track was also used to provide the forcing function, this being interpolated from a series of longitudinal and lateral co-ordinates.

The results were compared with experimental data for the case of a vehicle negotiating a discrete lateral kink in the rail, of the type which would be found at a switch. The irregularity was set in plain track, however, to avoid the complicated lateral track support conditions at a switch. The wheel/rail force histories showed that a large lateral impact of both wheelsets occurred on the rail as the vehicle negotiated the lateral kink. This caused the wheelsets to oscillate laterally on the track at a frequency of approximately 13 Hz, it was noted that at the leading wheelset the vibration was relatively lightly damped while that at the trailing wheelset was heavily damped. This was claimed to be due to the creep forces on the adjacent wheel which caused little damping if the friction
limit was already exceeded. This effect was predicted well by the theory, as was the rest of the response.

The latter vehicle model described, represents the model to be used for the calculation of lateral track forces in this thesis. However for the calculation of the forces at real switches and crossings, and particularly the forces causing permanent lateral ballast displacements a better model of the track is likely to be required.

1.4 Deterioration of Track Geometry

By far the majority of work which has been carried out on this subject has concerned the deterioration of vertical track geometry. The reason for this being that on the majority of track it is the vertical alignment which requires maintenance first and the lateral alignment is corrected at the same time by means of automatic track maintenance machinery. This is not exclusively so, however, and there are some areas of track where the reverse is true, a particular case being the switch, which can require frequent attention to geometry. Furthermore as new maintenance methods are introduced, for example pneumatic ballast injection (35) which offers improvements in the vertical durability of the track but not in the lateral durability, the present situation may not necessarily prevail.

In order to study the effects of deterioration of track geometry it is necessary to be able to define what represents good quality and what represents poor quality track. In 1978 Gilchrist (36) described the system used on British Railways (BR) where the standard deviation of lateral and vertical profiles is used, being first filtered to remove long wavelength effects. Reasonable correlation is shown between this measure and passenger ride as described in the paper. Furthermore an instrumented coach is available which can measure these quantities at speed (37) to determine when maintenance is required.

In 1973 the Office for Research and Experiments (ORE) of the International Union of Railways (38) studied the rate of change of vertical track level as a function of traffic on the railways of some of the various member countries. The researchers found that the characteristics of the mean settlement initially changed rapidly before converging to a rate of settlement
which was linear when plotted against the log of the amount of traffic. A relationship of the following form was proposed:

\[ \sigma_e(T) = b_1 + b_0 \log \left( \frac{T}{2 \times 10^6} \right) \]

where \( \sigma_e \) is the standard deviation of the settlement. The coefficient \( b_0 \) is the settlement after \( 2 \times 10^6 t \) (2 MGT) and \( T \) is the total traffic. General conclusions were also made, notably that reducing the sleeper spacing and increasing the rail vertical moment of inertia were both thought to be beneficial in reducing the rate of deterioration. On repeated loading triaxial test on ballast samples in the laboratory, Shenton (39) showed that a similar law to the one shown above was valid from the first loading cycle, these tests also showed that the frequency of application of the load had negligible effect on this relationship.

The experimental and theoretical work of Selig (40) in the USA also showed essentially the same trend both in triaxial tests and in site trials. The author also used a multi-layered elastic model of the foundation to predict rates of settlement (under uniform loading conditions) based upon the properties of the component layers.

Shenton (41) more recently re-examined all the available data including the effect of sleeper spacing, sleeper type, axle load and tamping machine lift during maintenance to arrive at the following empirical relationship below which is obtained by neglecting the last term of equation 5.1.

\[ S = K_s \left( 0.69 + 0.028L \right) N^{0.2} + 2.7 \times 10^{-6} N \]

where \( S \) is the settlement in mm, \( K_s \) is a sleeper factor (typically 1.1 for BR concrete sleepers), \( A_e \) is the equivalent axle load which can also be calculated for mixed traffic, and \( N \) is the number of cycles. This formula differs slightly from that proposed by ORE in that it is asymptotic to some constant rate of settlement in line with the majority of test data (Fig 1.3).

Lane (42) described a computer simulation model of the deterioration of vertical track geometry due to traffic. A logarithmic relationship between load and settlement similar to that proposed by ORE (38) was used but applied to each sleeper individually in the vicinity of a prescribed irregularity. The dynamic loads at a dipped rail weld were predicted by means of a two degree of
The freedom model of the vehicle/track system representing the sprung mass and the combined unsprung mass and lumped track mass.

The paper considered the effects of various parameters such as speed, axle load and unsprung mass on the rate of deterioration of the vertical track geometry. The natural shape of the rail was also considered and it was demonstrated that this could be transferred into irregularities in the running surface of the track. The method was also used to suggest how limits on the allowable size of irregularities in the track could be defined.

Concerning lateral deterioration of geometry the reason for studying this problem in the past has mainly been for the consideration of loss of alignment on curves, and the need to define limits on the lateral forces due to trains curving at
increased speeds. One of the earliest studies of note was in 1967 and due to Prud'homme (43) who reported the results of a series of tests performed by the French Railways (S.N.C.F) along with some theoretical work. A vehicle able to provide variable vertical and lateral loads was used to study the conditions required to generate permanent lateral displacements of the track. The author showed that when lateral load was plotted against lateral displacement for varying vertical load an essentially bi-linear response was found in that the displacement grew rapidly above some critical load (Fig 1.4). As might be expected the largest permanent displacements resulted if the critical load was exceeded, representing sliding of the sleepers on the ballast, some small residual displacement still occurred, however, if the critical load was not reached. An empirical relationship was also derived for the value of the critical lateral load as follows:

\[ L = A(P + P_0) \]
Where $L$ is the critical lateral load, $A$ is a coefficient depending upon the degree of compaction of the ballast, but generally having a value between 0.3 and 0.6, $P$ is the vertical load, and $P_0$ is a constant generally having a value of about 4 tonne.

Other conclusions made from the work were that the thermal compressive load in the track had little effect upon the results, at least for small displacements. The rail fastenings were also found not to have an influence.

The results obtained later by BR (44) were not entirely consistent with those obtained by Prud'homme. In a similar series of test on moving vehicles with varying vertical and lateral loads two main features were studied, namely the rate of increase of permanent displacement with number of load cycles and the permanent displacement after 50 passes as a function of the lateral to vertical load ratio.

The former showed that the amount of additional permanent settlement decreased with increasing number of cycles, at least up to 300 cycles for which an empirical relationship was derived as follows:

$$\frac{\delta}{\delta_0} \propto \left( \frac{n}{n_0} \right)^{0.25}$$

Where $\delta$ is the permanent settlement and $n$ the number of cycles, and the subscript 0 representing the value at 50 cycles.

Sufficient data was not available however to validate this law beyond 300 cycles, furthermore the effect of mixed loads was not studied. The permanent displacement at 50 cycles was found to relate best to the ratio $L/P$ (lateral force/vertical force) for which a relationship of the following form fitted the experimental data:

$$\frac{\delta}{\delta_0} \propto 2.1^{10L/P}$$

These relationships were therefore used to define limiting $L/P$ ratios for vehicles on curves and to influence the design of new vehicles.
The work described above, therefore, represents some of the more significant contributions to the subject of vehicle/track interaction. There are many reasons for wanting to study the subject and it is clear that no one model is ideal for all situations. In the work which follows the wheel/rail forces are studied with particular reference to track damage aspects. Moreover the requirements for track and vehicle models for this particular subject are considered in detail, and the results obtained for a particularly important area of study into track damage, namely deterioration of track geometry.
2. FORMULATION OF THE EQUATIONS OF MOTION FOR RAILWAY TRACK

2.1 Track Model for Vertical Dynamic Response

In order to adequately represent the track it is necessary to evaluate which components of a complex model are required to include the most important dynamic characteristics of the structure. The analysis will begin, therefore, with the most complex model which is likely to be required and an attempt can then be made to simplify this where possible to one which can be analysed more readily.

Fig 2.1 shows a view of a typical railway track containing all the main components. Starting from the top it is obvious that the two rails can be reasonably considered to act as infinite prismatic beams. Where concrete sleepers are employed (which is the current BR standard) synthetic or natural rubber pads are used between rail and sleeper and these provide resilience which is assumed to be modelled adequately, within certain limits, by a linear spring with parallel viscous or hysteretic damping.

The rail is discretely supported by the sleeper at intervals of typically 0.7m on main line BR track. Grassie et al (45) showed firstly that the effect of this discrete support was negligible except in the vicinity of the 'pinned-pinned' resonance of the rails at a frequency of approximately 750 Hz, and also that the first mode of the track vibration which involved bending of the sleeper was at approximately 200 Hz (Fig. 2.1). It will be assumed therefore, for this application, that the sleeper layer can be treated as a uniform layer providing only mass. Rotation of the sleeper in elevation and the rocking stiffness of the pads are also neglected.

A significant part of the track resilience is shown to come from the ballast/foundation layer. Experimental (46) and theoretical (47) work has shown that it is reasonable to assume that the mass of the foundation involved in the vibration is small compared to the mass of the track, and that the ballast layer can be reasonably modelled as a spring/viscous damper combination. Ballast is normally found to behave as a stiffening spring, however, particularly at small loads and for this reason a tangent is taken to the ballast load/deflection curve at the static axle load to represent the equivalent linear spring constant.
Fig. 2.1 Simplified View of Typical Railway Track
It is implied from this, therefore, that the track profile is the statically loaded profile.

Finally as only symmetric irregularity shapes in the track will be considered torsion of the track about a longitudinal axis involving rocking of the vehicle will be neglected, thus meaning that in effect the vehicle/track system can be treated as a single rail problem. Based upon these considerations, therefore, the model shown in Fig. 2.2 results.

In order to gain an understanding of the characteristics of this model and any simpler models, the vehicle will be neglected at this stage and the problem addressed will be that of the track only with a moving harmonic load. A similar problem to this has been approached by other authors, i.e. that of a moving harmonic load on an infinite single layer beam on Winkler foundation (1,2,3,4), but all these authors have simplified the problem or restricted the solution to special cases in order to solve the equations. In the context considered here, the Winkler foundation is defined as a continuous foundation giving a force proportional to the displacement applied plus the damping force, which will generally be proportional to the velocity.
Fryba (1) and Kenney (2) solved the problem of a moving quasi-static load on an infinite beam on Winkler foundation with damping, Mathews (3,4) solved the same problem and also the case of the stationary harmonic load with damping.

2.2 Analysis of the Two Layer Beam on Winkler Foundation

Two simultaneous equations of motion result for the coupled rail sleeper system which are derived from Newton's second law and simple bending theory:

\[ EI \frac{\partial^4 z_1}{\partial x^4} + m \frac{\partial^2 z_1}{\partial t^2} + k_p (z_1 - z_2) + c_p \left( \frac{\partial z_1}{\partial t} - \frac{\partial z_2}{\partial t} \right) = P_0 e^{i\omega t} \delta(x - vt) \]  

\[ m_s \frac{\partial^2 z_2}{\partial t^2} - k_p (z_1 - z_2) + k_f z_2 - c_p \left( \frac{\partial z_1}{\partial t} - \frac{\partial z_2}{\partial t} \right) + c_f \frac{\partial z_2}{\partial t} = 0 \]

\( P_0 e^{i\omega t} \) is the harmonic load using the complex notation and \( \delta(x vt) \) is the delta-dirac function representing a point load moving with velocity \( v \).

As a first stage in the solution a transformation is made to a co-ordinate system which effectively moves with the load at velocity \( v \) by the substitution:

\[ r = x - vt \]

such a transformation is quite commonly used for such problems for example Ref. (3) and thus enables partial derivatives with respect to \( z \) to be replaced viz:

\[ \frac{\partial z}{\partial t} = \frac{\partial z}{\partial r} \frac{\partial r}{\partial t} + \frac{\partial z}{\partial t} \]

\[ \frac{\partial^2 z}{\partial t^2} = \frac{\partial^2 z}{\partial r^2} \frac{\partial r}{\partial t} - \frac{\partial z}{\partial r} \frac{\partial r}{\partial t} + \frac{\partial^2 z}{\partial t^2} \]

\[ \text{and} \quad \frac{\partial z}{\partial x} = \frac{\partial z}{\partial r} \]

Note that the transformed derivatives on the right hand side of these equations are now functions of \( r \) and \( t \). Substituting in 2.2 gives:
The displacements $z$ are now expanded by means of the Fourier transform as follows:

$$z_j = \frac{1}{2\pi} \int_{-\infty}^{\infty} q_j(s) e^{isr} dr$$

where the transform of $z_j$, $q_j(s)$ is given by:

$$q_j(s) = \int_{-\infty}^{\infty} z_j e^{-isr} dr$$

The R.H.S. of equation 2.3 can also be transformed in this manner. Thus carrying out the transformation of equations 2.3 and 2.4 gives:

$$ EI s^4 q_1 + m_r \left( \frac{d^2 q_1}{dt^2} - 2ivs \frac{dq_1}{dt} - v^2 s^2 q_1 \right) + c_p \left( \frac{dq_1}{dt} - ivsq_1 - \frac{dq_2}{dt} + ivsq_2 \right) + k_p (q_1 - q_2) = P_0 e^{i\omega t}$$

$$ m_s \left( \frac{d^2 q_2}{dt^2} - 2ivs \frac{dq_2}{dt} - v^2 s^2 q_2 \right) - c_p \left( \frac{dq_1}{dt} - ivsq_1 - \frac{dq_2}{dt} + ivsq_2 \right) + c_f \left( \frac{dq_2}{dt} - ivsq_2 \right) - k_p (q_1 - q_2) + k_f q_2 = 0$$

This results in two second order simultaneous differential equations for which the particular integral is required (for the steady state solution). This is achieved by assuming the following solution:
\[ q_1(r,t) = q_a(r) e^{i\omega t} \]
\[ q_2(r,t) = q_b(r) e^{i\omega t} \]

Substituting in equations 2.6 and 2.7 above gives:

\[ EIs^4 q_a + m_r (-\omega^2 q_a + 2\omega s q_a - \nu^2 s q_a) + c_p (i\omega q_a - iusq_a - i\omega q_b + iusq_b) + k_p (q_a - q_b) = P_0 \]
\[ m_s (-\omega^2 q_b + 2\omega s q_b - \nu^2 s q_b) - c_p (i\omega q_a - iusq_a - i\omega q_b + iusq_b) - k_p (q_a - q_b) + k_f q_b + c_f (i\omega q_a - iusq_a) = 0 \]

On collecting terms this rearranges to:

\[ (EIs^4 - m_r \omega^2 + 2\nu s m_r - m_r \nu^2 s^2 + ic_p \omega - ic_p vs + k_p)q_a + (-ic_p \omega + ic_p vs - k_p)q_a = P_0 \]
\[ (-ic_p \omega + ic_p vs - k_p)q_a \]
\[ + (m_s \omega^2 + 2\nu s m_s - m_s \nu^2 s^2 + ic_p \omega - ic_p vs + k_p + k_f + ic_f \omega - ic_f vs)q_b = 0 \]

The two equations above are therefore simultaneous equations in \( q_a \) and \( q_b \) of the form:

\[ Aq_a + Bq_b = P_0 \]
\[ Bq_a + Cq_b = 0 \]

Having the solution:

\[ q_a = \frac{CP_0}{AC - B^2} \]
\[ q_b = \frac{-BP_0}{AC - B^2} \]
In order to proceed to the inversion of \( p_a \) and \( p_b \) it will be necessary to factorise the expression \( AC - B^2 \). This clearly is a sixth order polynomial equation in \( s \) with complex coefficients. There is clearly no easy way to solve this analytically. However standard computer programs exist for such problems and use is made of one such program to evaluate the six complex roots of this expression which can be expressed as \( A_k \) for \( k = 1 \) to \( 6 \) where \( A_k = a_k + ib_k \)

It is then necessary to substitute back into equations 2.8 and 2.9 to get back to the transformed variables \( q_1 \) and \( q_2 \)

\[
q_1 = \frac{(EIs_4 - m_r^2 + 2 \omega \nu s_m^r - m_r \nu^2 s^2 + ic \omega - ic \nu s + k)}{(s-A_1)(s-A_2)(s-A_3)(s-A_4)(s-A_5)(s-A_6)} P_0 e^{iot} \tag{2.10}
\]

\[
q_2 = \frac{(c \nu - c \nu s + k)}{(s-A_1)(s-A_2)(s-A_3)(s-A_4)(s-A_5)(s-A_6)} P_0 e^{iot} \tag{2.11}
\]

The inversion of these expressions can be obtained by the method of contour integration and the theory of residues which states that the integral required for inversion is given by:-

\[
I = 2\pi i \sum \text{Residues at the poles}
\]

where the residue at each pole \( (A_k) \) is the term in \( (s-A_k) \) in the Laurent expansion at the pole.

The required integral for \( z_1 \) is from 2.5:-

\[
z_1 = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{(EIs_4 - m_r^2 + 2 \omega \nu s_m^r - m_r \nu^2 s^2 + ic \omega - ic \nu s + k)}{(s-A_1)(s-A_2)(s-A_3)(s-A_4)(s-A_5)(s-A_6)} P_0 e^{iot} e^{isr} \]

for which the residue at pole \( A_1 \) is:-

\[
\frac{(EIA_1^4 - m_r^2 + 2 \omega \nu s_m A_1 - m_r \nu^2 A_1^2 + ic \omega - ic \nu s + k)}{(A_1-A_2)(A_1-A_3)(A_1-A_4)(A_1-A_5)(A_1-A_6)} P_0 e^{iot} e^{i\omega A_1} \]

The full integral can thus be evaluated in this manner and results in the equations:-
\[ z_1 = \sum_{k=1}^{6} iB_k e^{i\omega t} e^{(i\alpha_k - b_k)r} \]  \hspace{1cm} (2.12)

\[ z_2 = \sum_{k=1}^{6} iC_k e^{i\omega t} e^{(i\alpha_k - b_k)r} \]  \hspace{1cm} (2.13)

Where \( B_k \) and \( C_k \) are complex coefficients.

It can be seen from these expressions that the solution for the displacements consists of the product of a harmonic term in time, a harmonic term in distance, and an exponential term in distance. It is apparent from this that for positions in front of the moving load (i.e. \( r > 0 \)) the coefficients \( b_k \) need to be positive for waves to decay with distance. The reverse is also true for positions behind the moving load. Thus for positions ahead of the moving load, only those roots of the denominators of equations 2.8 and 2.9 having positive imaginary parts need to be considered, and for positions behind the load those roots with negative imaginary parts are considered.

The solution of equations 2.12 and 2.13 has been achieved using a computer with a programming language capable of handling complex numbers. The roots of these polynomial equations with complex coefficients have been obtained using a standard subroutine from the Numerical Analysis Group library (C02ADF) which uses the method of Grant and Hitchins (48). In order to evaluate more usefully the results of this model, however, a similar model is also developed for the more simple single layer beam on Winkler foundation which has been studied in the past, as described earlier. Analytical solutions are available for the following cases:

(i) A moving harmonic load on an undamped beam

(ii) A moving quasi-static load on a damped beam

(iii) A stationary harmonic load on a damped beam

The equations will therefore be developed for the single layer infinite beam case to give comparison with the more complex model and establish if the increased complexity is necessary. The opportunity will also be taken to include the effect of shear displacement and rotatory inertia which can be expected to
have an effect at higher frequencies, it can then be established if these need to be included in the two layer model.

2.3 Analysis of the Single Layer Beam on Winkler Foundation

The equation of motion for a beam on a uniform elastic foundation is shown below in which the pad layer is neglected.

\[
EI \frac{\partial^4 z}{\partial x^4} + m \frac{\partial^2 z}{\partial t^2} + c \frac{\partial z}{\partial t} + k \frac{\partial^2 z}{\partial t^2} - \rho I \frac{\partial^4 z}{\partial x^4 \partial t^2} = P_0 e^{j \omega t} \delta(x - vt)
\]

The term \(-\rho I \partial^4 z/\partial x^2 \partial t^2\) is due to rotatory inertia which allows for cross sections rotating in addition to displacing vertically. The terms involving \(k'\) are terms to allow for shear displacements of the beam which allow for plane sections ceasing to remain plane. The constant \(k'\) is a coefficient dependent on the shape of the cross section and is usually known as Timoshenko’s shear coefficient. The derivation of these terms is fully described in Reference (49). It should be noted that the equation shown above is obtained from a pair of simultaneous equations in the coupled vertical and shear displacements from which it is possible to eliminate the latter.

Solution of this equation can proceed as in the previous case noting that in the transformation to the moving co-ordinate system:

\[
\frac{\partial^4 z}{\partial x^4 \partial t^2} = v^2 \frac{\partial^4 z}{\partial r^4} - 2v \frac{\partial^4 z}{\partial r^3 \partial t} + \frac{\partial^4 z}{\partial r^2 \partial t^2}
\]

\[
\frac{\partial^4 z}{\partial t^4} = v^4 \frac{\partial^4 z}{\partial r^4} - 4v^3 \frac{\partial^3 z}{\partial r^3 \partial t} + 6v^2 \frac{\partial^3 z}{\partial r^2 \partial t^2} - 4v \frac{\partial^3 z}{\partial r \partial t^3} + \frac{\partial^4 z}{\partial t^4}
\]

For this equation a single fourth order equation in the transformed displacement variable results which can once again be solved by a computer based algorithm to provide a complete solution. A flow chart for the process used in both cases is shown in Fig 2.3. The solutions for this model were first verified against the analytical solutions for the special cases described earlier, it can also be shown that the results of the two layer model converge to the same results as the single layer model for large values of stiffness in the upper layer.
INPUT PARAMETERS

FOR REQUIRED SPEEDS

FOR REQUIRED FREQUENCIES

CALCULATE TERMS (COMPLEX) OF DENOMINATOR OF 2.8

SOLVE USING STANDARD ROUTINE TO OBTAIN POLES

CALCULATE RESIDUE AT EACH POLE WITH POSITIVE IMAGINARY COEFF.

CALCULATE DISPLACEMENTS FROM EXPRESSIONS 2.12, 2.13

OUTPUT RESULTS

END

Fig 2.3 Flow Chart for Beam on Winkler Foundation Model Solutions
2.4 Railway Track Subject to Moving Harmonic Load

The first stage of the comparison will attempt to answer two main questions, namely is it necessary to consider loads on the railway track to be moving and is it important to consider the shear and rotatory inertia terms in the response of the rail. In order to achieve this the frequency response functions for the models will be obtained for a range of speeds of the moving load, both with and without shear and rotatory inertia included.

The parameters to be used in the investigation are given below. These represent those parameters which are considered to be applicable to 'average' track and have been obtained from a variety of experimental data, for example Ref. (46). All parameters are quoted per rail. Due to variations in foundation conditions there is clearly a significant amount of scatter in the values of some of the quantities which have been measured, however the conclusions which will be drawn have been verified over a range of values. Due to the importance of the damping present, two values of track damping have been considered which are referred to as 'standard' and 'reduced' damping.

\[
\begin{align*}
EI & = 4.87 \times 10^6 \text{Nm}^2 \\
PR & = 1 \text{N} \\
m_r & = 56 \text{kg/m} \\
m_s & = 250 \text{kg/m} \\
k_p & = 300 \times 10^6 \text{N/m}^2 \\
c_p & = 0.04 \times 10^6 \text{Ns/m}^2 \\
k_f & = 70 \times 10^6 \text{N/m}^2 \\
c_f & = 0.15 \times 10^6 \text{Ns/m}^2 \text{(standard damping)} \\
c_f & = 0.05 \times 10^6 \text{Ns/m}^2 \text{(reduced damping)} \\
\rho & = 7843 \text{kg/m}^3 \\
k' & = 0.34 \\
I & = 2352. \times 10^{-8} \text{m}^4
\end{align*}
\]

Single Layer Model: This case has been studied in the past by other authors, for example by Mathews (3,4). It is possible to show analytically when the load is quasi-static and damping is neglected that the roots of the equation for the single layer beam on Winkler foundation model show an instability. This occurs at a particular speed, usually called the critical speed. This is not itself a resonance phenomenon because the load is not harmonic, but is shown to correspond to a situation where the velocity of the load reaches the minimum speed at which waves propagate along the beam, and therefore shows a certain analogy with the sound barrier effect of objects approaching the speed of sound in air. As this speed is approached the maximum displacement of the beam approaches infinity in the undamped case, the critical speed is given by:-
For the typical track parameters quoted above this results in a critical speed of 347 m/s (780 mph) which will certainly not be achieved by railways in the foreseeable future, however it will be necessary to establish if the response of the track at some fraction of this speed is significantly different to the response due to a stationary load.

A fundamental natural frequency of the beam is also defined which represents a rigid body bounce mode of the beam occurring without bending. This natural frequency is therefore defined by \( \omega = (k_f/m_t)^{0.5} \) by analogy with single degree of freedom systems. Mathews also showed how the critical speed and the fundamental natural frequency of the beam are inter-related and this relationship (again evaluated for the case of zero damping) is reproduced in graphical form in Fig. 2.4. This suggests that at half the critical speed the natural frequency of the beam is reduced to 60% of the stationary load value.

The results calculated for the solution of the single layer beam on Winkler foundation are studied first. These are presented in the form of frequency response curves at a range of speeds between zero and the critical speed. The sleeper/ballast force is also calculated because this is a major factor when considering track geometry deterioration. It should be noted as the speed increases the position of maximum displacement no longer occurs directly beneath the load but occurs at some point further behind, the position of maximum sleeper/ballast force is also not necessarily coincident with the load position. Taking the two layer model as an example this force can be assumed to be:

\[
P_b = klz_2^2 + lc \frac{dz_2}{dt}
\]

That is to say that the sleeper/ballast force is the sum of the effective ballast spring and damper forces. The parameter \( l \) is the sleeper spacing which gives the force as an effective value per sleeper, although the model is a continuous one. The displacement \( z_2 \) is given by equation 2.13 which is solved incrementally within the program to obtain a maximum value of \( P_b \).
Fig. 2.4 Relationship Between Fundamental Natural Frequency and Critical Speed

Fig 2.5 shows the results of the calculations for the single layer model for the cases of 'standard' and 'reduced' track damping. The ratio of the dynamic displacement beneath the load to the static displacement is presented as well as the ratio of maximum sleeper/ballast force to the applied force. The results obtained with the shear and rotatory inertia terms included are also shown.

The response curves where shear effects are neglected, and particularly the curve for zero speed, show results very characteristic of a single degree of freedom oscillator. The resonant peak is not large in the standard damping case, as might be expected for the relatively large amount of damping which is found in the railway track. As speed increases the displacement amplitude at the
Fig. 2.5 Response of Single Layer Track Model with Moving Harmonic Load
loaded point is also shown to decrease and a reduction in the resonant frequency is also noticed in line with the suggestion of Mathews. The displacement beneath a quasi-static load is shown to decrease with speed in the case of standard damping and increase with speed in the case of reduced damping, as does the ballast force. This effect can also be seen by studying the plots of displacement against distance from the load for the quasi-static case in Fig 2.6, again for standard and reduced track damping. The maximum displacement is shown to increase slightly with increasing speed, but the displacement beneath the load is slightly greater at 25% $v_c$ and significantly less at $v_c$. The deflected shape is very similar to the stationary case at the lower speed but the position of maximum displacement is shown to retreat behind the load at high speed and a large precession wave is shown to appear ahead of the load.

![Fig. 2.6 Deflected Shape of Single Layer Beam on Winkler Foundation with Moving Quasi-Static Load](image-url)
Returning to Fig 2.5 it can be seen that the maximum ballast force at resonance tends to decrease with speed for both values of damping chosen, this is because as the speed increases more of the wheel/rail force is reacted by the track inertia and hence less in the spring and damper forces. This conclusion does not take into account of course that the wheel/rail forces from a vehicle will increase with speed.

The shear and rotatory inertia effects are only shown to begin to have an influence above 1000 Hz. At this point all the curves merge together showing no effect of longitudinal speed. The curves without shear and rotatory inertia are in fact shown to agree to within 5% at 5600 Hz. This frequency appears to be very high and is certainly well in excess of the likely frequency range of interest and validity limits of the model. The reasons why the effect of shear and rotatory inertia terms should be of such apparently minor importance can be better understood by reconsidering equation 2.12 which has an equivalent for the single layer model:

\[ z = \sum_{k=1}^{4} iB_k e^{i\omega t} e^{(ia_k - b_k)r} \]

This can be re-written:

\[ z = \sum_{k=1}^{4} iB_k e^{-b_k r} e^{i\omega t(\cos a_k r + i \sin a_k r)} \]

Thus coefficient \( b_k \) describes the rate of decay with distance of waves, \( a_k \), the wavelength of the rail profile (with respect to the moving coordinate \( r \)) and the modulus of the complex number \( B_k \) defines the maximum amplitude. For waves travelling ahead of the load (\( b_k \) positive as described earlier). There are two roots \( a_k + ib_k \) and two corresponding values of \( b_k \). These are shown for selected frequency values in Table 2.1 for zero longitudinal speed.

The dynamic profile of the rail can therefore be considered as consisting of two component waves. For zero excitation frequency these are essentially identical in form with \( a_k \) and \( b_k \) equivalent to the value \( \beta \) for a static beam on elastic foundation. As the frequency increases one of the component waves grows in wavelength and decays more rapidly while the second component wavelength does the reverse. At high frequencies it can be seen that the shorter waves are significantly altered by including shear and rotatory inertia terms.
but the longer waves are not thus resulting in a reduced effect on the overall beam response.

<table>
<thead>
<tr>
<th>FREQUENCY</th>
<th>EXCL. SHEAR AND R.I.</th>
<th>INCL. SHEAR AND R.I.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$</td>
<td>B_k</td>
</tr>
<tr>
<td>0 Hz</td>
<td>0.71 1.38 1.38</td>
<td>0.71 1.38 1.38</td>
</tr>
<tr>
<td></td>
<td>0.71 -1.38 1.38</td>
<td></td>
</tr>
<tr>
<td>100 Hz</td>
<td>0.51 0.58 2.09</td>
<td>0.51 0.57 2.10</td>
</tr>
<tr>
<td></td>
<td>0.51 -2.09 0.58</td>
<td></td>
</tr>
<tr>
<td>5600 Hz</td>
<td>5.6x10^{-4} 0.06 16.74</td>
<td>3.6x10^{-4} 0.04 22.88</td>
</tr>
<tr>
<td></td>
<td>5.6x10^{-4} -16.74 0.06</td>
<td>3.6x10^{-4} -22.88 0.04</td>
</tr>
</tbody>
</table>

**Table 2.1 Selected Values of Equation 2.12**

A certain symmetry is also noticed in the results presented, this is due to the consideration of a stationary load. This symmetry is lost for non-zero longitudinal speed but the conclusions concerning frequency effects remain the same.

The final feature to notice is the additional resonance which appears in the response resulting from the inclusion of the shear displacement terms. This peak represents a maximum response of the shear displacement which is shown to have a small effect on both vertical displacement and sleeper/ballast force. This occurs at an extremely high frequency, however, (around 12000 Hz) and is therefore only of peripheral interest for the current problem.

**Two Layer Model:** The effect played by the rail pads can now be studied using the two layer model with moving load. The values of track stiffness and damping are taken to be the same as the previous example, and two cases of rail pad parameters are also considered. The 'standard' pad is the one which is most commonly used on BR and the parameters have been calculated to be representative under a pre-load equivalent to a static axle load. The 'soft' pad is typical of some newer pads which are under consideration as a means of
reducing high frequency loads due to wheel flat impacts and corrugations which are thought to cause damage to concrete sleepers. The parameters used therefore are as follows:

- **Standard Pad** - \( k_p = 300 \times 10^6 \text{ N/m}^2 \) \( c_p = 40 \times 10^3 \text{ Ns/m}^2 \)
- **Soft Pad** - \( k_p = 75 \times 10^6 \text{ N/m}^2 \) \( c_p = 20 \times 10^3 \text{ Ns/m}^2 \)

Rail pad materials are often argued to have a damping characteristic closer to hysteretic damping. For this reason the standard pad case is also repeated having a hysteretic damping characteristic with a loss factor of 0.1, this being a typical value for the pad materials used. This is achieved in the equations for the system by replacing the viscous damping constant \( c \) by \( H/\omega \). It should be noted, however, that very little work has been carried out to evaluate the properties of these materials as they relate to railway use. In fact measurement of the properties is not easy as they tend to show a dependence on many factors including compressive load, temperature and frequency.

For the two layer case an analytical expression for the critical speed of the new model cannot be easily derived. However it is reasonable to assume that the critical speed will be slightly less for the two layer system due to the increased flexibility, what is of more importance is the track response at likely railway speeds. The speeds considered are therefore the same as before, representing 0, 25, 50, 75 and 100% of the critical speed of the single layer case. The maximum sleeper/ballast force can be obtained in the same way as for the previous case but calculated instead from the response of the sleeper layer. The results are shown in Figs 2.7 - 2.9. In this case the response curves are similar to those of a two degree of freedom system having a second resonant peak at a higher frequency. Similar conclusions can be drawn to those obtained previously in that the response at both resonant peaks is reducing with increasing speed, the frequency of maximum response is also reduced more at the first resonant peak than at the second as the speed increases. The response up to the first resonant peak is very similar to that for the single layer model, bearing in mind that the curves are normalised to the static displacement which will be slightly different in each case. This is also true of the sleeper ballast force where the effect of the pads can be seen to cause a small additional peak in the response above the first resonant peak and then to cause a large attenuation above this frequency.
Fig. 2.7 Response of Two Layer Model with Moving Harmonic Load:
Standard Pad Stiffness
Fig. 2.8 Response of Two Layer Model with Moving Harmonic Load: Reduced Pad Stiffness
Fig. 2.9 Response of Two Layer Model with Moving Harmonic Load:
Hysteretic Pad Damping
The effect of considering hysteretic damping as an alternative is negligible at the first resonance as it is straightforward to show that at this frequency the amount of effective pad damping is very similar to the viscous case. The higher frequency resonant peak is however shown to be more prominent and results in an increased transfer of rail head force to ballast force at these frequencies. The difference in response as a function of speed, however, is still shown to be small, although the ballast forces at very low frequencies are shown to be less than with the viscous pads at very high speeds. It is concluded, therefore, that the effect of the damping characteristic is most noticeable at the higher frequencies, and this should be borne in mind when considering the track response at such frequencies.

The present maximum operating speed on BR is 125 mph (200 kph) with a proposed increase in the future to 140 mph (225 kph). This latter represents a value of 16% of the critical speed calculated for the example case. The case of 25% considered already therefore represents a useful margin in excess of current speeds. The 'error' which would be incurred by neglecting the longitudinal speed for 'standard' conditions can therefore be summarised for the case of 16% and 25% of critical speed as shown in Table 2.2

<table>
<thead>
<tr>
<th>SPEED (kph)</th>
<th>$\frac{u}{u_{crit}}$</th>
<th>ERROR</th>
<th>MAX. RESPONSE</th>
<th>MAX. BALLAST FORCE</th>
</tr>
</thead>
<tbody>
<tr>
<td>225</td>
<td>16%</td>
<td>1.0%</td>
<td>1.3%</td>
<td></td>
</tr>
<tr>
<td>310</td>
<td>25%</td>
<td>2.2%</td>
<td>3.1%</td>
<td></td>
</tr>
</tbody>
</table>

Table 2.2 Errors Anticipated due to Neglecting Longitudinal Speed

These conclusions are based of course upon unit loads, variations in the characteristic response of the track would also cause the vehicle/track interaction forces to change.

It is considered reasonable therefore to accept that the load is stationary, particularly as to do so would only marginally underpredict the majority of dynamic loads on the track, and overpredict those in the region of resonance.
2.5 Choice of Model for Vertical Vehicle/Track Interaction

The track models developed so far are shown to differ in the response at high frequencies. A model is required which at least predicts to a reasonable level of accuracy the forces seen by the ballast due to vehicle loads. These will depend upon both the spectrum of forces seen at the railhead and the transfer function of rail head force to sleeper/ballast force, the latter part of which has already been derived. As it is at the higher frequencies that the models differ it is reasonable for this comparison to study the response of a vehicle unsprung mass to a known spectrum of track irregularities in order to choose a suitable model. The vehicle response to the higher frequencies is known to be negligible.

The two layer beam on Winkler foundation model will be considered as well as the single layer model. A further model which can also be studied is a lumped parameter model having one degree of freedom derived initially from an equivalent single layer model, the equivalent parameters for which were derived from the methods described in Appendix I.

The model to be studied therefore is an unsprung mass negotiating a known rail surface profile. To facilitate the solution of this model and also to maintain a good representation of the high frequency response, the wheel is supported on the rail by the non-linear Hertzian contact. For this exercise a linear tangent stiffness will be taken at an assumed nominal wheel load.

The Hertzian load deflection relationship is normally assumed to be of the following form:

\[ P = \left(\frac{y}{G_H}\right)^{3/2} \]

The constant \( G_H \) can be calculated for contacting elastic bodies having known maximum and minimum curvature in two planes (11). A typical value for railway wheel to rail contact is \( G_H = 0.295 \times 10^{-7} \text{m.N}^{-2/3} \). The effective tangent stiffness at preload \( P \) is therefore given by:

\[ \frac{dP}{dy} = k_H = \frac{3}{2} \left(\frac{y}{G_H}\right)^{1/2} \frac{1}{G_H} \]
For a 10 tonne wheel load this gives an effective stiffness of \( k_H = 2.3 \times 10^9 \text{ N/m} \).

The unsprung mass can be combined with the track model by considering the receptance of the two component systems. No specific track model is implied in this analysis, therefore any model for which the receptance is known could be used. The response is required to a prescribed track irregularity which in this instance is described in the frequency domain. The component systems are shown in Fig 2.10.

![Fig 2.10. Combined Receptance of Unsprung Mass and Track.](image)

The equation of motion (in the frequency domain) for the lower system is given by:

\[
M_w \ddot{z}_u + k_H (z_u - z_t - s_t) = 0
\]

Assume a solution \( z_u = Z_u e^{i\omega t}, z_t = Z_t e^{i\omega t} \) when \( s_t = S e^{i\omega t} \),

\[
-M_w \omega^2 Z_u + k_H (Z_u - Z_t) = S k_H
\]

\[
\frac{1}{H_u} Z_u - k_H Z_t = S k_H \quad \text{where} \quad H_u = \frac{1}{k_H - M_w \omega^2}
\]
Eliminating $P_c$ from 2.15 and 2.17.

$$Z_t = H_t k_H (Z_u - Z_t - S)$$

and rearranging gives:

$$Z_t = \frac{H_t k_H (Z_u - S)}{1 + H_t k_H}$$

Substituting in 2.16 yields the expression:

$$Z_t = \frac{S k_H U}{1 + H_t k_H - H_u k_H^2}$$

from which:

$$Z_u = \frac{S k_H U}{1 + H_t k_H - H_u k_H^2}$$

Substituting in 2.16 gives:

$$Z_t = S \left( \frac{1}{1 + H_t k_H - H_u k_H^2} - 1 \right)$$

and substituting these expressions into 2.17 gives:

$$P_c = k_H S \left( \frac{-1 + H_u k_H}{1 + H_t k_H - H_u k_H^2} \right)$$

$P_c/S$ is the transfer function of track irregularity to wheel/rail contact force. If $S$ is in power spectral density (PSD) form the PSD of $P_c$ is obtained by multiplying by $|P_c/S|^2$. In a similar manner the PSD of the sleeper/ballast force can also be obtained from the transfer function already discovered between rail head force and sleeper/ballast force.
The solution to the problem of a wheelset negotiating a known spectrum of random rail surface irregularities can therefore be calculated based upon the solution already derived. A typical spectrum of the vertical rail profile is thus required in order to calculate the spectrum of the track response. Analysis of typical rail profiles from track recording cars (37) shows that there is a large component of random roughness in the profiles and also a significant contribution from the rail length and its harmonics which appear as peaks in the power spectrum. A typical example of this is shown in Fig 2.11 with a regression curve fitted to the random component which has the form:

\[
S(\lambda) = \left( \frac{0.448\lambda^2}{1+1.71 \times 10^{-5} \times \lambda^5} + 0.063\lambda^{1.317} \right) \times 10^{-8} \ m^2/m
\]

The discrete peaks coincide with the rail length (18.3m) and its harmonics (9.15m, 6.1m, 4.575m...) and have a particular phase relationship. The response to these will be considered later when the time domain solution is developed.

In order to complete this analysis and discover the important features of the track models it is worthwhile to consider some additional parameters of interest, i.e.:

i) **Rail Head Displacement:**
   This is calculated as before but the response is due to a unit harmonic irregularity in the rail profile rather than a unit force.

ii) **Wheelset Displacement:**
    This is calculated on the same basis as the rail head displacement.

iii) **Phase Angle of Ballast Force to Input Profile:**
    This is important when considering track deterioration, as an irregularity will only grow when the forces occurring cause settlements which are in phase with the original shape or partially so. An out of phase situation would be expected to cause an irregularity to reduce in size. The phase lag can be considered to be composed of two parts, a phase difference between the input shape and the rail head force as defined by equation 2.21, and a phase difference between the rail head force and the sleeper/ballast force as defined by equation 2.14. A phase difference is implied by these equations as they are expressions in complex numbers. If the real
iv) **Response to Random Rail Irregularities:**

The response to the typical spectrum of random irregularities is derived as described above. The power spectra of rail head force and sleeper/ballast force are evaluated.

The single layer and double layer beam on Winkler foundation models are analysed as well as the equivalent lumped parameter model. The parameters considered are also the same as before, i.e. standard and soft pad stiffness, but the track damping value was taken as standard. The response curves are plotted this time on linear axes up to a maximum frequency of 600Hz. In order
to produce a forcing function which is a function of frequency the input spectrum has to be converted. This requires the speed of the vehicle to be defined and the examples produced were based upon a speed of 125 mph (55.9m/s) with an unsprung mass of 1000 kg. Ordinates of the power spectrum therefore need to be multiplied by $\lambda^2/v$ to convert to units of mm$^2$/Hz and the horizontal scale is multiplied by $v$ to convert to Hz.

The results are taken in sequence, Fig 2.12 shows the wheelset displacement due to a harmonic rail irregularity of unit amplitude. At low frequencies the wheelset follows the undulating profile with little movement of the track. At a frequency around 50Hz the wheelset reaches the first resonance, at this frequency the wheelset bounces vertically on the total series stiffness of the track, and the displacement is a maximum. The frequency at which this occurs varies little from model to model but is lowest as would be expected for the soft rail pad case. The effect of including the pad layer is apparently to reduce the amount of damping in this mode and the amplitude of wheelset oscillation is thus greatest in the two layer model, particularly when the pad stiffness is also low. In other words the relatively low damping in the pad layer reduces the overall amount of damping in the first mode of vibration.

Above this frequency the wheelset vibration amplitude begins to reduce with frequency due to it's large inertia relative to that of the track. This also accounts for the fact that the second mode of vibration of the track which is a feature of the response of the two layer model is barely apparent in the wheelset response, being a well damped mode with the largest amplitude of vibration occurring in the track layer.

At the higher frequencies the responses of the various models begin to differ significantly as the Hertzian contact stiffness comes in to play. A mode of vibration exists at quite a high frequency in which wheelset and rail move in anti-phase on the Hertzian contact spring, the lumped parameter model predicts a frequency of approximately 450Hz for this mode. This can be considered in more detail when the rail response to a unit amplitude irregularity is studied which is shown in Fig 2.13. The rail does not undergo any motion at low frequencies but does experience large amplitudes of oscillation at the first resonance in line with those of the wheelset. There is slightly more evidence of the second track mode for the two layer model around 100-150Hz in the case of the standard pads but this is still quite small. A very large amplitude of vibration is apparent however at a high frequency for the lumped parameter
Fig. 2.12 Wheel Displacement for Unit Rail Irregularity vs. Frequency
Fig. 2.13 Rail Displacement for Unit Rail Irregularity vs. Frequency
model. It is interesting that this mode of vibration becomes less apparent in the case of the single layer beam on Winkler foundation model and at a higher frequency. It dissapears altogether (to occur at a higher frequency still) in the case of the two layer model.

Fig 2.15 shows the phase angle between the track profile (effectively the forcing function) and the sleeper/ballast force, zero phase indicating that a maximum downward force is coinciding with a low point in the track and a positive angle indicating a phase lag for the ballast force. The curves show that the phase lag increases rapidly as the first resonance is approached around 50 Hz and then becomes larger than 180° (π rad) as the vibration changes to the higher mode. The phase angles predicted for all the models are very similar below this point. The reason for differences in the phase responses of the model is more easily illustrated by the phase difference between rail head force and rail head displacement as shown in Fig 2.14. This phase angle can be interpreted as an indicator of the energy dissipated by the dynamic system or alternatively as the energy input. This is shown to be the product of force and velocity integrated over a period of time. As velocity is 90° out of phase with displacement this function is shown to be a maximum when θ = 90° and zero for θ = 0° or 180°. In the case of the single degree of freedom lumped parameter model therefore the phase lag is zero at zero frequency and asymptotic to 180° at high frequencies. Maximum energy absorption occurs at resonance when θ = 90°. In the case of the infinite beam models energy is dissipated along the rails at frequencies above the resonant frequency and the phase angle is therefore not asymptotic to 180° at high frequencies, but is shown to be asymptotic to an angle of 135° (1.5π rad) reflecting this fact.

Fig 2.16 shows the PSD of wheel/rail contact force as calculated for the random track profile. It is clear straight away that the majority of the forcing which occurs at the rail head occurs at the 50 Hz resonance, and that this is more lightly damped and at a slightly lower frequency for the two layer model with soft pads. The lumped parameter model does also show a large component of wheel/rail force corresponding to the higher mode of wheelset/track vibration, and it must be concluded that this is incorrectly represented from the results of the other models, and reflects the expected inaccuracy of a lumped parameter model at high frequencies.

In Fig 2.17 the PSD of the predicted sleeper ballast force is presented for the same spectrum of random rail profile. The common feature of all the models
Fig. 2.14 Phase Angle: Vertical Force - Vertical Rail Displacement
Fig. 2.15 Phase Angle: Vertical Force - Sleeper/Ballast Force
Fig. 2.16 Spectrum of Wheel/Rail Force on Typical Track

(Vehicle Speed = 125mph)
Fig. 2.17 Spectrum of Sleeper/Ballast Force on Typical Track
(Vehicle Speed = 125mph)
is once again the resonant peak just below 50 Hz at which a large proportion of the sleeper/ballast force is predicted to occur. The height of the peak is apparently determined by the effective amount of damping in this important mode of vibration. The two layer model with rail pads has the least damping as mentioned earlier.

If the single layer model is likened to the two layer model with very stiff pads it can be seen that the peak value of sleeper/ballast force at the wheelset resonance increases with reducing pad stiffness, reflecting the reduction in the effective damping. Large amounts of energy are predicted to be transmitted to the ballast with the lumped parameter model, although the wheel/rail force component occurring around 450 Hz is very largely attenuated.

Referring once again to the phase relationship between irregularity and sleeper/ballast force shown in Fig. 2.15 it can be concluded that, as the great majority of the force component seen by the ballast occurs at a frequency less than 100 Hz, the models all give a similar representation of the phase angle. It can be concluded therefore that predicted sleeper/ballast forces would at least show the correct phase relationship if any one of the models were used for predictions of deterioration of track geometry.

2.6 Summary of Vertical Track Models

The effect of increasing the level of complexity of the models is shown to have most effect at higher frequencies which is to be expected. The pad layer is shown to have most effect on the level of damping which is apparent at the fundamental mode of vibration, in which the wheelset and the track vibrate in phase. It would be possible however to calculate an equivalent level of stiffness and damping for a single layer lumped parameter or distributed parameter model such that the response in this mode would be predicted to a reasonable level of accuracy. An approximate analytical expression is derived for this in Appendix I or alternatively, these could also be obtained intuitively from the observed response in experimental results.

The response at the higher frequencies is apparently predicted most accurately by the two layer model. In terms of track deterioration this model shows that the ballast does not experience much force component with a frequency greater than 100 Hz. The simplest lumped parameter model shows a large component of force having a mean frequency of 450 Hz which is essentially
spurious, although the single layer model, which is representative of track with very stiff pads shows some forcing at 500 Hz.

For the best representation of the higher frequency forces for the random irregularity case the two layer model would be preferable, therefore. It should also be unnecessary to model sleeper bending modes as the first of these occurs around 200 Hz. An improved version of the lumped parameter track model can also be achieved if rigid contact is assumed between wheel and rail and equivalent parameters are calculated based upon both pad and ballast layer as described in Appendix I. To model this situation with the analysis already performed the Hertzian contact stiffness was made very large, although in the following chapter a vehicle/track interaction model in which this requirement can be excluded altogether is described. When calculating such equivalent parameters for the standard track case the results presented in Fig. 2.18 are obtained. This is shown to give much improved results for the prediction of wheel/rail forces but the sleeper/ballast forces are still not well represented due to the fact that a separate sleeper layer does not exist. A later chapter will deal with the requirements of a model for the prediction of vertical deterioration of geometry.

2.7 Track Models for Lateral Dynamic Response

Reconsidering Fig 2.1 under the assumption of a lateral load on one rail, it can be suggested that this has certain similarities with the vertical case, being an infinite pair of beams resiliently supported laterally by movement of the sleepers in the ballast. It has been shown from the results of the experiments performed in connection with this project (Appendix III) that it is reasonable to assume a linear spring/viscous dashpot combination when the friction limit is not exceeded. It is also reasonable to assume that the bending displacement of each rail is the same, therefore the track under these conditions would behave laterally as a beam on Winkler foundation. Some lateral displacement of the rail head relative to the foot would be anticipated in the loaded rail, however, and it is assumed that this occurs due to the rail head bending laterally as a beam with resilience provided by the rail web. The foot is assumed to be rigidly connected to the sleeper which is usually the case in switch and crossing work for which the model is mainly intended. This is quite a simplifying approximation but it will be shown that anything more sophisticated is not justified. The model proposed is shown in Fig 2.19.
Fig. 2.18 Response of Lumped Parameter Model with Rigid Wheel/Rail Contact
An equation of motion similar to the vertical two layer model will therefore result with the exception that both layers have bending rigidity, i.e.:

\[ EI_h \frac{\partial^4 y_1}{\partial x^4} + m_h \frac{\partial^2 y_1}{\partial t^2} + k_w (y_1 - y_2) + c_w \left( \frac{\partial y_1}{\partial t} - \frac{\partial y_2}{\partial t} \right) = F_0 e^{i \omega t} \delta(x) \]  

-2.23
The load is assumed to be stationary in this case as it is easy to show once again that the approximate critical speed is well in excess of likely railway speeds and the effect of the moving load would therefore be small as shown for the vertical case. The shear and rotatory inertia terms can also be neglected on the same basis as before.

The solution of the above equations can proceed in exactly the same manner as for the vertical case except that no moving coordinate system is required.

The first step is to carry out the Fourier transform as before, i.e.:

$$y_j = \frac{1}{2\pi} \int_{-\infty}^{\infty} p_j(s) e^{isx} dx, j = 1, 2$$

This is followed by a substitution of the form:

$$p_1(x, t) = p_a(x) e^{i\omega t}$$
$$p_2(x, t) = p_b(x) e^{i\omega t}$$

resulting in the equations:

$$(EI, s^4 - m_k \omega^2 + ic_k \omega + k_w)p_a + (ic_\omega \omega - k_w)p_b = F_0$$  \[2.25\]

$$(EI, s^4 - m_l \omega^2 + ic_\omega \omega + ic_k \omega + k_b + k_w)p_b + (-ic_\omega \omega - k_w)p_a = 0$$  \[2.26\]

The expressions resulting for $p_a$ and $p_b$ can be expressed in identical form to 2.8 and 2.9. The denominator in this case however results in an eighth order polynomial in $s$ having four roots with positive real parts. The solution for the roots and subsequent inversion is exactly as for the vertical case already described.
2.8 Lateral Harmonic Excitation of Railway Tracks

An example of the results which are obtained for the case of a railway track excited laterally by a unit harmonic load can be produced in the same way as for the vertical case. In the case of a vehicle loading situation two quite different loads may be applied at each rail head, however the total response can be obtained by superposition in the linear case.

The parameters used are presented below which once again represent the values thought to be typical for railway track. The parameters for the upper layer of the model representing bending of the rail head are derived from track in which the rail foot was clamped as at a switch and the way in which these were obtained is described in Appendix III.

\[
\begin{align*}
EI_h &= 0.266 \times 10^6 \text{ Nm}^2 \\
EI_l &= 1.44 \times 10^6 \text{ Nm}^2 \\
m_h &= 24.7 \text{ kg/m} \\
m_t &= 587 \text{ kg/m} \\
c_w &= 7.25 \times 10^3 \text{ Ns/m}^2 \\
c_b &= 0.167 \times 10^6 \text{ Ns/m}^2 \\
k_w &= 53.2 \times 10^6 \text{ N/m}^2 \\
k_b &= 25.8 \times 10^6 \text{ N/m}^2 \\
F_0 &= 1 \text{ N}
\end{align*}
\]

The frequency response function obtained using these parameters is shown in Fig 2.20, this shows a response somewhat similar to the vertical two layer model which is to be anticipated. The frequency of the first resonant peak is found to be rather lower than the vertical case due to the lower lateral stiffness, also the second resonance is apparently more lightly damped and shows a large amplitude of motion of the rail head relative to the foot. This is obviously due to the small amount of damping assumed for the head relative to foot motion.

Fig 2.22 is a repeat calculation of the combined response of the track with a wheelset for which case the PSD of the lateral track geometry is taken to be:

\[
G(\lambda) = \frac{3.8}{10^6} \left( \frac{1}{1 + (\lambda/25)^6} \right)^2 \lambda^{0.9} + \frac{54}{10^{12}} \lambda^2 \text{ m}^2/\text{m}
\]

This again was taken from a large sample of measured track profiles and represents the random component as shown in Fig 2.21. Once again the response to discrete irregularities as represented by the peaks in the spectrum is studied later by means of a time domain solution.
The mechanism of lateral contact of wheel and rail is obviously more complicated than the vertical case and also non-linear. For this reason the results obtained from this exercise are only illustrative, however they are reasonably representative of the case in which a wheel is in flange contact with one rail. Under these conditions the wheel in flange contact is forced to negotiate irregularities in the lateral profile while the other wheel produces little force due to lateral oscillations, because the contact angle is small.

The results obtained are once again rather similar to the vertical case with the largest response occurring at the fundamental natural frequency in which all the components vibrate in phase. It is interesting though that the second natural frequency in which the two layers of the track vibrate in anti-phase becomes completely subdued with the wheelset present and obviously becomes...
very highly damped. The highest natural frequency is also beyond the frequency range considered and likely to be well damped.

The predicted PSD of the track response shows all the ballast force occurring at a frequency around 20 Hz with very little excitation above 100 Hz.

This apparent simplicity of the overall response, therefore, suggests that some simplification of the model may be possible. An obvious possibility is a lumped parameter model derived in a similar manner to the one obtained for vertical track modelling. A second alternative is to replace the rail head to foot flexibility which forms the upper layer of the model by a lumped spring/damper
Fig. 2.22 Response of Two Layer Lateral Track Model for Typical Track Profile
system. The main advantage of this would be in the case of the finite element model which will be described for the prediction of the forces and geometric deterioration of track, in which it is desirable to predict the forces at individual sleeper positions. The effect of this in the model is that the contact spring and rail flexibility spring are just treated as one spring having the appropriate series stiffness value. The results of this model and a single degree of freedom lumped parameter model calculated in the same way as for the vertical case are shown in Fig 2.23 and 2.24 respectively. It can be seen that a good representation of the power spectra of both the wheel/rail and sleeper/ballast force is obtained in both cases. This is because the inertia effects are quite small at the low frequencies at which the response occurs.

2.9 Finite Element Model of a Track Switch

The models developed so far assume that the track parameters are uniform. One of the main locations on the railway track where high lateral forces occur, however, is the switch or turnout.

Because of the practical difficulties of manufacturing a turnout which gives a tangential departure from the straight there is always a kink at this point, (the switch toe) the angle of which is defined as the entry angle. Also throughout the switch from this point onwards there is a deficiency of cant as switches are usually laid out on the flat. Flange contact can therefore occur for long periods and high lateral forces are likely to result which will be influenced by the dynamic response of the track.

The results of the high lateral forces which occur in these situations are high wear, damage to components and a loss of geometry which requires regular maintenance. A model was therefore required to predict the forces which occur at switches. Unfortunately the track parameters at a switch do not remain constant due to the varying construction of the component parts. The method chosen to handle this more complicated problem, therefore, was the finite element method which is capable of handling the case of varying parameters.

Fig 2.25 shows a diagram of a typical switch. As a wheelset negotiates the switch toe the outer rail is at this point pressing against the through straight rail (the stock rail). This rail is held down by means of baseplates onto timber bearers but is cut away on one side to accomodate the moving rail (the switch rail). The switch rail at this point sits on lubricated slide base plates and is held
Fig. 2.23 Response of Simplified Two Layer Lateral Track Model
Fig. 2.24 Response of Lumped Parameter Two Layer Lateral Track Model
Fig. 2-25 Diagram of Typical Switch

- Timber Bearers
- Heel Blocks
- Slide Baseplates
- Planing Length
- Switch Toe
against the stock rail by the switch locking mechanism. The slide base plates continue for sufficient distance to allow the switch rail to bend, which depends upon the length of switch. At points where the rail does not rest against the stock rail, however, the switch rail is supported by a small upstand on the slide baseplate which supports the rail foot laterally, and a lateral support on the rail web.

Beyond this point the rails become very firmly fixed by heel blocks. These are solid castings which are bolted through firmly holding both rails. Conventional rail fastenings are then used from this point up to and beyond the crossing nose. It can be expected, therefore, that the lateral stiffness of the rails would vary significantly through the switch. The stiffness of the bearers in the ballast would also be expected to vary to a certain extent with distance as the bearers become longer, however this effect is neglected. The effective lateral stiffness would also be governed by the lateral rail bending rigidity which also varies through the switch.

A schematic diagram of the model developed is therefore shown in Fig 2.26. Both rails are assumed to remain parallel, (i.e. no additional local bending of the rails takes place between sleepers) this means that a single beam is sufficient to model the lateral bending. Lumped masses are provided at each bearer position and a spring and damper are also attached at these points. Torsional rotational effects at the sleeper positions are also neglected. Obviously this type of model has to be of finite length but end conditions are provided to match those of a static semi-infinite beam on Winkler foundation and the length of model is chosen to be sufficiently large to avoid end effects. More sophisticated boundary conditions could be provided to model the infinite beam more closely but these were not considered to be justified for this relatively low frequency case.

The nodes for the discretisation are therefore chosen to be at bearer positions. All the mass is assumed to be lumped at the bearer position, although this is not a requirement it is approximately true for the case of a railway track and facilitates inversion of the mass matrix which is a requirement of the solution method. Each mass is therefore calculated as the sum of the baseplate and sleeper masses plus the mass of the rails in each adjacent half span. The model is chosen such that the two sides of the vehicle are equally spaced about the centre of the model. As the leading or trailing wheelset enters another span a new bay is added at the front or taken from the back as appropriate meaning in
Fig. 2.26 Finite Element Switch Model
effect that the model moves with the vehicle. Furthermore the model also has
ultimate limits beyond which constant parameters exist and a lumped
parameter model is considered to be satisfactory. A transition of two sleeper
bays is provided from one model to the other, in which the response is a weighted
average of the response of both models.

A general element of the model (i) having a length of one sleeper bay is
therefore shown in Fig 2.27, having applied end moments \( M_{i-1}, M_i \) and shear
forces \( V_{i-1}, V_i \) respectively.

![Fig. 2.27 Forces on a General Span i](image)

Thus the force balance equations for the element are shown as follows (in
matrix notation):

\[
\begin{bmatrix}
V_{i-1} \\
N_{i-1} \\
V_i \\
M_i
\end{bmatrix}
= \begin{bmatrix}
\frac{12EI_i}{l^3} & \frac{6EI_i}{l^2} & -\frac{12EI_i}{l^2} & \frac{6EI_i}{l^2} \\
\frac{6EI_i}{l^2} & \frac{4EI_i}{l} & -\frac{6EI_i}{l} & \frac{2EI_i}{l} \\
-\frac{12EI_i}{l^3} & -\frac{6EI_i}{l^2} & \frac{12EI_i}{l^2} & -\frac{6EI_i}{l^2} \\
\frac{6EI_i}{l^2} & \frac{2EI_i}{l} & -\frac{6EI_i}{l^2} & \frac{4EI_i}{l}
\end{bmatrix}
\begin{bmatrix}
\gamma_{i-1} \\
\theta_{i-1} \\
\gamma_i \\
\theta_i
\end{bmatrix}
\]

\[ i.e. \ P = [K]\delta \]
The stiffness matrix \([K]\) can be assembled, with those of each of the other sleeper bays, into an overall stiffness matrix, to which the stiffness of the lateral spring at each sleeper and the end stiffness can also be added. From analysis of a semi-infinite beam on Winkler foundation it is straightforward to show that the stiffness matrix to be added at the ends is:

\[
\begin{bmatrix}
  k_b & k_b \\
  \beta & 2\beta^2 \\
  k_b & k_b \\
  2\beta^2 & 2\beta^3
\end{bmatrix}
\]

Mass and damping matrices are also formed but these are simply diagonal matrices having values only in the odd diagonal locations (i.e. no torsional mass or damping). This results therefore in the familiar equation:

\[
[M]\ddot{Y} + [C]Y + [K]Y = Q
\]

**Calculation of the Force Vector:** In order to calculate the force vector \(Q\) it is necessary to evaluate a point load on a general span (Fig 2.28). The force vector can be calculated such that the amount of work done by the equivalent load vector is equal to the work done by the point load as described by Zienkiewicz (51) resulting in the expression:

\[
Q = -\int_0^l [N]^T \{q\}
\]

Where \([N]\) is the matrix which relates general intermediate displacements to displacements at the nodes and \(\{q\}\) is the vector of forces in the direction of each degree of freedom at the nodes i.e.:

\[
[N] = \\
\begin{bmatrix}
  \frac{2x^3}{l^3} - \frac{3x^2}{l^2} + 1, & \frac{x^3}{l^2} - \frac{2x^2}{l} + x, & -\frac{2x^2}{l^3} + \frac{3x^2}{l^2} & \frac{x^3}{l^2} - \frac{x^2}{l} \\
  \frac{6x^2}{l^3} - \frac{6x}{l^2}, & \frac{3x^2}{l^2} - \frac{4x}{l} + 1, & -\frac{6x^2}{l^3} + \frac{6x}{l^2}, & \frac{3x^2}{l^2} - \frac{2x}{l}
\end{bmatrix}
\]
and \( \{q\} = \begin{bmatrix} F \\ 0 \end{bmatrix} \)

Hence \( Q = [N]_{x=a}^T \{q\} \)

The solution of the equations of motion for the finite element track model is by numerical time stepping integration. A frequency domain solution would be possible for this model but as it is used in the interaction with the non-linear vehicle model via non-linear contact conditions this method cannot be used.

![Fig. 2.28 Point Load on General Span](image)

A numerical solution to the response of the normal modes of the structure would also be possible but this precludes the use of non-linearities in the model which may be required at a later date.

The time stepping solution used is Euler's method. The main reason for using this particular method was that it was also implemented in the lateral vehicle dynamics model adopted and is simple to introduce. The method has also become known as the 'pantograph' method within British Rail because of its use in a program to predict the response of pantographs and catenary wires on electric locomotives. Work has also been done to verify it's accuracy and stability (52).

The equations used in the time stepping procedure are as follows:

\[
\dot{y}_{t+\Delta t} = \dot{y}_t + \ddot{y}_{t+\Delta t} \Delta t
\]
\[ y_{t+\Delta t} = y_t + \dot{y}_t \Delta t \]

As constant acceleration and a linear change of velocity during the time step are implied by the first equation a third term might be expected in the second equation. However it is found that this term does not reduce the error and is therefore neglected.

For the general matrix equation below:

\[ [M]\ddot{Y} + [C]\dot{Y} + [K]Y = Q \]

it is possible to calculate \( \ddot{Y} \) at a new time step i.e. :-

\[ \ddot{Y}_{t+\Delta t} = [M]^{-1}(Q_{t+\Delta t} - [C]\dot{Y}_{t+\Delta t} + [K]Y_{t+\Delta t}) \]

this enables the cycle to be completed and a new cycle to begin.

It can be seen that it is necessary to find the inverse of the mass matrix which need only be calculated once. This is particularly easy if the mass matrix has only diagonal terms.

A maximum time step of \( \Delta t = 2/\omega \) is usually found to satisfy the criteria of stability and accuracy where \( \omega \) is the largest natural frequency of the system. In the case of the interaction of two separate models (i.e. vehicle and track) it is not easy to establish the largest natural frequency, particularly when non-linearities exist. Furthermore in the contact equations between wheel and rail spurious roots exist with high effective natural frequencies. The effect of these is to demand a finer time step than might be anticipated. However as these roots do not participate in the response the fine time step required does result in good accuracy of the predictions of the lower frequencies whenever the calculation is stable.
3. MODELLING OF VEHICLE/TRACK INTERACTION

3.1 Vehicle Model for Vertical Response

The typical vehicle which will be considered is illustrated in Fig 3.1, this is a four axle bogie vehicle having ten vertical degrees of freedom. By the choice of appropriate parameters this model is capable of representing a wide range of rail vehicles including locomotives, passenger coaches and freight vehicles. Locomotives and passenger coaches generally have fairly sophisticated suspensions, with primary and secondary springs assisted by hydraulic dampers having an essentially viscous characteristic. Freight vehicles, however, are more usually provided with primary or secondary springs only and damping is achieved by means of sliding friction surfaces, thus providing a relatively cheap suspension.

In order to consider both cases, therefore, general damping forces are considered in the model and separate equations developed for the viscous and friction damped cases. Furthermore a component of resilience is considered in
series with springs and dampers which is provided as a design feature in some vehicles, or may result due to local deformations of body components. This stiffness is generally quite large but can modify damping forces at high frequencies and is shown to be particularly important in the friction damped case. Although this feature is not considered in the general derivation of the vehicle/track interaction equations below, it is included and described in full at a later stage.

It is assumed that the body and bogie frames are rigid in bending and that no coupling exists between the vertical, and the lateral and longitudinal modes of vibration of the vehicle. Both rails are also assumed to have the same vertical profile thus meaning that roll components can be neglected, and the wheelsets are assumed to remain in contact with the rails at all times.

By application of Newton's second law the equations of motion for the vehicle on the track can be derived as follows:-

For the body:-

\[
M_b \ddot{z}_b + k_s (2z_b - z_{fl} - z_{ft}) + Q_{st} + Q_{st} = 0
\]

\[
I_b \ddot{\phi}_b + b_k s (2\phi_b - z_{fl} + z_{ft}) + b_s Q_{st} - b_s Q_{st} = 0
\]

For the leading bogie:-

\[
M_{fl} \ddot{z}_{fl} + Q_{pl1} + Q_{pl2} + k_p (2z_{fl} - z_{wl1} - z_{wl2}) - Q_{st} - k_s (z_b + \phi_b - z_{fl}) = 0
\]

\[
I_{fl} \ddot{\phi}_{fl} + a_k_p (2a_{fl} - z_{wl1} + z_{wl2}) + a_s Q_{pl1} - a_s Q_{pl2} = 0
\]

\[
M_{wl1} \ddot{z}_{wl1} - Q_{pl1} - k (z_{fl} + a_{fl} - z_{wl1}) = -P_{fl1}
\]

\[
M_{wl2} \ddot{z}_{wl2} - Q_{pl2} - k_i (z_{fl} - a_{fl} - z_{wl2}) = -P_{fl2}
\]

and similarly for the trailing bogie.

The damping forces \(Q\) in the viscous damping case can be represented as follows:-

\[
Q_{st} = c_s (\dot{z}_b + \dot{\phi}_b - \dot{z}_{fl})
\]
$Q_{st} = c_{st}(\dot{z}_b - b\dot{\phi}_b - \dot{z}_f)$

$Q_{pt1} = c_{pt1}(\dot{z}_f + a\dot{\phi}_f - \dot{z}_{w1})$

$Q_{pt2} = c_{pt2}(\dot{z}_f - a\dot{\phi}_f - \dot{z}_{w2})$

similarly $Q_{pt1}, Q_{pt2}$. For friction damping:

$$\frac{F_s(\ddot{z}_b + b\ddot{\phi}_b - \dot{z}_f)}{\dot{z}_b + b\dot{\phi}_b - \dot{z}_f}$$

$$\text{etc.}$$

The above can be assembled to give the following equations for a viscous damped vehicle shown in matrix notation:

$$[M]\ddot{Z} + [C]\dot{Z} + [K]Z = -P$$  

At this point both the displacement vector of absolute displacements $Z$ and its derivatives are unknown as is the wheel rail contact force vector $P$. This latter can be described in terms of the track displacement equations but it is first necessary to relate the dynamic track displacement and wheel displacement to the track irregularity profile.

**Vehicle Displacements Expressed Relative To Track Based Co-ordinates:**

The displacements of the vehicle can be expressed in terms of the co-ordinate system shown in Fig 3.2, from which the following relationships can be derived for the relative displacements $\eta$ and $\epsilon$:

$$\eta_b = z_b - (z_{u1} + z_{u2} + z_{t1} + z_{t2})/4$$

$$\epsilon_b = \phi_b + (z_{u1} + z_{u2})/4 - (z_{t1} + z_{t2})/4$$

$$\eta_f = z_f - (z_{u1} + z_{u2})$$

$$\epsilon_f = \phi_f - (z_{t1} - z_{t2})/2$$

$$\eta_{w1} = z_{w1} - z_{t1}$$
\[
\eta_{wl2} = z_{wl2} - z_{lt2}
\]

also \( P = f(\eta_w, \dot{\eta}_w, \ddot{\eta}_w) \)

and similarly the trailing bogie displacements.

The four relative wheelset displacements \( \eta_{wl1}, \eta_{wl2}, \eta_{wt1}, \eta_{wt2} \) are now equal to the dynamic track displacement plus any deformation of the wheel/rail contact locally.

The above expressions can now be substituted into equation 3.1 to eliminate the absolute displacements and yield a new equation in terms of the relative displacement vector \( Z_r \), where \( Z_r = [\eta_b, \dot{\eta}_b, \ddot{\eta}_b, \ldots \text{etc.}] \)

\[
[M]Z_{\ddot{r}} + [C]Z_{\dot{r}} + [K]Z_r = -P - [M]Z_t \quad -3.2
\]

This is similar in form to equation 3.1 but has an additional forcing term involving the track irregularity co-ordinate accelerations. The force vector \( P \) can also be related to the dynamic wheelset displacements via the track model as described earlier, the displacements being consistent with the new co-ordinate system.

Expressing the equations in relative displacement terms has a particular advantage when a computer based numerical solution is considered, this being

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Fig. 3.2 Co-ordinate System Used for Vertical Vehicle/Track Interaction

The above expressions can now be substituted into equation 3.1 to eliminate the absolute displacements and yield a new equation in terms of the relative displacement vector \( Z_r \), where \( Z_r = [\eta_b, \dot{\eta}_b, \ddot{\eta}_b, \ldots \text{etc.}] \)

\[
[M]Z_{\ddot{r}} + [C]Z_{\dot{r}} + [K]Z_r = -P - [M]Z_t \quad -3.2
\]

This is similar in form to equation 3.1 but has an additional forcing term involving the track irregularity co-ordinate accelerations. The force vector \( P \) can also be related to the dynamic wheelset displacements via the track model as described earlier, the displacements being consistent with the new co-ordinate system.

Expressing the equations in relative displacement terms has a particular advantage when a computer based numerical solution is considered, this being
that the numbers being handled remain of similar magnitude. When dealing with absolute displacements subtraction of large numbers to obtain small suspension displacements can result in significant round off errors.

3.2 Formulation of the Equations for Vertical Vehicle/Track Interaction

Lumped Parameter Track Model: A solution of equation 3.2 is required for a defined vehicle, speed and track geometry. With the solution of the track models so far being in the frequency domain it would be possible also to solve the vehicle equations in the frequency domain. However the solution for friction damping cannot be obtained in the frequency domain and also the wheel rail contact relationship is non-linear. For these reasons a numerical time stepping method of solution was chosen as being most suitable.

It is required first of all to compare the results of the lumped and distributed parameter track models. For the former the Hertzian contact stiffness, which is high, will be neglected for the reasons studied in Chapter 2. For a single degree of freedom model at each wheel, therefore, with lumped parameters as derived in Appendix I. It is possible to describe the equations of motion as follows:

\[ M_{f_t} \ddot{z}_t + C_f \dot{z}_t + K_f z_t = P_e \]

noting that at the leading wheelset \( z_t = \eta_{wtt}, \text{etc} \)

This can also be described as a matrix equation describing all wheelsets using the same displacement vectors as equation 3.2:

\[ [M]_L \ddot{Z}_r + [C]_L \dot{Z}_r + [K]_L Z_r = P \]

which can now be substituted into equation 3.2 to yield:

\[ ([M] + [M]_L) \ddot{Z}_r + ([C] + [C]_L) \dot{Z}_r + ([K] + [K]_L) Z_r = -[M] \ddot{z}_t \]

The total matrices now obtained on the left hand side are the same as the overall mass, stiffness and damping matrices which would have been obtained by assembling the complete system (including the track) normally. The displacement vectors however would have related to absolute displacements. The right hand side expression is the vehicle mass matrix multiplied by the
second derivative with respect to time of the track input vector. It should be noted that the lumped track mass components do not appear in the forcing term.

**Distributed parameter track model:** Solution of the vehicle/track interaction problem when a more complicated distributed parameter model is used is achieved by means of the unit impulse response and convolution integral. The track displacement at any wheel is given by:

\[ z = \int_0^\infty I(t) P_c(t-\tau) d\tau \]

Where \( I(t) \) is the unit impulse response function and \( P_c(t-\tau) \) the wheel/rail contact force at time \( (t-\tau) \). Furthermore it is possible to consider the effect of adjacent wheels viz:

\[ z_1 = \int_0^\infty [I_{11}(t) P_{c1}(t-\tau) + I_{12}(t) P_{c2}(t-\tau)] d\tau \]

Where \( I_{jk}(\tau) \) is the impulse response function at \( j \) due to a unit load at \( k \). This is only incorporated to include the effect of the adjacent wheel of a bogie and is in fact shown to have a very minor effect on the vehicle/track combinations considered to date.

The impulse response function, when used with the time stepping integration technique to be employed is required as a series of discrete estimates with spacing equal to the time step used. This is easily obtained from the complex frequency response function derived in Chapter 2 by means of the discrete Fourier transform which relates the two functions as described for example in Ref. 50. The impulse response function is the inverse transform of the complex frequency response and it therefore given by:

\[ I(k,\Delta t) = \sum_{j=0}^{N-1} z_0(j,\Delta f) e^{(12mj/kN)} \]

This discrete Fourier transform has finite limits in frequency and time. The upper frequency limit can be chosen from the analysis of Chapter 2 depending upon the model being considered and whether for example wheel/rail force or sleeper/ballast force is of interest. Choice of a suitable time limit at which to truncate the impulse response function is also necessary and will
depend upon the amount of damping in the track model. It will normally be acceptable to truncate the response at the point at which the amplitude of vibration has reduced to say 5% of the initial amplitude. The frequency interval $\Delta f$ is chosen to give sufficient resolution of the frequency response function.

Equation 3.4 therefore represents a method of calculating the track displacement due to a known force history which is all that is required to implement the time stepping integration method. The force however needs to be related to the relative displacement of wheelset and track. At this point the non-linear contact stiffness can be introduced. This is explained below in the description of the time stepping method which has been used. A method of solving the vehicle equations which has particular advantages for the problem being considered is the component element method (53). This was found to be particularly suitable because of the way the dynamic equations are handled, allowing the configuration of the model to be easily changed. This method was implemented along with central difference equations to carry out the numerical integration. This approach proved to be quite straightforward to implement and well behaved with regard to numerical stability.

### 3.3 Solution of the Vehicle/Track Equations by the Component Element Method.

The preceding analysis described how the vehicle and track models could be combined and the effect of the irregular track profile included. A particular method of solution of the equations which was found to work very well for the models considered was the Component Element Method. When using this method it is not necessary to assemble directly the system matrices required by equation 3.2, instead Newton's second law is applied individually to each component mass at each time step. The assumption is made that applied forces (including spring and damper forces) do not change during a time step. This enables the acceleration of the individual mass, which will remain constant for the duration of a time step, to be calculated from the sum of the applied forces. Finite difference equations can then be used to calculate the displacements which would result at the end of a time step and a modified set of applied forces calculated in order to repeat the cycle.

The equations of motion are effectively created for the model by defining the so called coupling matrix which is a two dimensional matrix defining the
inter-relationship between each mass and each force producing element (i.e. spring, viscous damper or friction damper)

The calculation therefore proceeds as follows, given a mass $m$ upon which total forces $Q$ act at time $T$:

$$m \ddot{z}_T = Q_T$$

Hence $\ddot{z}_T = \frac{Q_T}{m}$

A central difference equation relating acceleration and displacement can then be written as follows:

$$\ddot{z}_T = \frac{(z_{T-1} - 2z_T + z_{T+1})}{(\Delta t)^2}$$

Hence $z_{T+1} = 2z_T - z_{T-1} + \ddot{z}_T(\Delta t)^2$

thus the displacement at a new time step can be calculated given a knowledge of the total forces acting upon any individual mass and the displacements of the mass at the two preceding time steps. To complete the time stepping cycle it is therefore necessary to be able to derive the force in each force element from its displacement (or velocity which can also be calculated) and sum all the forces acting upon any individual mass. The force elements used in the vehicle models are presented below:

i) Spring: For spring elements the force is given by:

$$P = k x_T$$

where $k$ is the spring constant and $x_T$ is the displacement across the spring at the current time step.

ii) Viscous Damper: For a viscous damper the force is given by:

$$P = c \dot{x}_T$$
where \( c \) is the viscous damping rate and \( x_T \) is the velocity across the damper. The velocity can be derived from the displacement history by the backward difference expression:

\[
\dot{x}_T = \frac{(3x_T - 4x_{T-1} + x_{T-2})}{2\Delta t}
\]

(iii) Friction Damper: The friction damping elements used in railway vehicles are assumed to display a characteristic as shown in Fig 3.3.

![Friction Damper Characteristic](image)

Fig 3.3 Friction Damper Characteristic

The origin may have any initial position \((d_f)\) due to previous displacements which have not returned it to zero. As load is applied the damper is initially assumed to follow the characteristic of a stiff linear spring. This is due to flexibility of the damper mountings or any of the series components which may be deliberately or unavoidably present.

When the damper load reaches the limiting value of the friction, \( F \), sliding of the friction surfaces occurs and the load is then assumed to remain constant. If the loading cycle therefore produced a path along OA, a subsequent removal of the load would result in the path AD being followed. This would result in a new temporary origin D and \( d_f \) would become the distance \( O_aD \). Loading on the negative cycle would also produce a similar effect.

Different force expressions are therefore required for each of the different regimes viz:-
\[ P = F \text{ if } k_f(x_T - d_f) > F \]

also if this condition is satisfied a movement of the origin takes place.

\[ d_{f(T+1)} = d_{f(T)} - \frac{F}{k_f} \]

A similar limiting condition is also required for the negative load case.

(iv) Series Stiffness Element: Another feature included in the vehicle model is a spring in series with viscous damping elements. These have the effect of reducing damper forces at high frequencies and are provided either due to local stiffness of the vehicle structure or, particularly with passenger vehicles, as a deliberate feature usually in the form of rubber bushes. The arrangement is illustrated in Fig 3.4.

![Viscous Damper with Series Stiffness](image)

**Fig. 3.4 Viscous Damper with Series Stiffness**

Resolving the forces in the line of force \( P \):

\[ c(x_1 - x_3) = k(x_3 - x_2) = P \]

\[ giving \ x_3 = \frac{P}{k} + x_2 \]

Using the backward difference expression 3.5:-
\[
\dot{x}_3 = \frac{(3x_{3(T)} - 4x_{3(T-1)} + x_{3(T-2)})}{2\Delta t}
\]

similarly \(\dot{x}_1\)

\[
\therefore P = \frac{c(3(x_1(T) - x_{3(T)}) - 4(x_1(T-1) - x_{3(T-1)}) + x_1(T-2) - x_{3(T-2)})}{2\Delta t}
\]

Substituting for \(x_3\) gives:

\[
P = \frac{c(3(x_1(T) - x_{3(T)}) - \frac{P}{k}) - 4(x_1(T-1) - x_{3(T-1)}) - \frac{P(T-1)}{k} + x_1(T-2) - x_{3(T-2)}) - \frac{P(T-2)}{k}}{2\Delta t}
\]

Collecting terms in \(P\) and substituting for \(X = x_1 - x_2\) gives:

\[
P = \frac{c(3X_T - 4(X_{T-1}) - \frac{P(T-1)}{k}) + X_{T-2} - \frac{P(T-2)}{k}}{2\Delta t + \frac{3c}{k}}
\]

Thus the force in the spring damper combination can be calculated from the total displacement and the force at the current and previous two time steps.

(v) Wheel Rail Contact: The deflection in the wheel/rail contact zone is assumed to obey the theory of Hertz which results in a relationship for displacement in terms of the force as follows:

\[
P = \left(\frac{x}{G_H}\right)^{22}
\]

where \(G_H\) is the Hertzian contact coefficient which can be calculated (11). For typical wheel/rail contact conditions \(G_H\) has a value of around \(2.95 \times 10^{-8}\) m/N^{22}. 
Coupling Matrices: The coupling matrix as referred to earlier describes the coupling between force elements and masses for each degree of freedom. The coupling matrix for the combined vehicle/lumped track model is shown in Fig 3.5.

With a matrix described in this way it is quite simple to change the vehicle model or suspension type. The computer program developed for this study therefore assembles the coupling matrix according to whether the vehicle has primary or secondary suspension (or both in the example as above) and defines the number of degrees of freedom and number of elements accordingly. The total number of elements used to describe the dynamic model has been kept to a minimum by grouping springs and dampers together such that one element describes a group of components. Only three different types are required namely a spring with parallel viscous damper and series stiffness, a spring with parallel friction damper and series stiffness, and a non-linear spring for wheel/rail contact. The track spring and damper in the case of the lumped model is treated as the first type with large series stiffness.

3.4 Description of the Track Profile

The most obvious way to describe the track profile would seem to be as a series of x and z co-ordinates (preferably at constant separation in the x direction). This is also the most practical way to record track profiles in the field and equipment exists to do this. There are some potential problems with such a method, however, which will be investigated.

Firstly as shown in equation 3.2 the input to the model is the second derivative of the profile with respect to time (which can be derived directly from the track curvature for a vehicle running at constant speed). Some form of interpolation method is therefore required to derive the curvature at intermediate points.

A second problem with sampling the track profile at discrete intervals is that this will impose some upper frequency (or shortest wavelength) limit to the input spectrum and this would need to be consistent with the valid frequency range of the vehicle/track model. Furthermore at the short wavelengths the irregularities are of smaller amplitude (although possibly still of significant
Fig. 3.5 Elements of Lumped Parameter Vehicle/Track Model with Coupling Matrix
curvature amplitude) and extreme measuring accuracy is called for if these wavelengths are important.

**Interpolation of the Track Co-ordinates:** Numerous methods exist for interpolation. Spline methods can be used, however these require all the points to be fitted at once and the curve can tend to oscillate between the data points, this is clearly not acceptable. Methods also exist to interpolate between just a few of the points, however these cause discontinuities in the first and second derivatives when one of the sample points is passed. The following method was therefore developed to try to overcome these problems.

![Diagram of curve fitting by overlapping second order curves](image)

**Fig. 3.6 Curve Fitting by Overlapping Second Order Curves**

For an intermediate position between two sample points a second order curve is fitted to two overlapping groups of three points. These are as shown in Fig 3.6. The result is that two estimates are made of the intermediate point, the required value is then taken to be a weighted average of the two estimates as follows:

The two estimates (with the centre point taken as a datum) are:

\[ Z_{t1} = a_1(x-x_n)^2 + b_1(x-x_n) + c_1 \]  \hspace{1cm} (3.6)

\[ Z_{t2} = a_2(x-x_{n+1})^2 + b_2(x-x_{n+1}) + c_2 \]  \hspace{1cm} (3.7)

in the case of data equally spaced at an interval \( l \), substituting in equation 3.6. for the unknown co-ordinate values gives:
\[ z_{n-1} = a_1 l^2 - b_1 l + c_1 \]
\[ z_n = 0 + 0 + c_1 \]
\[ z_{n+1} = a_1 l^2 + b_1 l + c_1 \]

which can be solved to give:

\[ a_1 = \frac{-2(z_n - z_{n-1} - z_{n+1})}{2l^2} \]
\[ b_1 = \frac{-z_{n-1} - z_{n+1}}{2l} \]
\[ c_1 = z_n \]

similarly \( a_2, b_2, c_2 \) can be obtained and a weighted average of the two estimates taken, giving:

\[ Z_t = \frac{(x - x_n)}{(x_{n+1} - x_n)} (z_{t2} - z_{t1}) + z_{t1} \]

from which the second derivative with respect to \( x \) can also be derived and is shown to be:

\[ \frac{d^2 Z_t}{dx^2} = 2a_1 - 6(x - x_n) \frac{(a_1 - a_2)}{(x_{n+1} - x_n)} - 2 \frac{(b_1 - b_2)}{(x_{n+1} - x_n)} \]

The second derivative with respect to time can then be easily derived for a vehicle travelling at constant velocity.

The method outlined above results in a curve having a continuous first derivative but which can still have steps in the second derivative, which is shown above to vary linearly between sample points. The effect of these steps is to introduce a component of the forcing function having a wavelength equal to twice that of the sample points and the shorter wavelength harmonics thereof.
This effect is very small in practise, however, and is found not to have a significant effect on the results.

**Choice of time step and data spacing:** The numerical methods used for the step by step integration of the dynamic equations by the central difference method require a minimum time step for stability and accuracy. A guide figure is usually taken as $\Delta t = 2/\omega$ (53) where $\omega$ is the highest natural frequency of vibration of the system. This can be verified in the model by trial and error to see if reducing the time step further causes the results to change. A further consideration with the model described above is the spacing of the sample points which itself has two aspects, namely:

i) The data points need to be sufficiently finely spaced to represent all the frequencies to which the vehicle/track system will respond.

ii) The time step needs to divide the space between sample points into sufficient intervals such that the profile is accurately represented. This is because the curvature as estimated is assumed to remain constant during a time step.

In order to resolve point (i) Fig 3.7 shows the transfer function calculated for the interpolation function derived above. As the wavelength of the true profile becomes small compared to the spacing of the discrete estimates an error is introduced due to the fact that the sample points may not coincide with or be close to, the peaks in the profile. The curve fitting routine compensates for this to some extent but not completely. The amplitude of the estimated profile therefore varies as the estimates move from being coincident with the peaks to non-coincident (unless the sample spacing is exactly a harmonic of the wavelength in which case the relationship would remain fixed). The transfer function can therefore be considered to cover a range as indicated by the minimum and maximum values plotted in Fig 3.7 and the mean amplitude will be half way in between. This aliasing effect also causes some of the content of the shorter wavelength component of the profile to appear as a longer wavelength and below the point at which $\lambda/l = 2$ in Fig 3.7 all the energy is 'folded' to a longer wavelength. This latter aspect does not generally cause a problem due to the fact that the amplitude is reducing with wavelength. To accurately model the vehicle response it is therefore necessary to ensure that the maximum frequency at which a significant response of the model occurs is adequately represented. This is obviously also dependent on the vehicle speed. From the frequency
domain analysis of Chapter 2 for the lumped parameter model the wheel/rail force is found to be below 10% of the maximum resonant value above about 75 Hz for average parameters.

From Fig 3.7 a figure of 3.4 points per cycle is therefore suggested for a mean error of 10% of the amplitude at this value, which corresponds to a measurements interval of 0.08 m for a frequency of 75 Hz at 20.1 m/sec. A value of 0.1 m is used in practise and has been found to be satisfactory for most cases, particularly as higher speeds are normally considered. The required measurement accuracy can be implied by considering a typical spectrum of vertical track geometry as shown in Chapter 2 from which it can be calculated that the r.m.s amplitude of wavelengths in the range 0 - 0.2 m is 0.005 mm. An
allowable error of 10% at this wavelength would therefore require an accuracy of measurements of around 0.001 mm to ensure that spurious waves at this amplitude were not introduced.

Large time step - poor resolution

True curvature

Discrete estimates

Small time step - improved resolution

Fig. 3.8 Interpolation of Curvature Between Sample Points

The problem referred to in point (ii) is illustrated in Fig 3.8. Clearly if the time step is large compared to the true spacing of the sample points, significant errors could result and the perceived track shape would be rather different to the intended one. The true profile as seen by the vehicle can be calculated by double integration of the stepped curvature profile shown in Fig 3.8 and compared to
the input profile. It is found that a time step of one fifth of the time spacing of the points is sufficient to give accurate results.

When choosing the time step, therefore, for the interaction problem it is necessary to consider both the stability and accuracy of the dynamic equations and also accurate representation of the input irregularity shape. It is usually found that a time step of 0.5 msec is suitable for quite a wide range of cases including non-linear vehicles.

Defining the Track Profile for Infinite Track Models: When an infinite model is considered and the Hertzian contact effects are included, the natural frequencies of the vehicle track system extend much higher as has been demonstrated. Clearly the method described above would not be suitable unless extreme measuring accuracy used at very small longitudinal spacings was used.

These models are therefore restricted to design cases of track profile where the track shape can be defined by functions. The obvious example of this is the dipped rail weld or joint where an angular irregularity is found. A typical shape for this type of irregularity is as shown in Fig 3.9a, having a parabolic approach to the weld and an angular discontinuity at the weld for which the curvature profile is also shown (Fig 3.9b). The second derivative with respect to time on the parabolic part is given by:

\[
\frac{d^2z}{dt^2} = 2\left(\frac{d}{l^2} - \frac{a}{2l}\right)v^2 = \text{const.}
\]

The angular change results in an impulse in the curvature of magnitude \(a\). When implemented in the time stepping method (in which the curvature remains constant during the time step) this is accommodated by providing a second derivative with respect to time of \(au/\Delta t\) having a duration of \(\Delta t\).

3.5 A simplified vehicle model

The vehicle model described so far is a full four axle bogie vehicle having ten degrees of freedom. However if the body pitch and bounce frequencies were the same, the sum of these two modes of vibration would have the effect of isolating the two ends of the vehicle and make each end behave as if half the body mass were lumped over each bogie (ie the radius of gyration in pitch equal
to the bogie spacing). In fact for most railway vehicles the geometry is such that this condition is approximately met and it is found that there is little difference in the results obtained by modelling half of the vehicle and thus halving the number of degrees of freedom.

It is found, however, that good agreement is not obtained for friction damped vehicles. This is because if the friction limit is exceeded at one end of the vehicle and not at the other this causes different effective suspension stiffness to be apparent and a non-symmetric condition results. Under these circumstances the pitch and bounce modes do not have the same natural frequency and the simplified model is not valid. For friction damped vehicles, therefore, where the magnitudes of the forces are important rather than just the characteristics of the response, the full model should be used. The simpler model, however, with appropriate parameters can be used to model two axle vehicles with viscous or friction damped suspension by treating this case as a half bogie vehicle with primary suspension only.
This concept of effective suspension stiffness for friction damped vehicles is considered further in the following chapter where the results of the models are studied.

3.6 Vehicle Model for Lateral Response

Use was made of an existing vehicle model (34) which was developed to study the transient curving response of vehicles and from which the vehicle and wheel/rail contact equations described below were obtained. The model was originally developed based upon the assumption of rigid track but as a result of the initial conclusions of this work was modified to include the track resilience. This was achieved by including a simple spring at each wheelset location which was considered adequate for predicting the vehicle response. Extended models of the track were therefore required to model in more detail the response of the track. Fig 3.10 shows a diagram of the model which is of a two axle vehicle. The co-ordinate system, as with the vertical dynamics model is expressed relative to the track profile, in this case to the track centre line. The relationship between the lateral co-ordinates and the contact co-ordinates is shown in Fig 3.11.

As with the vertical model no coupling is assumed between lateral and vertical modes of vibration through the vehicle, however due to the inclined contact at the rail head, vertical forces have a component acting laterally. For this reason the bounce, pitch and roll responses of the vehicle body are used to predict the vertical forces. Modes of vertical vibration having higher frequencies than these are neglected for this application on the basis that they will have only a second order effect on the overall response.

The equations of motion for the vehicle which has eight lateral and three vertical degrees of freedom are therefore as follows:

For each wheelset \((i = 1,2)\):

Lateral:

\[
M \ddot{y}_i = (F_{yi} + F_{ri}) - (P_{yi} + P_{ri})\theta_i + S_i - \frac{M_w v^2}{R_i}
\]

Yaw:

\[
I_w \ddot{\theta}_i = (T_{yi} + T_{ri})\gamma_0 + Q_i - \frac{I_w v^2}{2a} \left( \frac{1}{R_1} - \frac{1}{R_2} \right)
\]
Fig 3.10 Model of Two Axle Vehicle for Lateral Response
Fig. 3.11 Co-ordinate System used for Lateral Vehicle/Track Interaction Models
Rotation:
\[ J_{\omega_{i}}^{\ddot{\omega}_{i}} = (T_{1r_{i}}-T_{1l_{i}})\dot{r}_{0}+(T_{1l_{i}}-T_{1r_{i}})^{\Delta r} \]

Also for the vehicle body:
Lateral:
\[ M_{b_{i}}^{\ddot{\omega}_{i}} = -(S_{1}+S_{2})-\frac{M_{b}v^{2}}{2a} \left( \frac{1}{R_{1}} + \frac{1}{R_{2}} \right) \]

Yaw:
\[ I_{b_{i}}^{\ddot{\phi}_{b}} = -(Q_{1}+Q_{2})-(S_{1}-S_{2})a+Q_{i}-\frac{I_{b_{i}}v^{2}}{2a} \left( \frac{1}{R_{1}} - \frac{1}{R_{2}} \right) \]

Roll:
\[ J_{b_{i}}^{\ddot{\theta}_{b}} = (U_{b_{1}}+U_{b_{2}})+(S_{1}+S_{2})(h_{b}-r_{0}) \]

Vertical:
\[ M_{b_{i}}^{\ddot{\omega}_{i}} = -(V_{1}+V_{2})+M_{b_{i}}g \]

Pitch:
\[ K_{b_{i}}^{\ddot{\theta}_{b}} = (V_{1}-V_{2})a \]

where
- \( S_{i} \) = lateral suspension force
- \( Q_{i} \) = Yaw suspension torque
- \( U_{b_{i}} \) = Roll suspension torque
- \( V_{i} \) = Vertical suspension force
- \( F_{li},F_{ri} \) = Lateral contact forces in plane of rail surfaces
- \( P_{li},P_{ri} \) = Contact forces normal to plane of rail surfaces
- \( 1/R_{i},\theta_{i},z_{ti} \) = Lateral curvature, cross level and vertical track alignment

All at wheelset \( i \)

The vertical contact forces can similarly be expressed as follows:

\[ P_{li} = \left[ M_{\omega_{ti}}\ddot{\omega}_{ti} - \frac{I_{\omega_{ti}}^{\ddot{\omega}_{ti}}}{l_{0}} - (F_{li} + F_{ri})(\theta_{i} - \frac{r_{0}}{l_{0}}) - V_{i} + \frac{U_{i}}{l_{0}} - \bar{M}_{\omega_{ti}}g \right]/2 \]

\[ P_{ri} = \left[ M_{\omega_{ti}}\ddot{\omega}_{ti} + \frac{I_{\omega_{ti}}^{\ddot{\omega}_{ti}}}{l_{0}} - (F_{li} + F_{ri})(\theta_{i} + \frac{r_{0}}{l_{0}}) - V_{i} - \frac{U_{i}}{l_{0}} - \bar{M}_{\omega_{ti}}g \right]/2 \]
Forces in the Plane of Contact: The wheel-rail contact forces in the plane of contact are calculated using the methods described by Elkins and Gostling (25) which enable creepages and hence the lateral and longitudinal forces to be calculated when the wheel-rail contact angles, rolling radius difference and the size and shape of the contact patch are known. These quantities are all non-linear functions of the lateral positions of the wheelset relative to the two rails, and the vertical load. For this application these values are tabulated based upon calculations made from measured wheel and rail profiles. The creepages which take place between wheel and rail at wheelset $i$ are given (in local contact coordinates) by:

\[
\begin{align*}
Longitudinal: \quad y_{1ri} &= \frac{\dot{r}_{ri}}{v} + 1 + \frac{l_0}{R_i} \frac{l_0 \psi_i^*}{v} \\

y_{1li} &= \frac{\dot{r}_{li}}{v} + 1 + \frac{l_0}{R_i} \frac{l_0 \psi_i^*}{v} \\

\text{Lateral:} \quad y_{2ri} &= \frac{(y_i^* + \dot{r}_{ri} \psi_i^*)}{(v \cos \delta_{ri})} \\

y_{2li} &= \frac{(y_i^* + \dot{r}_{li} \psi_i^*)}{(v \cos \delta_{li})} \\

\text{Spin:} \quad \omega_{3ri} &= \frac{\dot{r}_{i} \sin \delta_{ri}}{v} \\

\omega_{3li} &= \frac{\dot{r}_{i} \sin \delta_{li}}{v}
\end{align*}
\]

Where $y_i^* = y_i - r_0 \theta_i$, $\psi_i^* = \psi_i - \psi_{ci}$

With a knowledge of the three components of creepage it is then possible to calculate the forces in the plane of contact (creepage forces) from the theory due to Kalker (24). These are non-linear functions also and are used as a series of non-dimensionalised tabulated values which are interpolated at each time step in the solution.
3.7 Lateral Track Displacements

As described above the lateral forces generated at each wheel are functions of the contact angle between wheel and rail and the creepages. The latter are in turn functions of other wheel/rail contact conditions such as rolling radius difference and size and shape of contact area. These, along with the contact angle, can be calculated for known vehicle/track combinations be they design case or measured profiles. These are also tabulated and interpolated when the calculation is made, typical examples being shown in Fig 3.12. As the forces required, therefore, are all ultimately functions of the relative lateral position.

![Fig. 3.12 Lateral Wheel/Rail Contact Functions from Measured Wheelsets and Track](image-url)
and velocity of wheel and rail, lateral displacement of the track can also be incorporated at this stage. The relative lateral position of wheel and rail can be expressed as:

\[ y = y^* - y_r \]  \hspace{1cm} (3.8)

Where \( y^* \) is the position of the wheelset relative to the undisturbed track centreline and \( y_r \) is the dynamic displacement of the railhead. A similar expression can also be written for the relative lateral velocity, however it is discovered as described later that this effect can be neglected.

In practise the functions of lateral wheelset positions described are calculated for rigid wheelsets on rigid track (i.e. the gauge does not change) and not for individual wheel and rails. This is not a significant problem however as can be appreciated by studying the curves for contact angle in Fig 3.12. Significant lateral forces causing gauge widening would only occur under flange contact conditions, i.e. with the wheelset displaced to the steep part of the curve to left or right. Under these conditions the opposite wheel is always on the tread resulting in small contact angles and small lateral forces. Thus equation 3.8 is applied only at the wheel closest to flange contact.

Rotation of the rail head is also neglected as the angular change is small in comparison to the contact angle of the wheel under lateral force.

When implemented into a time stepping method of solution the problem proceeds as follows:

i) Begin with known wheel/rail contact forces

ii) Calculate the vehicle displacements and accelerations due to known forces plus the effective forces due to the track co-ordinate accelerations.

iii) Calculate track displacements and accelerations from contact forces.

iv) Calculate relative displacements of vehicle and track from equation 3.8.

v) Calculate creep forces, dynamic forces and contact angle forces to establish new set of wheel/rail contact forces.

vi) Continue to i).
The way in which the wheel/rail forces are related to the forces applied to the track model is illustrated in Fig 3.13. The dynamic track model can be considered to comprise the sleepers and the lateral ballast resilience. The forces which act upon the model are therefore the total of the wheel/rail forces acting upon each rail ($F_{ll}$ and $F_{rl}$ at each wheelset). The assumption that the resilience of each rail can be treated as a single spring which was investigated in Chapter 2 can then be used to evaluate relative displacements of the railhead and sleeper. For the reasons described above it is only necessary to calculate the railhead displacement at the wheel which is closest to flange contact and the value $y_r$ referred to above is equal to the sleeper displacement plus the appropriate railhead displacement with due attention given to the direction of displacement.

![Fig. 3.13 Vehicle Forces on the Track Model](image)

The problem therefore remains of calculating the displacement of the track model with a forcing function defined at intervals $\Delta t$. This can be obtained either by a similar time stepping integration technique to the method adopted for the lumped parameter model and for the finite element model as already described, or by the impulse response method. The latter method is used only for the infinite beam model of the track for which the impulse response is derived from the solutions of Chapter 2 (via. the inverse Fourier transform of the frequency response function as for the vertical case), equation 3.4 can then be used.
3.8 Description of the Track Profile

The principal input to the dynamic model is the lateral curvature of the track as the vehicle co-ordinates are once again expressed relative to the track. The response of the vehicle body is also governed by vertical profile and cant and so the second derivatives of these functions are also required. As with the vertical model the method of describing the track geometry is by discrete measurements of the three parameters required to define the track profile against longitudinal distance. A similar interpolation routine to the one used for the vertical model is also used to obtain estimates of second derivatives at intermediate points. The frequency response calculations of Chapter 2 can once again be used to indicate a suitable spacing at which data should be sampled. The natural frequency at which the dominant wheelset mode of vibration is anticipated is somewhat lower than for the vertical case and choosing the same cut-off criterion as used before would suggest that an upper limit of frequency of around 30 Hz would be sufficient. For a vehicle travelling at 45 mph (20.1 m/sec) this would indicate a sampling interval of 0.2 m.

In the case of the lateral response, however, the higher frequency modes involving dynamic displacements of the track are only of importance under flange contact conditions, as the shorter wavelength, low amplitude irregularities have little effect upon the vehicle response under normal running. Thus in the case of discrete lateral irregularities, or situations where the regions of likely flange contact are known, description of the track geometry at fine spacing can be restricted to these locations only. In other locations measurements at sleeper spacing (typically 0.7 m) are usually sufficient to define the overall vehicle response.

3.9 Computer Program Flow Charts

The flow charts for the vertical and lateral vehicle/track interaction model computer programs are shown in Figs 3.14 and 3.15 respectively. In each case treatment of the different types of track model is shown as options within the program, whereas in practice separate programs are used having essentially the same core of main modules.
INPUT DATA
Vehicle Parameters
Track Parameters
Track Profile

INCREMENT TIME
CALCULATE LONGITUDINAL CO-ORDINATE AT EACH W/SET
INTERPOLATE TRACK PROFILE AT EACH WHEELSET POSITION TO OBTAIN Y CO-ORDINATE AND CURVATURE
TRANSFORM TO TRACK BASED CO-ORDINATE SYSTEM
CALC. FORCING FUNCTION = [M] Zt
CALCULATE FORCES ON MASSES FROM SPRINGS DAMPERS, FORCING TERMS, WHEEL/RAIL FORCES
CALCULATE ACCELERATION OF MASSES
CALCULATE DISPLACEMENTS OF MASSES FROM CENTRAL DIFFERENCE EXPRESSIONS
CALCULATE DISPLACEMENTS OF SPRINGS/DAMPERS
CALCULATE FORCES IN SPRINGS/DAMPERS
CALCULATE WHEEL/RAIL CONTACT FORCES

END OF MODEL?
no
yes

OUTPUT RESULTS

IF INFINITE MODEL
CALCULATE OR INPUT IMPULSE RESPONSE

OR CALCULATE IF DESIGN CASE PROFILE

FOR INFINITE TRACK MODELS CALCULATE TRACK DISPLACEMENTS FROM CONVOLUTION INTEGRAL

FOR INFINITE TRACK MODELS CALCULATE FORCES IN HERTZIAN CONTACT SPRING

Fig 3.14 Flow Chart for Vertical Vehicle/Track Interaction Model
INPUT DATA
Vehicle Parameters
Track Parameters
Track Profile
Wheel/Rail Contact Data
Creep Force/Creepage
Relationship Table

INCREMENT TIME

CALCULATE LONGITUDINAL CO-ORDINATE AT EACH W/SET

CALCULATE TRACK PROFILE DATA: LAT. POS, CANT, CURV

TRANSFORM TO TRACK BASED CO-ORDINATE SYSTEM

CALCULATE FORCING TERMS FROM TRACK PROFILE

CALC. REL. WHEEL/RAIL CONTACT POSITIONS

CALC. CONTACT ANGLES CONTACT CONDITIONS, ROLLING RADIUS DIFFERENCES FROM TABLES

CALCULATE CREEPPAGES

CALCULATE CREEP FORCES

CALCULATE LATERAL TRACK FORCES FROM CREEP/CONTACT ANGLE FORCES

CALCULATE SUSPENSION FORCES

CALCULATE VERTICAL WHEEL/RAIL FORCES

CALCULATE DISPLACEMENTS AND VELOCITIES OF MASSES AT NEW TIME STEP FROM APPLIED FORCES

END OF MODEL?

no

yes

OUTPUT RESULTS

CALCULATE TRACK DISPLACEMENTS

CALCULATE SLEEPER BAY NO FOR EACH WHEELSET

CALCULATE STIFFNESS, DAMPING, MASS MATRICES IF IN NEW BAY

CALCULATE FORCING FUNCTION

CALCULATE TRACK DISPLACEMENTS AT NEW TIME STEP

RETURN

RETURN

Fig 3.15 Flow Chart for Lateral Vehicle/Track Interaction Model
4. RESULTS

4.1 Vertical Vehicle/Track Interaction

The first vehicle chosen to illustrate the results of the vertical modelling is a 25 tonne axle load bogie freight vehicle as illustrated schematically in Fig 4.1. This vehicle would be typical of many using the BR network. The half vehicle model is used for this analysis as the leading and trailing bogies are shown to give sufficiently similar results for the conclusions to be drawn. The track parameters used were those presented in Chapter 2 as being typical of BR track and initially the lumped parameter track model is used.

\[ M_b = 88600 \text{ kg (tot.)} \]
\[ I_b = 0.906 \times 10^6 \text{ kg.m}^2 \]
\[ M_f = 2148 \text{ kg} \]
\[ I_f = 1611 \text{ kg.m}^2 \]
\[ k_{pp} = 4.82 \times 10^6 \text{ kN/m} \]
\[ F_p = 17.5 \text{ kN} \]
\[ M_w = 1776 \text{ kg} \]

Fig. 4.1 Typical 25 tonne Axle Load Freight Vehicle Configuration

In order to be able to interpret the results it is useful to first analyse the natural frequencies of the vehicle. In the case with lumped track parameters the (undamped) natural frequencies can be calculated from the stiffness and mass matrices for the model. As described earlier the stiffness matrix is not calculated explicitly for the model in question, but it is possible to show that the stiffness matrix can be derived from the coupling matrix which has already been derived for the vehicle being studied.
The stiffness matrix and mass matrix are given by:

\[
[K] = [B]K[B]^T
\]

\[
[M] = [I]M
\]

where \([B]\) is the coupling matrix as described earlier
\(K\) is the vector of stiffness for each element
\([I]\) is a unit matrix
\(M\) is the vector of masses for each degree of freedom.

Natural frequencies can then be calculated by Eigenvalue analysis as follows:

In matrix form (neglecting damping) the equations of motion are written as:

\[
[M]\ddot{Z} + [K]Z = 0
\]

If \(Z\) is assumed to be of the form:

\[
Z = A \sin \omega t
\]

Then

\[
-[M] \omega^2 + [K] = 0
\]

This is a conventional Eigenvalue problem for which the solution for the values of \(\omega\) is found where the determinant of the matrix \([K]-[M] \omega^2\) is zero. Numerical methods exist for calculating the Eigenvalues (natural frequencies) and Eigenvectors (mode shapes) for problems with many degrees of freedom. Use was therefore made of a commercially available computer program to solve this particular problem, namely the Numerical Analysis Group Library Program F02AEF. For the vehicle chosen (the half model) the results are shown in Fig 4.2.

It can be seen from the results that the natural frequencies are either well separated, in the case of vehicle modes, or coincident in the case of the wheelset modes. When the whole vehicle is considered the bogie and wheelset modes and body bounce mode are found at exactly the same frequencies as for the half vehicle model but a body pitch mode at 2.9 Hz is also introduced.
Body bounce 2.3 Hz

Bogie pitch 12.2 Hz

In phase wheelset bounce 46.5 Hz

Out of phase wheelset bounce 46.5 Hz

Fig. 4.2 Vehicle/Track Natural Frequencies and Normal Mode Shapes
In order, therefore, to study the response of the vehicle an idealised track profile is considered first of all which is that of a dipped welded joint.

4.2 Bogie Vehicle Negotiating a Dipped Weld

The track shape chosen has the attributes shown in Fig 4.3 with a semi-length \( l \) of 4.5 m and an irregularity angle \( \alpha \) of 10 mrad resulting in an overall depth \( d \) of 12 mm. The angle chosen would be considered to represent one of the most severe dips which would be encountered on welded track.

![Fig. 4.3 Idealised Dipped Weld Profile Used in Analysis](image)

The profile is specified at 0.5 m intervals with the spacing reduced to 0.1 m in the vicinity of the kink. When the distributed track parameter model is used a function describing the track curvature is used with an appropriate impulse in the curvature at the kink as described in Chapter 3.

Fig 4.4 shows typical output of the model for the vehicle considered at a design maximum speed of 60 mph (26.8 m/sec). The dynamic component of force at the leading and trailing wheelsets is plotted as well as the track profile as a function of distance. The results as shown are compared to what would be obtained when a single wheelset is modelled as has often been the case historically when calculating track forces. The result of this is shown in Fig 4.5.

The wheelset only model has one degree of freedom and this is seen to be excited at it's natural frequency when striking the weld. The damping provided by the track is shown to be quite large with the oscillation dying out within two or three cycles. No response is apparent to the longer wavelength irregularity of the dipping approach as this is evidently at too low a frequency. The result therefore is typical of the response of a single degree of freedom system to an
Fig. 4.4 Freight Vehicle Response at a Dipped Weld
(Lumped Parameter Track Model)
UNSPRUNG MASS FROM FREIGHT VEHICLE \( V = 26.6 \text{ M/SEC} \)
10MRAD PARABOLIC DIP OVER 9M.
TRACK GEOMETRY

**Fig. 4.5 Freight Vehicle Wheelset Response at a Dipped Weld**
(Lumped Parameter Track Model)
impulse. The peak positive (downward) force of this characteristic is usually called the $P_2$ force by railway engineers.

The small sharp unloading as the wheelset hits the kink is due to the curve fitting method which attempts to fit a smooth curve around the sharp change of angle resulting in a small error.

The results of the half vehicle model (Fig 4.4) show some important differences. The most immediately noticeable is that the leading and trailing wheelsets give different force histories. Considering firstly the leading wheelset there is now some unloading at the approach to the dip due to the inability of the body to respond at that frequency. The unloading in this case is approximately 20 kN per axle or 8% of the static axle load. As the leading wheelset hits the weld the wheelset natural frequency is excited as before. This part of the response looks very similar to the response when the wheelset only was studied but with a lower effective origin resulting in a smaller peak value being reached.

It is also noticeable that the trailing wheelset applies significant forces to the track when the leading wheelset reaches the kink. This is caused by the bogie pitching response, a similar effect being noticed from trailing wheelset to leading wheelset when the trailing wheelset reaches the kink.

The trailing wheelset impact ($P_2$) response is again similar to that of the single wheelset but this time with a higher origin due to the response of the vehicle body which is excited by the leading wheelset striking the weld. This tendency for the trailing wheelset to give higher forces than the leading wheelset is something which has been noticed in experimental results and has been impossible to explain using simple models.

Viewed in an approximate manner, however, it can be assumed that the body is excited at it's natural frequency when the leading wheelset strikes the kink. This would cause compression of the vehicle springs and a corresponding superimposed force at both leading and trailing wheelsets for one half cycle of oscillation. For the vehicle studied with a natural frequency of 2.3 Hz and a forward speed of 60 mph (26.8 m/sec) this distance would be 6 m, with the maximum occurring at 3m. Hence the present vehicle with an axle spacing of 2 m is close to the 'worst speed' for this effect and most bogie vehicles would be configured such that the trailing wheelset force was higher at dipped welds.
As the dynamic force decays, a long wavelength oscillation can be seen to change to a shorter wavelength, lightly damped oscillation. This is due to the friction in the suspension 're-locking' under body oscillation. The friction in the current vehicle studied has a minor effect due to the fact that a quite low value was used. Some friction damped vehicles however can be found with quite large values of friction in their suspensions for reasons such as corrosion, lack of maintenance or contamination by some of the materials transported. The effect of this will be considered in a later example.

**Effect of track model:** The bogie vehicle chosen is now used to illustrate the effect of some of the variations in the modelling which are shown in Fig. 4.6. The first case considered as shown in Fig. 4.6a is the solution with the single layer beam on Winkler foundation model and Hertzian contact conditions for the same case as above. To simplify the presentation, only the leading wheelset forces are shown.

The response is seen to be essentially the same as for the lumped parameter model but with an additional high frequency force peak at the weld. This is the force which has normally been called $P_1$ and corresponds to a mode of vibration in which the track vibrates on the Hertzian spring with little oscillation of the wheelset mass, in this case around 500 Hz. The result obtained is very much in agreement with the one obtained for the frequency domain analysis of Chapter 2. It could therefore be anticipated under these conditions that only 2% of the high frequency force would be seen by the ballast and something like 60% of the lower frequency $P_2$ force.

For the same case with standard pads introduced as shown in Fig. 4.6b the high frequency component is shown to be reduced dramatically. In fact the mode of vibration in which the rail vibrates on the Hertzian spring is not present but the mode in which the sleepers vibrate out of phase with the rail is apparent at a frequency of approximately 175 Hz. The component of sleeper/ballast force at this frequency would again be expected to be relatively small in comparison to the $P_2$ component, although may possibly be larger than the case without rail pads due to the lower frequency at which the vibration occurs.

The case with soft rail pads shown in Fig. 4.6c shows that virtually no high frequency force is visible. It is also apparent as suggested previously that the
Fig. 4.6 Effect of Track Model on Response at a Dipped Weld
rail pads have the effect of reducing the amount of damping in the P₂ mode of vibration.

The conclusions which can be made concerning the importance of the track models are therefore essentially the same as those obtained from the frequency domain analysis, namely that the models differ in their characteristics at high frequencies, and also that the amount of damping to lower frequency vibrations is affected. Because of the localised nature of the impact which occurs at a dipped weld it might also be expected that the high frequency impact forces could have an effect upon the differential ballast forces and hence the deterioration of track geometry. This is considered further in Chapter 5.

4.3 Comparison of Viscous Damped and Friction Damped Vehicles

To study this particular feature it is desirable to eliminate some of the other aspects of the vehicle response which are not of direct relevance to the problem being considered. For this reason a fictitious four wheeled vehicle is studied having its radius of gyration in pitch equal to the semi-axle spacing. The result of this as described earlier is to make the natural frequencies of body bounce and pitch identical and the vehicle behaves as two independent monocyles having half the body mass over each wheelset. Because the effective suspension stiffness can vary at each end of the vehicle as described previously this situation is not apparent for all of the time, but for the cases studied this effect is small.

The suspension parameters were chosen for the vehicle by following some simple design rules. A typical unsprung mass was chosen and the body mass was calculated to give an axle load of 25 tonne. The spring stiffness was chosen to give a natural frequency of the body bounce/pitch mode of 1.5 Hz, this being a representative value for such vehicle types. Damping rates for viscous damped vehicles normally give a ratio to critical damping of 25%, whereas in the case of friction damped vehicles a friction level of 10% of the static axle load is typical. The case is also considered of a friction damped vehicle in the worn state which can quite easily result in the friction level being doubled.

The series stiffness in the vehicle suspension was taken to have a value per axle of $k_s = 10^8$ N/M/axle. Once again this is thought to be a representative value for the type of vehicle being considered.
The resulting vehicles therefore are comparable and would represent two optional design approaches. The parameters resulting would be quite close to those of a large number of real vehicles of this type.

Clearly the friction damped vehicle exhibits a non-linear response, in fact the friction dampers have a bi-linear characteristic as discussed earlier. Under small amplitude vibrations the vehicle behaves as if it has an extremely stiff suspension while ever the friction limit is not exceeded, and it would be straightforward to calculate the natural frequencies which would be applicable to vibrations happening in this regime.

Similarly under continuous high amplitude oscillation the friction damper would be sliding for a large proportion of the time and it's effect on the vehicle response would be quite small. Under this condition the normal undamped natural frequencies for the vehicle/track system could reasonably be assumed to be applicable. For the friction damped vehicle being studied here, therefore, the natural frequencies for each of these states are calculated of which there are four (two equal pairs) these are shown in Table 4.1.

<table>
<thead>
<tr>
<th></th>
<th>Small amplitude (Friction locked)</th>
<th>Large amplitude (Friction exceeded)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Body Bounce</td>
<td>8.8 Hz</td>
<td>1.5 Hz</td>
</tr>
<tr>
<td>Wheelset Bounce</td>
<td>57.5 Hz</td>
<td>48.3 Hz</td>
</tr>
</tbody>
</table>

Table 4.1 Range of Natural Frequencies of Friction Damped Vehicle

For steady state excitation of intermediate amplitude it can be expected that the vehicle will exhibit non-linear vibrations having effective frequencies somewhere between the calculated extremes. It is also noticeable that the range of body frequencies is much greater (proportionally) than the range of wheelset frequencies. This is an important point which will be apparent later.
4.4 Response of Viscous and Friction Damped Vehicles to an Idealised Dipped Weld

Fig 4.7 shows the force history at the leading wheelset for each of the three vehicle cases considered, with all vehicles travelling at 60 mph (26.8 m/sec). The viscous damped vehicle shows an unloading on the dip approach as anticipated and the characteristic high frequency response of the unsprung mass as the wheelset strikes the kink. In fact the response shows very much the classical response of a damped two degree of freedom system to an impact, with the body response persisting for some distance.

The response of the friction damped vehicles, however, shows some important differences. The approach to the kink firstly excites an oscillatory response due to the higher natural frequency applicable when the friction is locked. At the kink a decaying oscillatory response of the wheelset is noticed which is similar to the viscous damped vehicle. It is noticeable, however that in the case with high friction levels there appears to be more damping in this mode. This may depend upon the exact instant of friction breakout and the direction of the forces on the wheelset at the time. The peak force reached under both levels of friction is found to be approximately 15% greater than the level for viscous damping.

The point made earlier that the frequency of vibration of P2 oscillation is little different in the friction case is reasonably apparent from these results, this also means that the force amplitude of such vibration is not greatly different for the friction damped vehicles.

The response subsequent to the wheelset impact is flat topped for one cycle suggesting that the friction limit is exceeded during this time. Subsequent to this the response settles down to a decaying sinusoidal oscillation with the level no longer exceeding the limiting friction level.

4.5 Comparison of Viscous Damped and Friction Damped Vehicles on Typical Track Profiles

The response of the vehicles is now studied on typical measured track profiles. Figs 4.8 and 4.9 show the force histories for the three suspension configurations on two typical measured track profiles. These profiles were measured by an overlapping straight edge technique to produce a profile defined
Fig. 4.7 Response at a Dipped Weld: Viscous and Friction Damped Vehicles
at 0.1 m intervals. The profiles represent reasonably good and reasonably poor track respectively, by the normal method of establishing quality of track which is the standard deviation of vertical profile.

On the good track (fig 4.8) the viscous damped vehicle gives generally a very low level of dynamic force. Large peaks do appear occasionally, however, and these are due to the welds, being mostly at approximately 18m intervals. There is shown to be very little long wavelength response of the vehicle body.

The friction damped vehicles however tend to produce a continuous oscillatory force which appears most of the time to reach or exceed a value equal to the friction level. The frequency also is seen to be close to the calculated maximum for body vibration of 8.8 Hz. Impacts at welds also cause the higher frequency peaks as before but it is noticeable that these do not appear to have any particular phase relationship with the underlying body oscillation, and that the amplitude of the impact force is rather similar for all the vehicles when the base point is ignored.

It is concluded, therefore, that even on good track there is sufficient excitation to cause forces up to the friction level even if the friction level is well in excess of the design level. This is due in part to the low level of damping effective for this mode of vibration, this seems reasonable however in view of the large mass and stiffness involved.

On the poor track (Fig 4.9) the viscous damped vehicle appears to give a generally higher level of dynamic force. Moreover there is also evidence of the vehicles body being excited at a long wavelength suggesting that the sample poor track profile is worse in both short and long wavelength content.

The friction damped vehicles once again show a significant oscillation between the positive and negative limits of friction force. It is superficially evident, however, that less of the time is spent executing motion which is close to simple harmonic motion suggesting that the friction limit is exceeded for more of the time, particularly in the case of the design case friction level.

To study the effect of the track quality on the frequency of response the results can be studied in the frequency domain. Figs 4.10 and 4.11 show power spectral density plots which were prepared using the time histories of leading
Fig. 4.8 Response on Good Track: Viscous and Friction Damped Vehicles
Fig. 4.9 Response on Poor Track: Viscous and Friction Damped Vehicles
Fig. 4.10 Power Spectra of Forces on Good Track
Fig. 4.11 Power Spectra of Forces on Poor Track
wheelset force for each vehicle considered on the good and poor track respectively.

As might be expected from the time histories the majority of the energy is to be found in the case of the friction damped vehicles at and around the body frequency which is effective when the friction is locked (calculated at 8.8Hz). It is also noticeable that this energy is shifted towards a lower frequency when either the track is poorer or the friction level is lower confirming that the friction is broken out for a longer proportion of the time. This also results in a corresponding increase in the frequency of the dominant vibration of the wheelset mode in accordance with the values previously calculated.

The viscous damped vehicle shows peaks in the response once again at the calculated natural frequencies resulting in a much lower body bounce frequency and significantly less energy at this frequency. The content of the wheelset vibration for this vehicle is shown to form a much more significant proportion of the total energy.

A conclusion which can be made concerning the performance of friction damped vehicles in comparison with viscous damped vehicles, therefore, is that they appear to be significantly worse on generally randomly irregular track. When negotiating discrete irregularities such as rail welds the impact forces are not necessarily much greater, but there may be an additional effect of the poor body response.

4.6 Experimental Verification

An experiment was performed in early 1984 primarily to compare the vertical forces at dipped welds for a range of freight vehicles. Although not involved with conducting the experiment the author performed all the analysis of the results. It was at that time that the vertical models described here were under development and use was made of the available data to perform comparisons with the theoretical results and provide information for improving the modelling.

A full description of the experimental procedure and analysis and comparison of the results is presented in Appendix II. The agreement between theory and experiment was generally quite good but the results showed the
important effect of friction damping on the forces produced and the difficulty in measuring track forces which take these into account.

Since that particular experiment and with a knowledge of results of the theoretical modelling more evidence has been obtained to suggest that the behaviour of friction damped vehicles in particular is in line with theoretical predictions made and that friction levels can far exceed the design levels resulting in very unfavourable track forces. New experiments are also planned to measure these in more detail. The possible effects of this on an important area of track damage, namely the deterioration of track geometry, are considered in Chapter 5.

4.7 Lateral Vehicle/Track Interaction

The results of the lateral modelling will be illustrated by considering two possible designs of freight vehicle. The first one to be considered will be a design representative of an older freight vehicle in that a very stiff yaw restraint of the wheelsets is provided. The vehicles has two axles with lateral and vertical suspension provided by springs and viscous dampers between the wheelsets and body as described in Section 3.6.

The second vehicle for comparison is considered to be of more modern design in that yaw relaxation of the wheelsets is provided for by means of spring restraint having sufficiently low stiffness to allow the wheelsets to adopt a more radial position in curves. All the other parameters are assumed to be the same. The parameters of the second vehicle are in fact those measured from a prototype high speed freight vehicle which was used in the experiments reported later. The parameters used for both the vehicles are presented in Table 4.2, being referred to as vehicles 'A' and 'B' respectively.

4.8 Response at a Discrete Lateral Irregularity

The first case to be considered will be that of the two vehicles negotiating a discrete lateral irregularity in the track which would represent a kinked rail or a misaligned weld or joint. For this exercise the irregularity is assumed to consist simply of two pairs of straight parallel rails which meet at an angle. Three angles are considered namely 5, 10 and 15 mrad in combination with vehicle speeds of 20, 40 and 60 mph (8.9, 17.9 and 26.8 m/s). The more severe combinations of speed and angle are unlikely to be found on plain track, but
with the exception of the very worst case, would be representative of the situation occurring at switches.

\begin{align*}
M_w &= \text{Wheelset mass (inc. subframe)} & 2.39 \text{ Mg} \\
I_w &= \text{Wheelset roll/yaw inertia} & 1.66 \text{ Mgm}^2 \\
J_w &= \text{Wheelset polar inertia} & 0.05 \text{ Mgm}^2 \\
m_b &= \text{Body mass} & 30.00 \text{ Mg} \\
I_b &= \text{Body yaw inertia} & 170.00 \text{ Mgm}^2 \\
J_b &= \text{Body roll inertia} & 51.00 \text{ Mgm}^2 \\
K_b &= \text{Body pitch inertia} & 240.00 \text{ Mgm}^2 \\
k_y &= \text{Lateral suspension stiffness/axle} & 0.43 \text{ MN/m} \\
k_{yA} &= \text{Yaw suspension stiffness/axle (Vehicle A)} & 15.00 \text{ MNm/rad} \\
k_{yB} &= \text{Yaw suspension stiffness/axle (Vehicle B)} & 3.10 \text{ MNm/rad} \\
k_z &= \text{Vertical suspension stiffness/axle} & 4.10 \text{ MN/m} \\
c_y &= \text{Lateral suspension damping/axle} & 0.056 \text{ MNs/m} \\
c_{yA} &= \text{Yaw suspension damping/axle} & 0.00 \text{ MNs/rad} \\
c_z &= \text{Vertical suspension damping/axle} & 0.028 \text{ MNs/m} \\
c_0 &= \text{Semi wheelbase} & 3.15 \text{ m} \\
l_0 &= \text{Semi gauge} & 0.76 \text{ m} \\
r_0 &= \text{Wheel radius} & 0.375 \text{ m} \\
h_0 &= \text{Height of body e.g. above rails} & 1.55 \text{ m} \\
& \text{Vertical spring semi separation} & 0.99 \text{ m} \\
& \text{Distance, rails - lateral spring on axles} & 0.76 \text{ m} \\
& \text{Distance, rails - lateral damper on axles} & 0.57 \text{ m} \\
& \text{Distance, rails - lateral spring on body} & 0.36 \text{ m}
\end{align*}

\textbf{Table 4.2 Vehicle Parameters for Lateral Modelling}

The most severe case is considered first as this is in many ways the simplest to understand, and illustrates what is happening most clearly. Fig 4.12 shows the time histories of some of the important parameters which are predicted by the model for the poorer curving vehicle which is described as vehicle 'A'. Considering firstly the leading wheelset forces it can be seen that as the wheelset passes the kink there is a short time interval during which the forces remain quite small. This is of course due to the flangeway clearance between wheel and rail and means that the lateral forces at this stage come mainly from the creep forces between wheel and rail with negligible contact normal force. Once the flangeway clearance is taken up, however, there is then a large impact of the wheelset on the outer rail which causes an oscillatory impact.
Fig. 4.12 Response of Vehicle A to 15mrad. Kink at 60mph
force. This oscillation is found to have a frequency of approximately 15 Hz which corresponds to the mode of vibration in which the wheelset oscillates laterally on the track resilience and therefore has an analogy in this respect to the vertical case. This is also true of the body response which causes a longer wavelength oscillatory force resulting from the excitation of the combined lateral and rotational modes of the vehicle body.

The force history of the inner wheel shows a somewhat different characteristic. As flange contact is not made on this wheel the majority of the lateral force is due to creepage effects. The force is therefore gauge spreading due to the relative angle of attack between wheel and rail which appears immediately as the wheelset passes the kink. It is noticed that the force level reaches an apparent plateau which is of course the friction limit, $\mu N$ which for the parameters studied can be calculated to be 21 kN. This value is not quite reached laterally in fact because there are also longitudinal creep forces acting at the same time.

When the creepages on the inner rail fall below the level at which sliding takes place this immediately increases the amount of damping effective to lateral oscillations of the wheelset. In other words when the friction limit on the inner wheel is already exceeded, lateral oscillation of the wheelset does not cause changes in the force level until the relative wheel and rail velocities fall to zero and sliding stops. As mentioned earlier the effect of lateral velocity of the rails on the creep forces is neglected, as to include them requires a much finer time step due to the high lateral track velocities which potentially can occur. In practise this has little effect upon the results because lateral oscillations of the track only occur when the creep forces are already saturated and flange contact occurs.

The response of the trailing wheelset is rather more complicated due to the fact that the response of the body has not settled and the angle of attack of the wheelset is influenced by the position of the front of the vehicle. In the case considered so far the trailing wheelset begins to exert forces on the track before the kink as the wheelset is yawed relative to the track. Some steering of the rear wheelset does take place, however, as it attempts to remain perpendicular to the track. As the wheelset passes the kink a large impact of the wheelset on the rail is avoided but the response of the body does push the wheelset into flange contact, resulting in a sharp peak to the underlying body force seen on the outer rail. The inner rail force is seen to vary very little in this case although at slower
speeds, on the severe kink, flange contact can occur on the inner rail resulting in a small impact force.

The second case to be considered in detail is that of the smallest kink angle taken at the largest speed, the results for which are shown in Fig 4.13. With the small kink angle the vehicle is almost able to steer around the irregularity without encountering a lateral impact of the flanges, in fact the trailing wheelset is seen to achieve this successfully. The leading wheelset however does impact lightly upon the outer rail causing a small dynamic augmentation of the curving forces which is well damped.

The other results over the range of cases considered will be presented in summary form, but first the two cases considered in detail will also be studied for the vehicle with improved steady state curving which is referred to as Vehicle 'B'. The results for the high speed and large kink angle are shown in Fig 4.14. The forces predicted for the leading wheelset are shown to be extremely similar to those produced for Vehicle A. This is because the superior curving ability of Vehicle B still does not prevent flange contact occurring very shortly after passing the kink and at the point of maximum impact the yaw angle in each case is very similar, in the subsequent response the yaw angle for Vehicle B does however reduce more quickly.

The trailing wheelset response is shown to have some important differences, as apparently for vehicle B two lateral impacts occur. The reason for this is due to the increased yaw flexibility which allows the trailing wheelset to remain closer to a perpendicular attitude to the rails as the vehicle body turns the corner. Thus at the kink the wheelset runs straight into flange contact for a brief period before being driven into flange contact once again by the body response. Under these conditions, therefore, the superior steady state curving vehicle actually gives the worst track forces, but this phenomenon is only shown to occur under this most severe case. For the case of the smaller angle as shown in Fig 4.15 vehicle B is able to negotiate the small kink angle without flange contact at either wheelset resulting in very low track forces.

The results of the forces obtained from all the cases studied are summarised in Fig 4.16 in the form of peak lateral forces versus speed. At the leading wheelset both vehicles are shown to produce the same lateral forces at the kink except for the smallest angle where Vehicle B is able to negotiate the kink without flange contact. At the trailing wheelset the forces are generally
Fig. 4.13 Response of Vehicle A to 5mrad. Kink at 60mph
Fig. 4.14 Response of Vehicle B to 15mrad. Kink at 60mph
Fig. 4.15 Response of Vehicle B to 5mrad. Kink at 60mph
Fig. 4.16 Effect of Kink Angle and Speed
smaller than the leading due to the inherent steering effect which the front end of the vehicle provides. This effect has the most benefit for Vehicle A. Large forces do occur, however, when the lateral response of the body drives the wheelset into flange contact thus producing an additive effect and a large increase in the impact force at the higher speeds. Furthermore, although the maximum force at the trailing wheelset for vehicle B is no greater than for vehicle A, two lateral impacts can occur for the former. The impact forces at the trailing outer wheel of both vehicles are also generally highly damped as the inner wheel creep forces do not saturate.

In the case of the leading wheelset where impacts of the wheelset and the track occur in a fairly straightforward manner the peak force is shown to increase approximately linearly with speed. This is therefore analogous to the vertical dipped weld case where all the modes of response of the vehicle and track are excited by the impact and the maximum force is generated by oscillations of the wheelset on the track resilience. This is also studied further when the response at a switch is considered.

The general conclusion which can be drawn from the study of the response of these two vehicles to this type of discrete irregularity seems to be that the responses are little different. This is in contrast to a steady state curving situation where vehicle B would be expected (and is shown in the following Section) to show superior performance.
4.9 Response at a Transition Curve

The response of the two vehicles considered in Secton 4.8 to a transition from straight track to a constant radius curve has also been calculated. A speed of 60 mph was chosen for which the design rules indicate that a minimum curve radius of 420 m having a cant of 150 mm can be negotiated at this speed. The transition curve was designed by adopting the usual practise of increasing the curvature and cant linearly with speed resulting in a cubic profile to the track. Adopting the limit for the maximum allowable rate of change of cant with time resulted in a design length for the transition curve of 72 m.

The responses at 60 mph of vehicles A and B are shown in Figs 4.17 and 4.18 respectively. It can be seen that for the design case geometry considered neither vehicle produces significant dynamic forces, the forces just increase gradually to their steady state values. As expected vehicle A produces the higher lateral forces (by approximately 30%) under these quasi-steady state conditions due primarily to the higher yaw angles relative to the track.

The leading wheelset of each vehicle is, however, shown to run into partial flange contact and it would be interesting to investigate the effect of irregularities in the track under these conditions, as there are shown to be minor dynamic forces due to small errors in the defined profile (as indicated by fluctuations in the curvature). To study this case an irregularity was added to the original design case profile. The shape of the irregularity was derived from the typical spectrum of lateral alignment referred to in Chapter 2, by generating a profile with the desired PSD, on the assumption that the phase of each component was randomly distributed. This resulted in a new profile with a random component having an additional root mean square error of 2 mm. The repeated results for vehicles A and B are shown in Figs 4.19 and 4.20. The effect of including the random irregularities is shown to be to produce dynamic forces due to impacts of the wheelsets and the rail which are largest at the leading wheelset. Also these are marginally worse for vehicle A because this vehicle moves more quickly into flange contact when entering the transition.

It is shown therefore that in cases like this the actual design profile of the track can produce quite small steady state and transient forces, whereas the dynamic forces which result when considering real track profiles can be much greater.
Fig. 4.17 Response of Vehicle A on Transition Curve
Fig. 4.18 Response of Vehicle B on Transition Curve
Fig. 4.19 Response of Vehicle A on Transition Curve with Random Irregularities
Fig. 4.20 Response of Vehicle B on Transition Curve with Random Irregularities
4.10 Response at a Switch

Fig 4.21 shows the typical features of a straightforward switch of the type used on BR. Because it is impractical to plane down a rail to give a tangential entry to the turnout direction an entry angle is unavoidable. This is followed by the planing radius, which is effective over the length of switch rail which is planed away, i.e. that part of the switch rail which is touching the stock rail. The radius then changes to the switch radius which is a tangential curve to the through straight track, and finally onto the turnout radius, which is chosen to give a crossing angle within a range of standard values and is often equal to the switch radius. The radius of the turnout curve can then begin.

![Diagram of switch geometry characteristics](image)

**Fig. 4.21 Typical Switch Geometry Characteristics**

For the prediction of forces at a switch, the finite element model was used, for which it is possible to define the sleeper lateral support stiffnesses individually and the rail head lateral stiffness at each sleeper position. This would require in practise an elaborate measuring technique to evaluate the necessary parameters due to the complicated structure and also the large loads which would need to be applied to obtain a realistic loading situation. In order to provide a first indication of the important parameters which might be needed the values were estimated from a knowledge of the support conditions which exist, and their likely relationship to plain track, for which much more experimental data exists.
In the region of the switch toe the ballast stiffness is likely to be very similar to plain track, as the sleepers are of standard length. All the rail support also comes from the stock rail, as the switch rail simply rests against the stock rail and so the lateral rail stiffness is also taken to be equivalent to that on plain line for cases where the rail foot is fixed. At the end of the planing the lateral rail stiffness is assumed to have reached twice its base value whereas the ballast stiffness has remained approximately the same. The lateral bending rigidity of the rails however is increased to that of three rails, the open switch rail being considered to be free.

Between the end of the planing and the heel blocks the rail is laterally restrained by propping the foot and web. This zone can therefore be expected also to possess an intermediate stiffness value. At the heel blocks the rail is held very rigidly indeed and thus will have a very high lateral stiffness. The number of rails effective in bending will also increase to four at this point. Beyond the heel blocks the track returns to normal construction and some rail flexibility will again be returned. The anticipated distribution of the parameters is therefore shown in Fig 4.22, the sleeper masses being derived from the weight of the sleeper, baseplates and rails. Although the increasing length of each timber bearer may increase the lateral ballast stiffness and damping slightly with distance, this effect is neglected.

<table>
<thead>
<tr>
<th>SWITCH TYPE</th>
<th>ENTRY ANGLE (mrad)</th>
<th>PLANING LENGTH (m)</th>
<th>PLANING RADIUS (m)</th>
<th>DISTANCE TO Toe TO HEEL (m)</th>
<th>SWITCH RADIUS (m)</th>
<th>SPEED LIMIT (mph)</th>
</tr>
</thead>
<tbody>
<tr>
<td>DV</td>
<td>6.455</td>
<td>5.2</td>
<td>367.05</td>
<td>12.4</td>
<td>331.7</td>
<td>30</td>
</tr>
<tr>
<td>GV</td>
<td>2.856</td>
<td>11.6</td>
<td>1826.30</td>
<td>27.9</td>
<td>1650.4</td>
<td>70</td>
</tr>
</tbody>
</table>

Table 4.3 Geometric Parameters for Two Typical Switches

The dynamic properties were derived in this way for two typical switches, one designed for 30 mph (13.4 m/sec) and a second designed for 70 mph (31.3 m/sec), for which the important geometric parameters are shown in Table 4.3. These are called types DV and GV respectively, the V indicating that the rails are mounted vertically instead of the normal rail inclination of 1 in 20 for plain line.
The results are presented for the vehicle referred to as vehicle B in the previous examples, which has a soft yaw suspension, the results for the other vehicle being not dramatically different as suggested by previous results. These are shown for the DV and GV switches respectively in Figs 4.23 and 4.24. There is a slight dynamic impact at the leading wheelset for the DV switch but apparently none in the case of the GV switch. There is also a very slight dynamic
Fig. 4.23 Response of Vehicle B on DV switch
Fig. 4.24 Response of Vehicle B on GV switch
augment in the area of the heel blocks due to the rapid increase in the lateral stiffness at this point. At the trailing wheelset there is very little evidence of dynamic forces as the wheelset avoids flange contact.

These limited theoretical results obtained for switches seems to suggest that the highest dynamic forces are to be expected at the lower speed switches. The reasons why this may be so can be obtained by simplifying the consideration of the lateral impact which occurs. Where flange contact does occur at switch entry this is due to the inability of the wheelset to negotiate the entry angle. Thus the wheelset strikes the rail laterally with an approach velocity of approximately $av$ where $a$ is the switch toe angle and $v$ the velocity of the vehicle. The mode of vibration which is principally excited by this impact is the oscillation of the wheelset on the track resilience which can be treated approximately as a single degree of freedom oscillator if the track mass is neglected. If the damping is also neglected in order to find the peak force which occurs due to the impact, it is straightforward to show that the peak force is given by:

$$P = av\sqrt{km} + \text{creep force}$$

Where $k$ is the effective track stiffness and $m$ the unsprung mass. Under the assumption that these two values are constant for different switches and that the creep force would also be constant the peak force would be governed by the quantity $av$. This value can be summarised for a range of switches having various design speeds as shown in Table 4.4. The formula presented also satisfies the apparent linear increase in force with angle and speed observed in the case of the lateral kinks studied earlier.

The design rules for switches used on BR are based upon some simple rules to establish equivalent curvatures and hence cant deficiency values for which certain limits are set for passenger comfort. These appear to result in a design which produces a lower value for the product $av$ at higher speeds and therefore lower forces might be anticipated.

Limited experimental data is available for the forces occurring at switches due to the difficulty in making such measurements. However Clark (54) did run a prototype high speed freight vehicle (for which the suspension parameters are those of vehicle B), fitted with load measuring wheelsets, through a variety of types of switch for which the measured dynamic impact forces are summarised
<table>
<thead>
<tr>
<th>Switch Type</th>
<th>Design Speed mph</th>
<th>Design Speed m/s</th>
<th>Entry Angle (mrad)</th>
<th>Product $\alpha v$</th>
</tr>
</thead>
<tbody>
<tr>
<td>CV</td>
<td>30</td>
<td>13.4</td>
<td>9.311</td>
<td>116.4</td>
</tr>
<tr>
<td>DV</td>
<td>35</td>
<td>15.6</td>
<td>6.455</td>
<td>89.7</td>
</tr>
<tr>
<td>EV</td>
<td>45</td>
<td>20.1</td>
<td>5.308</td>
<td>103.0</td>
</tr>
<tr>
<td>FV</td>
<td>55</td>
<td>24.6</td>
<td>4.467</td>
<td>105.4</td>
</tr>
<tr>
<td>GV</td>
<td>70</td>
<td>31.3</td>
<td>2.856</td>
<td>91.1</td>
</tr>
<tr>
<td>HV</td>
<td>100</td>
<td>44.7</td>
<td>1.576</td>
<td>70.0</td>
</tr>
</tbody>
</table>

Table 4.4 Product $\alpha v$ for a Range of Switches

in Fig 4.25. There is a certain amount of scatter in the results but these seem to suggest fairly strongly that the forces occurring at the switches do in fact increase with speed and that the force levels are higher than so far predicted for the cases considered.

It seems likely therefore, as with the case of the transition curve, that irregularities in the geometry could be affecting the forces produced. If such irregularities were of similar magnitude on all switches this would be expected to produce larger forces at the higher speeds. For this reason errors in the profile were introduced to the design geometry in the same way as for the transition curve, i.e. by adding to the profile an irregularity shape derived from typical track spectra. It is likely for the case of a switch that some of the irregularities may not be so random in nature due to wear or movement at the points of high forces which is a possibility that is not considered. The irregularity shape which has been added, however, is the same in principle as for the transition curve but with the addition of a purely random error at each bearer position to represent errors in construction of the switch. These are also assumed to have a mean of 0 mm and standard deviation of 2 mm.

The new results obtained for the DV and GV switches at their design speeds are shown in Figs 4.26 and 4.27 respectively. As with the case of the transition curve the force levels have been increased significantly by including track misalignments. These are now shown to produce much larger forces on the higher speed switch and the predicted maximum forces are more in line with the measured data, in fact exceeding those values by approximately 50% in both cases, suggesting that the irregularity sizes may have been slightly
overestimated. Due to the apparent importance of the additional irregularities at a switch, it would be worthwhile making measurements of existing switches to establish whether such levels of error in geometry are apparent.

4.11 Experimental Verification

Very little experimental work has been performed on the lateral dynamic forces at track irregularities. Therefore in the initial stages of development of the theoretical models an experiment was proposed in order to provide some experimental validation of the results obtained and assist in discovering the important features such models would require.
Fig. 4.26 Response of Vehicle B on DV Switch with Random Irregularities
Fig. 4.27 Response of Vehicle B on GV Switch
with Random Irregularities
An experiment was therefore proposed to measure the track forces at a discrete lateral kink, as this was considered to be a most likely source of high lateral dynamic forces. The kink to be evaluated was required to be representative of the lateral irregularity which would be found at a switch. An actual switch was ruled out, however, due to the severe difficulty of measuring forces and displacements at such a location, and the extra complication required in the modelling.

A kink was therefore placed in a piece of plain line on a BR Research Department test track such that the geometry was representative of that of a switch, and full instrumentation could be installed. This experiment was planned and executed by the author with on site assistance from staff of the BR Research Department and took place in late 1979.

The experimental procedure and the results of the analysis and comparison with the theoretical modelling are presented in Appendix III. The results obtained were shown to be quantitatively very similar to the initial theoretical predictions but the impact forces measured were significantly smaller in magnitude. Subsequent analysis discovered various improvements to the model which dramatically improved the agreement, these were as follows:

i) Initially the design case geometry was used based upon an angular bend in two straight rails at the kink. Inclusion of the actual measured geometry of the track was shown to make a significant difference to the predicted force levels. This also tended to confirm some of the theoretical findings that differences in the profiles of switch geometry from the design case geometry could significantly affect the forces.

ii) The lateral vehicle dampers were originally modelled as linear viscous ones. In fact the dampers were actually non-linear having a 'blow off' feature to provide a limited maximum force value. When this feature was incorporated into the model a reduction in the peak dynamic force and damping of the wheelset was noticed.

iii) A small effect was noticed due to the lateral flexibility of the wheel. When loaded in the laboratory the wheels were found to have an effective flexibility of wheel rim relative to the axle approximately equal to the lateral rail head bending stiffness. To incorporate this effect the rail head bending stiffness was adjusted to give the appropriate series stiffness value. This would not normally need to be included in practise, however, as the wheelsets used in the
experiments were spoked load measuring wheels which are less rigid than solid wheelsets.

The results obtained therefore were found ultimately to agree very well with the theoretical model for the plain rail case thus meaning that the track model could be extended to the more complex case of a switch with a fair degree of confidence.
5. APPLICATION TO DETERIORATION OF TRACK GEOMETRY

5.1 Introduction

Deterioration of the vertical and lateral geometry of the track takes place due to settlement and disturbance of the ballast and subgrade, predominantly as a result of traffic. Uniform settlement or uniform shift in the lateral case would not in itself cause a deterioration in the quality of the track geometry, but it is the differential movements which result in a worsening of passenger ride and an increase in dynamic loads. Differential displacements can occur due to the general random nature of the settlement process and also due to the lack of straightness of the rails (57), but the major contributor to the problem is the vehicle dynamic loads.

The result of this reduction of track quality as far as railway permanent way engineers are concerned is a large maintenance overhead. Main line railway tracks need attention to geometry typically every six or nine months which requires possession of the track for the maintenance machinery, thus disrupting services and incurring large costs. The main benefit to be gained, therefore, from studying the deterioration of geometry and finding ways in which it could be reduced, would be a saving of some of these costs. This could be achieved in a number of ways, for instance:

i) By discovering the elements of the track structure which affect the deterioration process and helping to improve track design.

ii) By doing the same for the vehicles and thus enabling the vehicle designer to evaluate the cost benefits of various suspension design options.

The track and vehicle design could also be optimised in this way by trading off cost savings in track maintenance against the cost of superior vehicle suspensions such as active suspensions.

In the past the deterioration of the vertical geometry has been the main area of concern. The reason for this being that under present conditions the vertical geometry has deteriorated the most rapidly and the lateral geometry has thus been maintained as a matter of course during vertical maintenance operations. Two reasons why this may not necessarily be the case in the future,
however, have initiated more interest in the consideration of lateral geometric deterioration also. The first reason is the possible introduction of tilting trains on the BR network which can curve at high cant deficiencies and hence with higher lateral forces. The second reason is the development of a new method of vertical track maintenance known as pneumatic ballast injection (35). Essentially this method uses measured quantities of fine ballast particles injected under compressed air pressure beneath a sleeper which has first been lifted to the desired height. This is in contrast to the existing tamping method which uses vibrating tines to squeeze ballast underneath the raised sleeper thus disturbing the already compacted ballast bed. The new method therefore offers an improvement in the vertical maintenance interval and may mean in future that maintenance cycles could be governed in more areas by considerations of lateral geometry.

Certain work has been done on the deterioration of geometry as described in Chapter 1, this has also related almost exclusively to considerations of vertical deterioration. The most sophisticated modelling to date appears to the work described in Ref. 42 in which the vehicle and track were modelled as unsprung mass and lumped spring, mass and damping respectively. This Chapter will therefore study some of the possible implications of the improved vehicle and track modelling upon rates of deterioration of geometry.

5.2 Prediction of the Deterioration of Vertical Track Geometry

A reasonably simple method has been developed for predicting the likely rate of deterioration of vertical track geometry under prescribed conditions. It is not intended as a method for predicting absolute values but more as a relative indicator in order to consider the various track and vehicle models proposed and suggest what areas of further work may be appropriate.

As a basis for the study the dynamic calculation of Chapter 3 is used which essentially is able to predict the wheel/rail forces which result from various vehicle and track combinations to prescribed track geometry. The work of Shenton (39) suggests that the rate of settlement of a sleeper is governed by the maximum dynamic load experienced by the sleeper and the number of applications of load. A four axle vehicle for example is considered to apply four separate loading cycles. The relationship between load and settlement \( (S) \) is given by:-
\[ S = K_s \frac{A_e^2}{20} ((0.69 + 0.028L)N^{0.2} + 2.7 \times 10^{-6}N) \] \hspace{1cm} -5.1

\( K_s \) is a known sleeper factor (typically 1.1 for BR concrete sleepers), \( A_e \) is an equivalent axle load, \( L \) is the lift at a tamping operation if applicable and \( N \) is the number of cycles.

The equivalent axle load, therefore, allows the effect of mixtures of traffic to be included, and in this case is used to calculate the effective axle load from the individual axles of the vehicle being studied. \( A_e \) can be calculated by equating amounts of settlement for a fixed number of cycles of one load to a different number of cycles of a different load, resulting in the expression:

\[ A_e = \left( \frac{A^5_{N_1} + A^5_{N_2} + A^5_{N_3} + \ldots}{N_1 + N_2 + N_3 + \ldots} \right)^{0.2} \] \hspace{1cm} -5.2

The main problem therefore represents one of finding the maximum load experienced by each individual sleeper. In the dynamic models developed so far this required two additional features to be included. The first of these was a calculation of the sleeper/ballast force beneath the load, which was calculated from the impulse response of the sleeper layer due to a load at the rail head. The second requirement was a knowledge of the distribution of the force along the track. This latter was assumed to be given by the static beam on Winkler foundation relationship. The analysis of Chapter 2 showed that this was reasonably accurate, in fact as the static force also needs to be taken into consideration it is generally the case that the maximum force occurs when the wheel is directly over the sleeper.

The maximum force at prescribed sleeper positions (which were taken to be 0.7 m apart in keeping with BR practise) was therefore calculated in this way and the settlement calculated from equations 5.2 and 5.1 for some prescribed number of axles. This allowed a new profile to be calculated representing the deteriorated profile after the specified traffic had passed. Strictly of course the new profile would result in different vehicle dynamic forces but this effect was neglected for this exercise. To include this feature would require that the calculation proceeded in a step by step manner taking small groups of axles at a time and recalculating the geometry after each step.
The pre and post traffic profiles were both transformed into the frequency domain by calculating the discrete Fourier spectrum of each profile. This implies periodic data, thus the profiles analysed were considered to be repeating continuously. In the case of a dipped rail weld taken over a single rail length this is of course very close to the truth, the only difference being that successive weld angles would vary.

The standard deviation of the profile before and after traffic was also calculated, which is the usual indicator of track quality used on BR. This, when long wavelengths are filtered out, is shown to give a reasonable indication of vehicle ride quality (36).

In the cases considered the track is assumed to possess uniform settlement qualities which would be more representative of a new ballast situation rather than the case of post tamping. The object of the exercise, however, is to study the vehicle effects upon the deterioration.

5.3 Results

Effect of Track Model at a Dipped Weld: Results were obtained first of all for the case of a dipped rail weld as studied earlier, based upon the 25t axle load freight vehicle. The vehicle was represented in all cases by the full vehicle model with the following track models also considered:-

i) Infinite track model without pads (single layer), 'standard' track parameters.

ii) Lumped parameter track model representing 'standard' track with standard rail pads and no Hertzian contact effects.

iii) Standard track parameters but with rail pad layer also included having 'standard' pad parameters.

iv) Standard track parameters with 'soft' pad layer.

The results of the calculations for the four cases are shown in Figs 5.1-5.4. Considering Fig 5.1 first of all the output is essentially the same as the standard force prediction model with various additions. Two track profiles are now presented representing the initial profile and the profile resulting after the
Fig. 5.1 Deterioration of Geometry at a Dipped Weld:
Single Layer Track Model
specified amount of traffic, the average settlement caused by the static axle load effect being first removed. In this case the examples were calculated using 200,000 vehicles representing 20 MGT (million gross tonnes) of 25t axle load traffic which is a typical annual figure for a heavily used main line.

The additional record on each wheelset force history plot (only the leading bogie results are plotted) is the history of maximum sleeper/ballast force at each position assuming that a sleeper is situated at each of these points. When the calculation of deterioration is made these forces are sampled at sleeper spacing intervals thus resulting in a slight dependence on the sleeper positions.

The final graph plotted is the spectrum (Fourier transform) of the initial and final profiles plotted against the spacial frequency in cycles/m with an indication of the respective standard deviations. Considering Fig 5.1, therefore, it can be seen that only a proportion of the peak force which occurs at the dip actually appears at the sleeper/ballast interface. Referring back to Chapter 2 it would be anticipated that approximately 50% of the force associated with wheelset vibration would be transmitted and a very much smaller percentage of the higher frequency component.

The resulting profile therefore shows a much increased vertical irregularity at the weld due to the high impact forces which occur there. When viewed in the frequency domain the profile is shown to have deteriorated at all spacial frequencies as would be anticipated. The shortest wavelength which is transferred to the ballast is around 0.8m (1.25 cycles/m) which is governed by the wavelength of bending of the beam on Winkler foundation and by the sleeper spacing.

In the case with lumped parameters as shown in Fig 5.2 the response is shown to be essentially similar but with the elimination of the high frequency response. This results in slightly lower ballast forces and a corresponding reduction in the amount of deterioration predicted.

The case with standard rail pads introduced (Fig 5.3) interestingly results in a greater amount of predicted deterioration of geometry than the previous two cases. The high frequency force amplitude is reduced for this case but the analysis of Chapter 2 showed that this would represent a mode of vibration having a resonance around 175 Hz and that this mode would result in some small transfer of force to the ballast level. This appears to be the case and the maximum ballast
Fig. 5.2 Deterioration of Geometry at a Dipped Weld: Lumped Parameter Track Model
Fig. 5.3 Deterioration of Geometry at a Dipped Weld: Two Layer Track Model with Standard Pads
force at the weld turns out to be approximately 20% greater at the leading wheelset than the case without rail pads at all. The amount of increase in standard deviation which results is also 50% higher.

In the case with soft rail pads (Fig. 5.4) the high frequency component of wheel/rail and track forces are virtually eliminated and the predicted deterioration of geometry is less in this case than for the case with standard pads. The deterioration is still greater in this case, however, than in that with no pads at all.

If the single layer model is considered as a two layer model with very stiff pads it appears that reducing the pad stiffness initially causes worse sleeper/ballast forces and hence greater deterioration of geometry, which then reduces with softer rail pads. This appears to be caused by two conflicting effects, the first is a reduction in the wheel/rail force amplitude as pad stiffness reduces, and the second is an increase in the transfer of wheel/rail force to the ballast as the highest natural frequency of vibration reduces with reducing pad stiffness. The effective amount of damping in the system may also be playing a part. It appears, therefore, that there will be a worst value of pad stiffness for deterioration of geometry considerations and the present standard pad stiffness may be close to this value.

**Effect of Traffic Types:** The model was also run with various types of traffic with the track model being kept the same (i.e. standard track parameters with standard pads). The vehicles considered were as follows:-

i) The typical primary suspension freight vehicle used previously having design case parameters and a maximum speed of 60 mph.

ii) The same vehicle in a 'worn' state having twice the design friction level.

iii) The same vehicle assumed to be fitted with viscous dampers providing a damping ratio of 0.25 to critical for the body bounce modes.

iv) A simple unsprung mass (equivalent to the freight vehicle unsprung mass) representing the effect of a much simplified vehicle model. This case is also of interest as it represents the
Fig. 5.4 Deterioration of Geometry at a Dipped Weld: Two Layer Track Model with Soft Pads
optimum level of vehicle mode isolation the vehicle designer could hope to achieve.

v) A high speed locomotive at a maximum speed of 125 mph.

vi) A passenger coach at a maximum speed of 125 mph.

The axle loads were 25t in the case of the freight vehicles, 17t for the locomotive and 8.5t for the passenger coach.

The results over a range of speeds are summarised in Fig 5.5 for the case once again of a dipped rail weld. Considering firstly the freight vehicles it can be
seen that an increasing rate of deterioration is seen with speed in most cases. The vehicle causing the least amount of increased track standard deviation was the unsprung mass only. The worst vehicle produced 30% more deterioration than this case illustrating once again the importance of considering the vehicle modes.

It is also apparent that the viscous damped vehicle produced the least amount of track deterioration of the real vehicles. The two friction damped vehicles gave the most deterioration, although interestingly these curves cross over such that the vehicle with the largest friction level produces the lesser deterioration at 60 mph but not at 40 mph. This appears to be caused by the non-linear response of these vehicles and the design case geometry considered which produces a very well defined forcing function unlike a true random track profile.

The range of track standard deviation results obtained, therefore, cover a range of about 30%. In terms of track maintenance requirements this range would be even greater due to the shallow slope of the standard deviation/number of cycles curve at large numbers of axles (Fig 5.6).

![Fig. 5.6 Vertical Track Geometry Related to Maintenance Interval](image)

The high speed passenger vehicles show a similar rate of deterioration to the freight vehicles at lower speeds and the amount of deterioration then
appears to continue broadly linearly with increasing speed. Despite the much lower axle load of the passenger coach the increase in standard deviation is not a great deal less than that of the locomotive indicating the dominance of the dynamic component of track force and the importance of unsprung mass (which is 37% less for the coach). Also at the highest speeds the unsprung mass vibration begins to appear more significantly in the track spectrum due to the longer wavelength and the reduced filtering effect of the track. Thus in order to increase speed still further it would be particularly beneficial to reduce the unsprung mass.

Besides considering the deterioration of geometry of an idealised dipped weld the method developed can also be used to predict the rate of deterioration of measured track profiles. The case considered as an illustrative example is a 55 m length (four 18.3 m rails) of average quality track. The two layer model with standard parameters was used as before although in this case the profile was defined by vertical co-ordinates at 0.1 m intervals thus reducing the effects of higher frequency loads. A reduced range of vehicles from the previous case was also considered.

The summary of the increase in standard deviation of the profile after 200,000 vehicles is shown in Fig 5.7. The initial standard deviation in this case was 2.55 mm. The amount of increase in standard deviation predicted in this case is generally less than in the case of the idealised weld profile, reflecting the rather severe nature of that particular irregularity. The vehicles, however, are generally ranked in the same order as for that situation.

The main difference with the typical track profile situation is the spectral content which contains a greater distribution of wavelengths having no particular phase relationship. Under these circumstances the fictitious vehicle consisting purely of an unsprung mass is significantly better than the other vehicles. The freight vehicle with a high friction level in this case causes some 290% additional deterioration. It is interesting to note also that the vehicle with the lower friction level is not dramatically better than the one with twice the design level of friction. It appears that at low speeds the friction is not broken out on either vehicle thus resulting in a similar response. At the highest speeds the friction is broken out on both vehicles and once again the response of each vehicle is rather similar. It is only therefore at intermediate speeds (and just what speed this will be is governed by the track quality) that the vehicles differ
The results for the two levels of friction damping at an intermediate speed (40 mph) are shown in Figs 5.8 and 5.9. It can be seen for this speed that the force amplitudes differ quite significantly and that this is reflected in the deteriorated track spectra which result. Also various peaks are seen to dominate the track spectra which appear at similar spacial frequencies in each case.

The dominant wavelength which appears in the spectrum appears to be that due to the body bounce mode on the locked friction damper at a frequency of 8.8 Hz or a wavelength in this case of 2 m (0.5 cycles/m). This also happens to
Fig. 5.8 Vertical Deterioration of Typical Track: Freight Vehicle

with High Friction at 40mph
Fig. 5.9 Vertical Deterioration of Typical Track: Freight Vehicle
with Design Friction at 40 mph
coincide with the wheel spacing on the bogies meaning that the force history from each wheelset is in phase, causing a particularly dominant wave of this wavelength to appear in the profile. Furthermore the non-linear summation function for the average force per vehicle and the filtering effect of the force distribution on the track result in harmonics of this wavelength appearing at higher spacial frequencies. The first is clearly apparent at 0.9 cycles/m but higher ones are filtered by the discrete sleeper spacing and will also be aliassed to appear at lower spacial frequencies.

A further peak in the spectrum appears at 0.36 cycles/m which corresponds to a frequency of 6.4 Hz or a wavelength of 2.8m. This does not correspond to any of the natural frequencies of the vehicle and thus also has appeared as a result of non-linear effects. In fact it can be shown that this is caused by summing in a non-linear fashion (by equation 5.2), four time histories which are essentially similar but shifted in phase. In a linear calculation no component of the vehicle dimensions would remain, but in this case a component of bogie wheel centres appears in the spectrum, the peak of which does not necessarily coincide with that dimension. This is also noticeable in some of the other deteriorated track spectra, including those from dipped welds.

The final case studied is that of the high speed locomotive at 125mph (55.9 m/sec) for which the results are shown in Fig 5.10. The most immediately noticeable feature of the deteriorated profile in this case is the large amount of settlement at the welds (at 8m, 27m and 45m) due to the combination of high speed and high unsprung mass. This results, as in the idealised weld case, in a significant component of deterioration at the longer wavelengths, more so than with the freight vehicles, and these longer wavelengths would be expected to have an effect upon passenger comfort. Harmonics of the rail lengths also appear further up the spectrum in line with the peaks normally found in measured track spectra. The natural frequency of the wheelset around 56 Hz also produces a small contribution to the total spectrum at 1 cycle/m.
Fig. 5.10 Vertical Deterioration of Typical Track: Passenger Locomotive at 125mph
5.4 Prediction of the Deterioration of Lateral Geometry

As described earlier there does not appear to have been any attempts to produce a theoretical model of lateral track deterioration in the past and thus the work presented here represents an initial look at the problem in order to obtain some preliminary results. The modelling is based around the theoretical methods developed for the prediction of the lateral forces and also upon the empirical relationship suggested from measurements made so far on the lateral movement of track due to traffic.

In a summary of the work performed on BR, Gilchrist (44) offered empirical relationships for the lateral permanent displacement of track as a function of lateral to vertical load ratio and also as a function of number of applications as presented in Chapter 1. These can be combined as follows:

\[ \delta \alpha 2.1^{10L/P}N^{0.25} \]

Where \( L/P \) is the lateral to vertical force ratio and \( N \) is the number of cycles. This expression has certain similarities with that used for the vertical case with the exception that a function of \( L/P \) describes the effect of force and that the power of \( N \) is 0.25 as opposed to 0.2 in the vertical case. From the data presented a mean constant of proportionality of 0.0324 was derived for \( \delta \) in mm.

For mixed lateral loads an equivalent value of the function \( 2.1^{L/P} \) can also be implied by analogy with the vertical case and is given by:

\[ 2.1^{10L/P}e = \left( \frac{2.1^{40L/P} + 2.1^{40L/P} + 2.1^{40L/P}}{N_1 + N_2 + N_3 + \ldots} \right)^{0.25} \]

The expressions given were therefore used in conjunction with the finite element lateral track model from which the lateral sleeper forces can easily be obtained. The vertical sleeper forces were calculated on the assumption of a static beam on Winkler foundation distribution from which the force can be calculated at a given sleeper for any position of a wheel. To this value the self weight forces were also added, and the permanent lateral sleeper movements obtained for each sleeper. The examples which have been produced are calculations of the deterioration at discrete lateral kinks and for this case the maximum lateral settlement was recorded as well as the standard deviation of
the settlements of all the sleepers. The standard deviation of the original profile was not calculated as this has little relevance for the case of a kink in two straight rails. Once again the effect of the new geometry in modifying the lateral forces was not considered in these initial calculations.

The calculations were once again performed for 200,000 vehicles, these being based upon the two four wheeled freight vehicles studied earlier with differing yaw suspensions.

5.5 Results

**Effect of Vehicle Parameters at a Lateral Kink:** The results from the model developed were calculated for the same range of kink angles and speeds as was studied in Chapter 4 with the same vehicles referred to as A and B. The forces predicted were therefore the same. An additional case which was also studied was the effect of a vehicle having reduced unsprung mass as this can be expected to be a contributory factor to the lateral deterioration of geometry as well as the vertical. The results presented in Fig 5.11 show a summary of the maximum permanent lateral displacement (which was always close to the kink) and the standard deviation of the resulting permanent displacement. The results obtained for vehicle B having a softer yaw suspension were in fact extremely similar to those for vehicle A, due to the lateral impact forces being very similar as shown earlier, for this reason only one set of results are presented for clarity.

The results show a rapid rise in lateral movement of the track with kink angle and with speed. The 5 mrad kink appears to be quite stable as far as speed effects are concerned. This is presumably because significant flange contact does not occur on any of the vehicles within the range of speeds considered. The suggestion that flange contact is an important consideration is also reinforced by the results obtained for the vehicle with reduced unsprung mass. These show that at the higher speed/larger angle combination benefits are obtained from a reduced unsprung mass due to the reduced dynamic load. The deteriorated profile predicted for vehicle A negotiating the 10mrad kink at 60 mph is shown in Fig 5.12.

The results obtained were also compared to the limits proposed by Prud'homme (43) as described in Chapter 1. The relationship for the critical lateral load \( L \) at which large lateral movements occur was given by:-

\[
L = A(P + P_0)
\]
Fig. 5.11 Summary of Predicted Deterioration of Geometry at Lateral Kinks
A is a coefficient as described in Chapter 1 which taken for the average conditions here was assumed to have a value of 0.45, \( P \) is the vertical load and \( P_0 \) is a constant generally having a value around 4 tonne. This gives for the case in question a value for \( L \) of approximately 95 kN. For the worst case considered above, the maximum lateral load was 60 kN and thus well inside this limit. The predictions of lateral movement made for this case, however, would be considered quite large by normal track standards. The results obtained, therefore, seem to conflict with the results of Prud'homme, which suggest that negligible lateral movements would be found below the critical load although measured BR data was also shown to conflict with those results.

A further case which was considered was track with a lateral kink as already considered but also having additional random track irregularities. The typical spectral density of lateral alignment described in Chapter 2 was used to create a lateral alignment profile having a standard deviation of 2 mm in a similar way to which the transition curve data was modified in Chapter 4. This was added to the original kinked straight rail profile and the deterioration calculation performed. Once again there was little difference between the vehicles A and B and so the former results are not presented. The summary of the results is shown in Fig 5.13. The results are in fact shown to be very similar to the ones obtained previously without random irregularities with the worst
Fig. 5.13 Summary of Predicted Deterioration of Geometry at Lateral Kinks with Random Irregularities
case showing a permanent lateral movement only 7% higher than the case without random irregularities. This small increase therefore suggests that the discrete irregularity is causing flange contact to occur and hence causes lateral track shifting. Random irregularities in this instance have little additional effect because they do not on their own cause additional flange contact, and hence large dynamic forces, to occur.
6. SUMMARY AND SUGGESTIONS FOR FURTHER WORK

Various models of vehicle and track have been studied and their suitability for evaluating the loads due to dynamic vehicle/track interaction forces have been investigated. The problem, whether viewed in the lateral or the vertical plane consists of a vehicle, having multiple degree of freedom, which travels along an infinite elastic, resiliently supported beam having surface irregularities. The fact that the vehicle is moving along the beam has been shown to have little effect on the response of the system, other than of course to provide the forcing input. Other features which are shown to have little influence upon the response in a railway track situation are the shear and rotatory inertia effects, it is only at extremely high frequencies that the bending wavelengths in the rail are sufficiently small to require consideration of these terms.

When considering the vertical response of the track, therefore, a model of an infinite beam on elastic support is considered. Resilience is provided by a rail pad layer between rail and sleepers, and by the ballast and foundation flexibility beneath the sleepers. When analysed in the frequency domain with a stationary harmonic load, such a model is shown to possess most of the features of a two degree of freedom system, displaying two resonant modes of vibration corresponding to an in phase and an out of phase response of rail and sleepers. Such a model appears to be capable of predicting the track response up to a frequency of around 200Hz, depending mainly upon the pad parameters. For many railway dynamics problems this is perfectly adequate, particularly when considering track foundation and component effects which are shown for typical track irregularities not to experience significant components of force with frequency higher than this level. For higher frequency considerations, however, such as wheel/rail noise and rail corrugation studies, models of the track including higher modes, for example of wheelsets and sleepers are likely to be required.

A time domain solution of the whole vertical vehicle/track system has also been obtained which allows non-linearities in the wheel/rail contact conditions and vehicle suspension to be included. When considering the vertical wheel/rail forces at the discrete irregularity at a rail weld, significant forces due to the impact of the unsprung mass are found, in keeping with the discoveries of other
workers in this area. The high frequency \( (P_1) \) component is strongly governed by the properties of the pad layer and with very soft pads is very nearly eliminated. Under these conditions good approximation to the wheel/rail forces can be obtained from equivalent lumped parameter models of the track which offer advantages in the computation of the results. Such simplified models are quite suited to the prediction of the forces from measured track profiles which is less easy with the more complex models.

Aspects of the response of the vehicles have also been shown to be important in terms of forces on the track. Modes of vibration of the vehicle components such as body and bogies do result in significant contributions to wheel/rail forces. In the case of dipped rail welds vertical excitation of the body of a bogie vehicle usually results in larger impact forces at the trailing wheelset than at the leading. Pitching of the bogie also applies forces either side of the weld irregularity. Furthermore a particular feature of freight vehicles which causes significant forces on the track is damping by friction. This is used almost universally on freight vehicles in one form or another and has particular consequences when poor design or state of wear allows friction levels to become large. Under these conditions lightly damped oscillatory forces up to the friction level are shown to prevail almost continuously. Interestingly the forces on poor track or at discrete irregularities are generally not significantly larger than those due to vehicles with better quality suspensions but on good track there is a large difference.

The predictions made from the vertical models developed have been used to study a particular track damage mechanism which is strongly governed by the vehicle/track interaction forces, namely the deterioration of geometry. By extending the modelling in a fairly simple manner to also predict the settlement from the track forces by the use of some empirical relationships it has been possible to indicate how some of the refinements of the modelling would affect a real railway problem. Predictions have been made once again for the differential settlement of a rail weld and also for a typical track profile.

It was shown that the best results are obtained when the rail pad layer is included in the track model. The results showed that for track deterioration considerations at a dipped weld soft pads or very stiff pads both in fact resulted in a lower rate of deterioration of geometry than pads with an intermediate stiffness value. Essentially this was because dynamic response of the sleeper at higher frequencies was the least under these conditions. The vehicle modes of
vibration were once again shown to be important and to make a significant contribution to the rate of deterioration. These modes could actually produce discrete peaks in the spectrum of track profile which would be most likely to occur in practice on sites where large numbers of similar vehicles run at the same speed. Certainly such apparent effects can be seen in some measured track spectra.

The non-linear additive effect of the settlement mechanism also tends to result in a component of wheel spacing appearing in the spectrum of the track profile. The interesting fact being that this wavelength becomes slightly corrupted due to the non-linear effects. As a large number of freight vehicles have 2 m bogie centres this might be expected to appear also in measured track spectra. Such discrete wavelengths have in fact been noticed, but these have normally been attributed in the past to rail rolling defects.

There is little doubt, therefore, that although simpler models of the vehicle and track can give some useful results, the most important features of the dynamic vehicle/track interaction problem can only be considered fully with the more elaborate models.

The track models developed for the lateral case have certain essential similarities with those used for the vertical modelling. The track can be considered to behave as a beam on elastic foundation laterally supported by the ballast resilience as in the vertical case, but the rail pad layer is replaced by a lateral flexibility of the rail head relative to the foot. In fact it is shown that the flexibility of the rail in this mode can be replaced quite simply by a lumped spring without any apparent loss of accuracy in the modelling, due to the low inertia and damping in the rail layer. Analysis of lateral models of the track have also suggested that the higher frequency response need not be as good as in the vertical case, the reason for this being that the higher mode of the track in which sleepers and rail head move in anti-phase is in practice very heavily damped. Very good representation of wheel/rail forces in plain track situations can therefore be obtained by the use of lumped parameter track models.

Non-linearities of the vehicle and wheel/rail contact conditions also exist in the lateral case and for this reason a time stepping integration method of solution has also been used for the lateral vehicle/track interaction modelling. When using such methods to consider the response of vehicle and track to discrete lateral irregularities certain similarities are found to the vertical case,
the most noticeable being that impact of the wheelset upon the lateral resilience of the track causes a decaying oscillatory dynamic force. This is somewhat similar in character to the vertical $P_2$ force but depending upon track conditions has a natural frequency of around 15 Hz in contrast to the vertical case which occurs around 40-50 Hz. Damping of this particular mode can come from the track, but also from creepage forces on the non flange contacting wheel, the influence of which becomes small if the lateral creepages are sufficiently large for sliding to take place.

The special case of a switch has been modelled by means of a finite element representation of the track, in order to reasonably accurately represent the variation of the parameters which would occur at such a complicated structure. The effects of assumed variations in these parameters are generally shown to be small for design case geometry. Where flange contact generally only occurs at the switch toe the anticipated force levels were also shown to be small when based upon design case geometry but the forces increased rapidly when errors of alignment were assumed. This was particularly true of the higher speed switches.

The work performed in this area would particularly benefit from measurement of parameters and geometry at a variety of switches. Dynamic parameters are very difficult to measure under realistic excitation conditions but static stiffness measurements would be a useful beginning. The geometry is also an important aspect, it is obviously not easy to install and maintain such components to fine tolerances, but it is anticipated that the methods developed can give a good indication of the likely benefits of improved geometry.

The models have also been extended in the lateral case to model deterioration of geometry in a fairly simple manner. The results obtained, however, give some useful pointers to the causes of unacceptable lateral track geometry. Essentially the steering ability of the vehicle is important to avoid lateral flange contact of the wheels. However, at discrete or short wavelength irregularities flange contact cannot be avoided and the steering ability of the vehicle has little effect. Under these conditions the dynamic impact forces, and hence the permanent lateral movement of the track is strongly influenced by the wheelset mass and track parameters. Under flange contact conditions the rate of track deterioration increases rapidly with irregularity size. This would occur of course at discrete irregularities in straight track and also most likely at
irregularities in curves negotiated at high cant deficiency in which case the outer wheels would already be in partial flange contact.

To study more thoroughly the lateral deterioration of geometry, and particularly the effect upon passenger ride, the model should be modified to handle bogie vehicles also. The connection between standard deviation of alignment and ride is also both difficult to define and not clear. The model offers the possibility to analyse accelerations of vehicle body and hence predict passenger ride quality more directly by defining a ride index for a given profile.

**Suggestions for Further Work:** Having gathered knowledge on the importance of some of the input parameters to the model, work is now required to establish more accurately some of these properties as they appear in railway vehicles and track. The properties of the pads for example are shown be quite important but little data actually exists on the characteristics of these materials as used in track. These are very difficult to measure of course because temperature, pre-load, frequency and amplitude of excitation can all effect the stiffness and damping properties of rubber and synthetic materials. On the vehicle side measurements are required of the friction levels in freight vehicle suspensions, as the limited amount of experimental data suggests that in practise the levels may be too high. This is shown theoretically to have an important effect on the track forces and track damage.

Validation exercises should also be performed to study some of the aspects of deterioration of geometry which have been discovered. The dynamic force model has in fact also been combined with a more complex deterioration model which includes effects of rail straightness, void generation beneath sleepers and change of dynamic loads with traffic. Although the amount of settlement predicted is expected to be more accurate with this model the general conclusions which can be drawn from the more complex model are the same as with the simpler model.

The vertical modelling of track forces and deterioration are also being used in financial models of track damage, not just for deterioration of geometry but also to predict the forces influencing rail and rail weld failures and sleeper damage which are other sources of track maintenance costs. The models were used to establish likely track maintenance cost changes of future traffic and speed options on the East Coast and West Coast Main Lines on which traction policy decisions have now been made.
There is no doubt that in the past more work has been performed on the vertical vehicle/track interaction than on the lateral. The work described has, however, gone some way towards redressing this balance. This modelling is also in need now of measurements of the track and vehicle parameters, particularly track geometry measurements at discrete irregularities and switches. Furthermore a part of the switch structure has been neglected up to now, namely the crossing at the intersection of the two rails. This is a combined vertical and lateral problem which would be a complex problem to approach, but again would offer potential benefits from improved design.

The model as developed can hopefully be used to influence the future design of switch and crossing works, particularly with respect to specifying tolerances on geometry. An experiment to measure parameters and forces at a switch with measured geometry would now be a worthwhile exercise.

A more thorough model of lateral deterioration may be useful. The present model does tend to agree in broad terms with measurements of subjective passenger ride (58) in that it appears to be an occasional large irregularity as opposed to the general continuous random irregularities which passengers object to. The model could therefore indicate, in the case of increased vertical maintenance intervals, how many more discrete irregularities are likely to increase to an unacceptable size. This type of occasional discrete irregularity would not be clearly detected by the present track measurement system (i.e. the measurement of filtered PSDs of 200 m sections) and it is therefore questionable whether such a method is an ideal measure of lateral track quality. Again this could be investigated by modelling.

It is clear, therefore, that there are plenty of potential applications for the work performed, and some have already been demonstrated. The annual maintenance and renewal costs for permanent way in 1984 on the BR network were over £350m, it is apparent from this, therefore, that there is plenty of scope for cost savings as a result of such improved knowledge. For this reason some of the developments of the modelling methods outlined are under way, but more importantly the results are now being used to influence vehicle and track design and maintenance policy for the future.
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APPENDIX I

DERIVATION OF PARAMETERS FOR LUMPED PARAMETER TRACK MODELS

I.1 Beam on Winkler Foundation

Equivalent lumped parameters can be obtained by comparing the solution for a beam on Winkler foundation to that for a single degree of freedom oscillator. For the former the differential equation is:

\[ EI \frac{d^4z}{dx^4} + m \frac{d^2z}{dt^2} + c \frac{dz}{dt} + k_f z = P_0 e^{i\omega t} \delta(x) \]

for which the particular solution can be assumed to be of the form:

\[ z(x,t) = z(x)e^{i\omega t} \]

Therefore substituting in I.1 gives:

\[ EI \frac{d^4z}{dx^4} + (k_f - m \omega^2 + i\omega c_f)z = P_0 \delta(x) \]

This is recognised as being equivalent to the equation of a static beam on Winkler foundation for which the standard solution can be used viz:

\[ z = \frac{F_0 \beta}{2k} e^{-\beta x} e^{i(\beta x + \sin \beta x)} \]

Where \( \beta = k_f - m \omega^2 + i\omega c_f \)

and \( \beta = \left( \frac{k}{4EI} \right)^{0.25} \)

from which it is possible to derive the receptance of the system \( H_1(\omega) \) as follows:

\[ H_1(\omega) = \frac{z(0,t)}{P_0 e^{i\omega t}} = \frac{\beta}{2k} \]
\[ i.e. \quad H_1(\omega) = \frac{1}{2} \left( \frac{1}{4EI(k_f - m_\omega^2 + i\omega \xi_f)^3} \right)^{1/4} \]

\[ = \frac{\beta}{2k} \left( 1 - \frac{m_\omega^2}{k_f} + \frac{i\omega \xi_f}{k_f} \right)^{-3/4} \quad \text{--- I.2} \]

where \( \beta \) is the static equivalent of \( \bar{\beta} \) usually known as the characteristic wavelength. The expression I.2 is now compared to what would be obtained for a single degree of freedom oscillator for which the equation of motion is given by:

\[ M_0 \dddot{x} + C_0 \dot{x} + K_0 x = P_0 e^{i\omega t} \]

The receptance for this system is easily derived and is given by:

\[ H_2(\omega) = \frac{1}{(K_0 - M_0 \omega^2 + i\omega C_0)} \]

Equivalent parameters can therefore be obtained by expanding equation I.2 in Maclaurin series form to give:

\[ H_1(\omega) = \frac{\beta}{2k} \left( 1 - \frac{3m_\omega^2}{4k_f} + \frac{3i\omega \xi_f}{4k_f} - \frac{3}{32} \left( -\frac{m_\omega^2}{k_f} + \frac{i\omega \xi_f}{k_f} \right)^2 + \ldots \right) \quad \text{--- I.3} \]

If terms higher than \( \omega^2 \) are neglected and the respective receptances are equated the following relationships can be established:

\[ K_f = \frac{2k_f}{\beta} \quad \text{--- I.4} \]

\[ M_f = \frac{3m_f}{2\beta} \quad \text{--- I.5} \]

\[ C_f = \frac{3c_f}{2\beta} \quad \text{--- I.6} \]
It can be shown for typical parameters (by comparing expressions 1.2 and 1.3) that at frequencies up to 100 Hz the agreement is better than 10% in both the real (dynamic stiffness) and imaginary (damping) terms.

### 1.2 Two Layer Beam on Winkler Foundation

In the case of the two layer model applied to the vertical case, derivation of equivalent parameters is not so straightforward. Equivalent parameters can be derived but explicit expressions such as those above cannot. An alternative approach can be taken, however, as the lumped model is only required to represent the fundamental mode of vibration of the track, which can be assumed to be approximated by the static deflected shape. Expression I.4 is also valid under static conditions and it is reasonable to assume that expressions I.4 to I.6 could be used to calculate lumped parameters if equivalent distributed parameters could be obtained for the two layer and single layer infinite models.

![Fig. I.1 Equivalent Single Layer and Two Layer Models](image)

Considering the two models shown in Fig I.1 equivalent expressions can be derived by considerations of the energy in the components of each system while undergoing simple harmonic motion. The strain energy in the springs over an elemental width $\delta x$ is given by:

\[
E = \frac{k \delta x z^2}{2}
\]

\[
= \frac{k_e \delta x z^2}{2} + \frac{k_p \delta x (z_1 - z_2)^2}{2} + \frac{k_f \delta x z_2^2}{2}
\]
Now \((z_1 - z_2) = \frac{k_f z}{(k_f + k_p)}\)

and \(z_2 = \frac{k_p z}{(k_f + k_p)}\)

\[
\frac{k_e \delta x z^2}{2} = k_f \delta x \frac{k_f^2 z^2}{(k_f + k_p)^2} + k_p \delta x \frac{k_p^2 z^2}{(k_f + k_p)^2}
\]

giving \(k_e = \frac{1}{\frac{1}{k_f} + \frac{1}{k_p}}\)

It is clear that this expression also satisfies static considerations. In the viscous damping case the energy dissipated per unit time, \(E = c \delta x \ddot{z}^2\)

\[
\because c_e \delta x \ddot{z}^2 = c_p \delta x \frac{k_f^2 \ddot{z}^2}{(k_f + k_p)^2} + c_p \delta x \frac{k_p^2 \ddot{z}^2}{(k_f + k_p)^2}
\]

giving \(c_e = \frac{c_p k_f^2 + c_p k_p^2}{(k_f + k_p)^2}\)

similarly by considering the kinetic energy of the masses:

\[
E = \frac{m \ddot{z}^2}{2}
\]

giving \(m_e = \frac{m_r k_f^2 + m_s k_p^2}{(k_f + k_p)^2}\)

These values can be substituted into expressions 1.4, 1.5 and 1.6 to give the equivalent lumped mass, stiffness and damping for the two layer system.
APPENDIX II

AN EXPERIMENT TO MEASURE THE VERTICAL DYNAMIC FORCES
AT A RANGE OF DIPPED WELDS

II.1 Introduction

The experiment was originally conceived to compare the track forces from a range of 25t axle load freight vehicles in order to establish if any particular vehicle was significantly worse than any other. This work was therefore performed by members of the Track Research Unit of the BR Research Department and the required results were obtained from the forces measured by load measuring baseplates installed in the vicinity of the rail welds. Measurements were also taken, however, of vehicle component displacements and accelerations which offered the opportunity to learn more about the way in which the vehicle/track forces are produced. For this reason the analysis presented below was performed by the author on the results obtained and a theoretical model developed which is described in the main text.

The tests basically consisted of a series of four dipped welds for which the geometry was measured and load measuring baseplates installed, over which four different types of freight vehicles were run at a range of speeds. The vehicles were nominally similar being 100t gross weight four axle bogie vehicles but having different bogie and suspension types. All the vehicles, however, were damped by friction.

II.2 Test Site

A site was found at which four consecutive welds were found to be dipped, and the range of angles was considered to be typical of those which might be found on the BR network in general. The vertical geometry in the region of each weld was measured using a straight edge and dial gauge system at a longitudinal spacing of 0.1m. The welds were referenced A, B, C and D in the direction of running and the dip angles at the welds were found to be 2 mrad, 4 mrad, 4 mrad and 7 mrad respectively.
II.3 Track Based Instruments

The track in the area of each weld was fitted with load measuring baseplates (55) which measure the load between rail and sleeper. Potentiometers were also installed at selected locations between the sleeper and a pile driven into the ballast in order to measure vertical displacement. This in conjunction with the force measurements enabled estimates of the vertical ballast stiffness to be made.

The measurements from the track based instruments were recorded onto digital tape by sampling at 500 scans/sec on site.

II.4 Test Train

The test train consisted of two freight locomotives hauling a range of 100t bogie vehicles (Fig II.1) which were grouped together into suspension types as follows:-

Secondary suspension bogies with primary rubber pads (S1) - Oil tankers Measuring Car.
Primary suspension bogies (P1) - Iron Ore.
Primary suspension bogies (P2) - Steel carrier Measuring Car.
Secondary suspension bogies (S2) - Oil tanker

The vehicles were all laden to their design axle loads, the axle spacings on the bogies were all identical but the bogie spacings were not necessarily the same. The design maximum speed of all of the vehicles was 60 mph.

II.5 Vehicle Based Instruments

One of the bogies of each of the different vehicle types was instrumented with accelerometers and displacement potentiometers. The layout of the instrumentation is shown in Fig II.1.

Recording of the vehicle based data was made onto analogue magnetic tape and digitised at a later date with a low pass filter at 80 Hz applied. In order to relate the measurements to locations on the track a signal from the track based
Fig. II.1 Test Vehicle Suspension Configurations and Instrumentation Layout
instruments was also recorded by means of a radio link. Additionally the train speed was recorded as a function of time.

II.6 Test Procedure

The test train was run through the site at a range of speeds, these being in nominally 10 mph increments up to the maximum of 60 mph. A run was also included at crawl speed. Over most of the speed range four runs were carried out at each speed and the whole train was turned at the half way point. Due to some problems with availability of vehicles and also instrumentation problems all of the runs were not necessarily available at every speed.

II.7 Analysis of Test Results

It was required to compare the wheel/rail forces predicted by the theoretical model with those measured by the experiments. On first inspection it may appear that the results from the load measuring baseplates would enable this, however that is not the case. The load measuring baseplates measure the load between rail and sleeper, and for a single axle, or for a vehicle with widely spaced axles, adjacent sleeper forces can be summed to give the dynamic wheel or axle load. In this experiment, however, the axles were only 2 m apart thus meaning that such an exercise would not be valid, meaning that the load measuring baseplates were only useful for evaluating the relative track forces from each vehicle.

In order to obtain a reasonable estimate of the wheel/rail forces, therefore, the vehicle based measurements were used. The measured accelerations of masses and displacements of springs were used to calculate the forces which would have been produced at the wheel/rail contact. This required a knowledge of the spring stiffnesses and masses of each vehicle which were taken to be the design values with friction forces being neglected. This is likely to produce an error if the values used were incorrect, but reasonable accuracy can be expected from such a calculation for the following reasons:

i) The largest component of the wheel/rail force at dipped welds is shown to be the $M^2$ term (unsprung mass x acceleration), the unsprung mass of each vehicle was known quite accurately.

ii) The effect of friction damping is to modify the response of the vehicle at the approach to the dip, but the friction does not significantly affect
the impact force as the wheelset is de-coupled at the impact response frequency. The error to be anticipated from neglecting the friction forces does not therefore exceed the effective amount of friction per axle which based upon design values is less than 10 kN/axle for each vehicle.

Thus the wheel/rail force can be calculated by straightforward application of Newton's second law of motion, and by equilibrium considerations.

The neglecting of the friction forces is obviously a restriction to the analysis, and it is anticipated that this would introduce an error into the reconstructed track forces. For the reason described in (i) above the friction force is generally not large at dipped weld impacts unless the track before the weld causes significant amounts of vertical body motion resulting in a pre-existing vertical force when the wheel hits the weld.

![Fig. II.2 Forces on Primary Suspension Vehicle](image)

The wheel/rail forces can therefore be calculated as shown below. For a primary suspension vehicle as illustrated in Fig II.2, resolving vertical forces on the wheelset gives:

\[
P_t = -M_w \ddot{z}_t + k z_s
\]
with the trailing wheelset similar.

\[ (z_t + z_t)Pt + Pt = Mb^2 + kz \]

By taking moments about the centre of the bogie frame:

\[ (Pt - Pt)a = -I_b \frac{(\ddot{x}_l - \ddot{x}_t)}{2a} \]

from which by elimination gives:

\[ Pt = -\left(\frac{Mb}{4} + \frac{I_b}{4a^2}\right)\ddot{x}_l - \left(\frac{Mb}{4} - \frac{I_b}{4a^2}\right)\ddot{x}_t + \frac{kz_s}{2} \]

For the vehicle with rubber pad primary suspension it is desirable to eliminate the pad force due to the anticipated non-linearity of the pad material. This is illustrated in Fig II.4.
From the equations for a secondary suspension vehicle the pad force $Q$ is given by:

$$Q_t = -\left(\frac{M_f}{4} + \frac{I_f}{4a^2}\right)z_{bt} - \left(\frac{M_f}{4} - \frac{I_f}{4a^2}\right)z_{bt} + \frac{kz_s}{2}$$

therefore by resolving forces on the wheelset:

$$P_t = -\left(\frac{M_f}{4} + \frac{I_f}{4a^2}\right)z_{wt} - \left(\frac{M_f}{4} - \frac{I_f}{4a^2}\right)z_{bt} + \frac{kz_s}{2} - M_w z_{wt}$$

and similarly $P_l$

II.8 Discussion of Results

The results obtained from the analysis of the vehicle based data were used to compare with the predictions made by the vehicle/track model. The parameters used for the modelling being design case ones for each of the four vehicles. The track stiffness was obtained by cross plotting sleeper vertical force
and displacement for samples of the instrumented sleepers, a typical example of such an exercise is shown in Fig II.5. The stiffness taken to be applicable for dynamic calculations was the tangent stiffness at the highest point of the curve as shown. The average value obtained was also verified by studying natural frequencies of response of locomotive wheelsets at the dipped joint impacts which also gave an indication of the damping present. The values obtained were used to calculate equivalent lumped track parameters by the methods of Appendix I.

![Typical Cross Plot of Vertical Sleeper Force and Displacement](image)

*Fig. II.5 Typical Cross Plot of Vertical Sleeper Force and Displacement*

An example of the time histories obtained from the predictions and from the experimental results is shown in Fig II.6 for the largest dip at the maximum speed. The results show the characteristic decaying oscillatory force due to the
Fig. II.6 Experimental and Theoretical Force Histories
impact, with reasonable agreement in this case of the amplitude, frequency and rate of damping. No high frequency vibration is noticed but it would not be anticipated that the method used would detect high frequency forces at the wheel/rail contact as these are quickly dissipated by inertia effects. Some longer wavelength oscillation away from the dip is predicted by the theory due to the locking of the friction dampers. Because forces in these are not measured in the experimental data these are not present in the experimental results. The agreement of the experimental and theoretical results is considered further in the following section.

II.9 Comparison with the Theoretical Results

The peak dynamic forces which were measured at each of the dips are presented and compared with the theoretical predictions in Figs II.7 to II.10, one of the medium sized dips and the largest dip are considered. The first thing that is apparent about the vehicle based results is the large amount of scatter, probably reflecting the rather variable and unrepeatable nature of friction damping.

A further consideration of the magnitude of the errors was made by calculating the percentage difference between experimental values and theoretically predicted values as shown in Table II.1. The analysis was also restricted to the highest three speeds (40, 50, 60 mph) to minimise the effect of the non-linear trend with speed. The errors tabulated represent the ratio of the difference in values to the measured value, a negative error indicating that the theoretical value was the smaller. An average percentage error was calculated over the range of speeds considered as well as the average of the absolute magnitude of the errors regardless of sign. Where the two numbers are the same this indicates that all the errors were of the same sign.

Table II.1 shows that some of the differences were quite large, particularly at the smallest dip angle. This tends to suggest, therefore, that the differences in the forces were a constant value, or at least not proportional to the speed, and this could well have been due to the unknown friction force. In fact the overall average of the absolute errors was 25% which, when considered with the overall average peak force of 49 kN suggests an average error of 12 kN, this being very close to the 10 kN error anticipated for design case friction levels.
Fig. II.7 Peak Forces at Dip B (4mrad.) - Leading Wheels
Fig. 11.8: Peak Forces at Dip B (mrad). - Trailing Wheels

VEHICLE P1

VEHICLE P2

VEHICLE S1

- THEORETICAL
- VEHICLE BASED

VEHICLE S2

AVERAGE PEAK FORCE PER WHEEL (kN)

SPEED (mph)
Fig. II.9 Peak Forces at Dip D (7 mrad.) - Leading Wheels
Fig. II.10 Peak Forces at Dip D (7mrad.) - Trailing Wheels
Vehicle S1 | Vehicle P1 | Vehicle P2 | Vehicle S2 | Mean | Mean abs. | Mean | Mean abs. | Mean | Mean abs. | Mean | Mean abs. | Tot.abs |
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<td></td>
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<td>-27</td>
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<tr>
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<td>25</td>
<td></td>
<td></td>
<td></td>
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</tr>
</tbody>
</table>

**Table II.1 Differences Between Experimental and Theoretical Results**

The results obtained from the largest dip give the smallest differences between theory and experiment (18% average) which suggests that when the P$_2$ component of force at the dip is significant, and the component due to friction effects proportionately smaller, the agreement is somewhat better.

The theoretical predictions generally produce a lower force than was measured, as indicated by the large number of negative errors in Table II.1. When studying the theoretical force histories as for example in Fig. II.6 it is apparent that the theoretical forces are generally unloading at the approach to the welds and that this would represent a negative force in the friction damper which would not be apparent in the experimental results. The resulting difference would, therefore, be in the direction measured.

In terms of variation between vehicles the differences between theory and experiment are generally the same, except that the vehicles with primary rubber pad suspension (S2) showed larger differences. This could indicate that either the friction levels were higher for this vehicle or that the rubber pad stiffness value used for the theoretical calculations was incorrect, or not valid for the range of loads experienced.
II.10 Conclusions

The experiment performed represented an opportunity to study the response of freight vehicles with friction damped suspensions to track irregularities, and compare the forces resulting. The impact of the unsprung mass of the vehicles at the weld irregularities investigated was shown to represent a significant part of the response as has been suggested by railway engineers for some time. It is also clear, however, that the dynamics of the vehicle can also have an effect upon the track forces and in this case the friction damping also appears to make a significant contribution.

The modelling of the track by a single degree of freedom lumped parameter oscillator at each wheel seems to be reasonable for predicting the response of the system at the frequencies at which the unsprung mass bounces, and those below, which include all the vehicle frequencies. Prediction of the higher frequency (P₁), type response would certainly require a better track model but the experimental measurement of these forces would also be much more complicated.

The agreement obtained with the theoretical predictions was reasonable but could have been improved with better knowledge of the characteristics of the friction dampers and their performance in the test. As a result of what was learnt in the experiment described here subsequent experiments are now planned to improve this knowledge.
APPENDIX III

AN EXPERIMENT TO MEASURE THE VEHICLE/TRACK RESPONSE 
TO A DISCRETE LATERAL IRREGULARITY.

III.1. Introduction

This work was planned in order to obtain experimental data on the 
response of a vehicle negotiating a discrete lateral irregularity of the type which 
might be found at a switch toe. Wheel/rail forces and wheel and track 
displacements were measured in the vicinity of a kink formed in plain rail. The 
experiment was performed on plain track as opposed to at a real switch in order 
to simplify the necessary instrumentation, but also to gain an idea of the 
fundamental features of the interaction which takes place between vehicle and 
track without the complicated structure which would be apparent at a switch.

The experiment was carried out in the initial stages of theoretical 
modelling of the lateral vehicle/track interaction problem in order to provide 
experimental data and also to isolate the important features of the track 
response which needed to be modelled.

III.2. Test Site

The tests were performed on a BR Research Dept. test track where the 
geometry of an existing transition curve was changed to match the geometry of a 
type 'E' straight planed switch, having a design speed of 40 mph. The essential 
feature of this type of switch being that the geometry at the switch toe 
essentially consists of two straight rails forming a kink. This was achieved in 
plain rail by bending the rails beyond yielding to form a knuckle using 
hydraulic bending equipment. The entry angle of this type of switch is 15.6 
mrad, followed by a straight length of 4.47m running tangentially in to a curve 
of 746m radius (Fig III.1). Adopting this particular shape meant therefore that 
the forces obtained from this test could be subsequently compared with results 
obtained from real switches. The track at the site was on granite ballast having 
timber sleepers with 109 lb/yd flat bottomed rail with BR1 type cast baseplates 
held by elastic spikes. The cant was removed throughout the site and the final 
geometry was obtained with the assistance of a tamping and lining machine.
III.3 Track Based Instruments

The instruments mounted on the track were as follows:-

i) O.R.E load measuring baseplates (55) on each rail on the sleeper preceding the kink and the subsequent four sleepers giving ten stations in all. These devices use a configuration of strain gauges which give a linear response to vertical and lateral load and can be calibrated in the laboratory before installation. A quite rigid connection between rail and sleeper is expected with these devices, however this is equally to be expected at a switch toe where rigid baseplates are also used.

ii) Displacement transducers mounted laterally on consecutive sleepers two before and seven following the kink. A hollow steel beam rigidly fastened to piles in the ballast formed the datum point to which the transducers were fastened. (Fig III.2)

iii) Displacement transducers mounted laterally between the rail head (on a glued bracket) and brackets rigidly attached to the sleepers. Three of these were mounted on the outer rail side and one on the inner rail side on the sleepers following the kink. (Fig III.2)

The output from the instruments was recorded continuously onto analogue tape during each pass of the test train and the data was subsequently digitized for analysis at a rate of 1000 scans/sec. Track switches were used to detect the approaching train and start and stop the tape recorders, an electronic timer being used in conjunction with the track switches to calculate train speed. A plan of the instrument layout is shown in Fig III.3.

III.4 Test Vehicle

The vehicle used for the test was a prototype two axle high speed freight vehicle described previously (Vehicle B) for which the suspension parameters had been measured. It was used laden resulting in an axle load of 17 tonnes. A list of the vehicle parameters applicable to the vehicle has already presented in Table 4.2. The configuration of the test train is also shown in Fig III.4.
Fig. 11.2 Lateral Displacement Measurement

Steel beam on concrete bored piles

Displacement potentiometers
Fig. III.3 Instrumentation Layout

"S" indicates strain gauge locations

Displacement potentiometers

Q.R.E. load measuring baseplates

Steel beam on concrete bored piles
Fig. III.4 Test Train Configuration
### III.5 Vehicle Based Instruments

The test vehicle was fitted with one load measuring wheelset (56) to give dynamic records of vertical, longitudinal and lateral forces. Three lateral proximity probes were also fitted to the same wheelset. These devices give continuous records of wheel position relative to the rail by means of a small disc wheel which contacts the gauge face of the rail at a small distance from the wheelset. Use of three proximity probes enables wheelset position, angle of attack relative to the track and dynamic gauge to be calculated.

The data from these devices was also recorded onto analogue tape with the exception of the load measuring wheel data which was processed into force records in real time and stored directly onto digital tape.

### III.6 Test Procedure

The test train ran over the site at speeds increasing in nominal 5 mph intervals up to a maximum speed of 45 mph, in one direction only. Tests were carried out with the load measuring wheel in both the leading and trailing positions. Also the effect of having the rails dry or wet was studied in order to vary the coefficient of friction. Tests were therefore performed on four separate days as follows:

- **Day 1** - LMW trailing, rails dry
- **Day 2** - LMW trailing, rails wet
- **Day 3** - LMW leading, rails wet
- **Day 4** - LMW leading, rails dry

### III.7 Results

**Measurements Of Track Geometry:** A survey of the track geometry was made in order to establish the true alignment of the test site. This showed that although the net change of angle of the kink was 15.6 mrad as required, some local curvature of the rails was also present increasing the effective angle to nearer 18.5 mrad. The straight length and curve radius were both found to be close to the required values.

Some slight variation in the gauge of the track was discovered and also a small degree of cant, but neither of these was considered large enough to be
significant, the measured geometry used for theoretical predictions was that of
the outer rail.

**Load Measuring Baseplates:** The results of the baseplates are capable of
being used in two ways. One way is to produce a continuous sum of the lateral
forces of all the baseplates on each rail. This sum is equal to the wheel/rail
lateral force (neglecting inertia effects of the rail) provided that no other wheel
is over or close to the row of base plates when the chosen wheel is being studied.
This was in fact the case for the long wheelbase vehicle used in the test and the
results obtained in this way were used firstly as a check against the load
measuring wheel results, where they agreed very well, and also to provide the
forces from the wheelset where vehicle based forces were not measured.

The second way in which the load measuring baseplate output can be used
is to give individual sleeper loads. This was used in this instance to obtain a
measure of lateral stiffness and damping associated with the lateral
displacements, the way in which this was approached is described later. Longer
time histories were available from the load measuring wheelset output so the
summed baseplate outputs were used to obtain the peak lateral forces which
occurred locally at the kink. Figs III.5-III.8 show the maximum forces on each
rail as a function of speed with the convention that a positive force is outward on
each rail.

**Load Measuring Wheelset:** As mentioned above this gave the history of forces
on each wheel throughout the site and was provided partly as a back up for the
load measuring baseplate measurements. The results of each were compared in
fact and found to agree very closely. Vertical forces were also obtained from the
load measuring wheelsets, which were shown to vary very little due to the
absence of significant vertical irregularities and minimal vehicle roll. Typical
examples of the lateral force histories from the load measuring wheels are
shown in figs III.9 and III.10.

**Lateral Proximity Probes:** The two most interesting parameters which can be
obtained from these devices are the lateral position of the wheelset relative to
the track, and the angle of attack, which were of use in forming the comparisons
with the theoretical results.
Fig. III.6 Peak Forces at Leading Right Hand Wheel
Fig. III.7 Peak Forces at Trailing Left Hand Wheel

Lateral Force (kN)
Fig. III.8 Peak Forces at Trailing Right Hand Wheel
Fig. III.9 Force Histories for Leading Wheelset (40mph)
It should be noted that in the immediate vicinity of the kink (±0.61m) the probes do not give a correct reading due to the fact that they are mounted away from the wheel/rail contact point.

**Lateral Track and Rail Head Displacements:** Samples of the maximum values of track lateral displacement and left hand rail head lateral displacement (relative to sleeper) which occurred are plotted in Figs III.11 and III.12, respectively as functions of speed. Small outward displacements were measured on the inner rail, however these were neglected for analysis purposes due to the limited number of measuring transducers on the inner rail. The displacements are shown to be approximately proportional to the lateral forces as might be anticipated.

**Cross Plots of Sleeper Force and Displacements to Obtain Ballast Stiffness:** The summation of the forces from two load measuring baseplates on the same sleeper gives the total lateral sleeper force. This can be cross plotted against lateral sleeper displacement which results in a hysteresis loop as the load comes onto the sleeper and is removed. The records from individual baseplates consisted in practice of a single cycle of load application and removal. Experimentation with a mathematical model of a simple oscillator, however, suggested that for the parameters applicable in this case, there would be little error in assuming that this cycle of force was equivalent to one cycle of a continuous harmonic force. A typical example of the result of doing this is shown in Fig III.13.

If the history of the applied force is therefore considered to be of the form

\[ P = P_0 \sin \omega t \]

then the displacement response (for viscous damping) will be of the form:

\[ x = x_0 \sin(\omega t - \phi) \]

Where \[ x_0 = \frac{P_0}{\sqrt{(k-m\omega^2)^2 + (c\omega)^2}} \]

and \[ \tan \phi = \frac{c\omega}{k-m\omega^2} \]

If force is now plotted against displacement an ellipse is produced (Fig III.14) having y intercept \( cwx_0 \) and slope (between points of maximum x) equal to
Fig. III.13 Typical Cross Plot of Lateral Sleeper Force and Displacement
Fig. III.14 Force vs. Displacement for Damped Single Degree of Freedom Oscillator
The parameter \( m \) in this case would be the mass of the sleeper including two baseplates. The results obtained from a range of such cross plots are presented in Table III.1.

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<th>PEAK DISP. (mm)</th>
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<th>DAMPING (MNs/m)</th>
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<td>11.9</td>
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<td>20.3</td>
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<td>17.7</td>
<td>19.8</td>
<td>0.129</td>
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<tr>
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<td>0.97</td>
<td>17.0</td>
<td>19.1</td>
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<td>19.5</td>
<td>0.62</td>
<td>22.6</td>
<td>24.7</td>
<td>0.208</td>
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<td><strong>MEAN VALUES</strong></td>
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<td></td>
<td><strong>20.1</strong></td>
<td><strong>0.130</strong></td>
</tr>
</tbody>
</table>

Table III.1 Experimental Measurements of Ballast Stiffness and Damping

**Rail Head Lateral Stiffness:** It is not possible to cross plot force against rail head displacement in the same way as for the sleeper motion as the load is moving and the measuring point is stationary. A reasonable approximation can be made, however, by dividing the maximum wheel/rail force by the maximum rail head displacement occurring at any of the measuring locations. This gives directly the lumped lateral stiffness subject to the assumption that the maximum spring force occurs at the same time as the maximum displacement (i.e., that the damping and rail inertia are both small).

The results of carrying out this exercise are presented in Fig III.15 where it can be seen that a reasonably linear relationship between force and displacements is obtained. This yielded an effective stiffness value of 43 MN/m.

**Coefficient of Friction between Wheel and Rail:** The effective coefficient of friction between wheel and rail can be calculated from the measured forces where the contact normal is vertical (or nearly so) if the value of maximum force in the horizontal plane divided by the vertical contact force reaches a constant limiting value indicating that sliding was taking place.
Fig. III.15 Peak Lateral Force vs. Peak Lateral Rail Displacement
This state appears to have been reached for most speeds on the leading right hand wheel as the vehicle passes the kink. This is evident from Fig III.6 and also from the force histories (Figs III.9 and III.10) where a 'flat topped' response is noticed. It has been assumed that the maximum lateral force divided by the static wheel load will reasonably accurately yield the coefficient of friction. This is subject to the assumptions that the longitudinal force is small and that the vertical contact force does not vary significantly from the static value. Both of these assumptions in fact are verified from the load measuring wheel records.

The mean values of coefficient of friction calculated for each of the four days therefore were as follows:-

- Day 1 (dry) - 0.33
- Day 2 (wet) - 0.18
- Day 3 (wet) - 0.18
- Day 4 (dry) - 0.22

### III.8 Discussion of Results

**Lateral Wheel/Rail Forces:** The maximum lateral forces encountered at the leading wheelset appear to vary reasonably linearly with speed. Hard flange contact seems to take place throughout the speed range on the outer wheel while the lateral force on the right hand wheel reaches a limiting value governed by the coefficient of friction. Both forces are in an outward direction on the rails (gauge spreading) which is generally the case for steady state curving conditions also.

The maximum lateral forces at the trailing wheelset were found not to vary linearly with speed, this was because flange contact did not occur at the lower speeds. This can be expected due to the fact that the front end of the vehicle produces an element of steering for the rear. At speeds above 10 m/sec however, flange impacts do occur and the maximum force rises quickly with speed giving higher forces than at the leading wheelset at 20 m/sec. It is also interesting to note that at the higher speeds the forces on the inner rail are inward as opposed to outward in the case of the leading wheelset.

Looking at the force history records for the leading left hand wheel (Figs III.9 and III.10) an oscillatory response is noticed as a result of the impact, which
was shown to have a frequency of approximately 15 Hz. This is qualitatively very similar to the theoretical predictions where the wheelset is caused to oscillate on the lateral track resilience.

At the trailing wheelset, where flange impact occurs, the response seems to be very heavily damped with only one peak visible in the force history. Clearly the amount of damping in the wheelset oscillation will be dependant upon the track damping but also upon the forces on the inner wheel. If the inner wheel is sliding laterally on the rail then oscillation may take place without altering the forces on the inner rail and will thus not provide damping for the oscillation. However, if the forces on the inner wheel are below the friction limit lateral movements will result in additional forces opposing the motion and will provide damping. As the lateral forces on the inner wheel of the trailing wheelset are quite small this is a likely reason for the apparently large damping.

It is noticed that the coefficient of friction had a slight effect on the forces, these were greater on both rails when the coefficient of friction was large. This is to be expected as a component of the lateral force on the outer rail is the lateral creep force, being equal to the lateral creep force on the inner rail. This also explains why the peak lateral force on the outer rail does not tend to zero at zero speed.

It is also noticeable that turning the vehicle had a small effect on the force levels, particularly at the trailing wheelset. This would most likely have been caused by misalignments of the vehicle wheelsets.

Lateral Track and Rail Displacements: The maximum track displacement measured was 1.0 mm and the maximum individual rail head displacement 1.3 mm. It is suggested therefore that both these sources of resilience would need to be included in a theoretical model.

No significant evidence was found in the experimental results of a higher mode of vibration which would be anticipated from a track with two major degrees of freedom subjected to an impact. This is also the case in the theoretical modelling and is presumably due to the high damping levels present.

All of the displacements measured were found to be apparently elastic, ie no residual displacement occurred. The track geometry was also measured again
after the experiment and found not to have changed, although in railway terms the amount of traffic passing the kink was very small.

**Coefficient of Friction:** The coefficient of friction was found to be lower when the rails were wet, as might be expected. The coefficient of friction on the two wet days was constant at 0.18, whereas the coefficient of friction on the two dry days varied significantly (0.33 and 0.22). This is quite a normal result, however, as coefficient of friction is strongly governed by atmospheric conditions and level of contamination of the rail head surface which may have been lightly rusted at the beginning of the tests.

**III.9 Comparison with the Theoretical Results**

The comparisons were achieved by using the lumped parameter lateral model described in Chapter 3. Certain refinements were also made to the model as a result of the discoveries made in order to improve the predictions mode, these are described in section 4.11.

A typical time history comparison is shown in Fig III.16 which was made for a test run at 45 mph (20.1m/sec) in wet conditions. The time histories presented were all derived from vehicle based (load measuring wheel) data. The agreement obtained is generally shown to be good, the lateral impact of the leading wheelset appears to be modelled quite well with apparently the correct frequency and damping rate. The longer wavelength body response also appears to be reasonably well represented with possibly slightly too little damping.

A feature which is not modelled particularly well, however is the longitudinal forces at the wheelsets. In fact this tends also to be a deficiency of steady state curving predictions and is therefore probably not attributable to errors in the dynamic model itself. A solution to this particular problem has not yet been found, however.

A summary of the peak force which occurred on each wheel at the kink is shown in Fig III.17 from the results of the third day when the rails were in a wet condition and a consistent set of results were obtained. These are shown to agree quite well also.

An analysis similar to the one used for the vertical force experiment was performed in that the difference between experimental and theoretical peak
Fig. III.16 Time History Comparison
Fig. III.17 Comparison of Peak Lateral Forces
impact force was described as a ratio of the measured force. The result of this is shown in Table III.2. In this case individual speeds were considered and once again a negative error indicates that the theoretical value was smaller.

<table>
<thead>
<tr>
<th>Speed (mph)</th>
<th>10</th>
<th>20</th>
<th>30</th>
<th>40</th>
<th>45</th>
<th>Totals (abs.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Leading</td>
<td>Error (kN)</td>
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<td>-2.15</td>
<td>-7.0</td>
<td>-1.4</td>
<td>0.2</td>
</tr>
<tr>
<td>Left</td>
<td>Error %</td>
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<td>-9</td>
<td>17</td>
<td>-3</td>
<td>0</td>
</tr>
<tr>
<td>Leading</td>
<td>Error (kN)</td>
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<td>2.0</td>
<td>-0.5</td>
<td>2.5</td>
<td>2.8</td>
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<tr>
<td>Right</td>
<td>Error %</td>
<td>11</td>
<td>14</td>
<td>3</td>
<td>18</td>
<td>20</td>
</tr>
<tr>
<td>Trailing</td>
<td>Error (kN)</td>
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<td>-3.7</td>
<td>0.6</td>
<td>8.7</td>
<td>-8.4</td>
</tr>
<tr>
<td>Left</td>
<td>Error %</td>
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<td>-38</td>
<td>6</td>
<td>37</td>
<td>16</td>
</tr>
<tr>
<td>Trailing</td>
<td>Error (kN)</td>
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<td>-4.7</td>
<td>4</td>
<td>1.9</td>
<td>-1</td>
</tr>
<tr>
<td>Right</td>
<td>Error %</td>
<td>-1</td>
<td>-63</td>
<td>-59</td>
<td>-22</td>
<td>14</td>
</tr>
<tr>
<td>Totals (abs.)</td>
<td></td>
<td>16</td>
<td>31</td>
<td>21</td>
<td>20</td>
<td>13</td>
</tr>
</tbody>
</table>

Table III.1 Differences Between Experimental and Theoretical Results

The agreement obtained is generally good, particularly at the higher speeds. In fact the largest errors in percentage terms were found at the trailing right hand wheel where the forces were extremely small, (always less than 8kN) and thus the errors were not large in absolute force terms. The best agreement was obtained at the leading wheelset where the response is governed by a straightforward impact, in contrast to the trailing wheelset where the response is more complicated. The trend in the trailing left hand wheel peak force to rise rapidly with speed at the higher speeds is reflected well in the theoretical results, however, and the magnitude of impacts at the trailing wheelset is shown both theoretically and experimentally to be sensitive to the input parameters. The force on this particular wheel was also altered quite noticeably when the vehicle was turned, although the coefficient of friction was essentially identical on days two and three, the peak force measured at the trailing left hand wheel increased from around 39 kN to around 67 kN as a result of turning the vehicle.

The overall error obtained in absolute percentage terms was 20%, which considering the contribution of the errors in the very small forces represents quite acceptable agreement for an experiment of this kind.
III.10 Conclusions

The experiment performed was able to identify the main characteristics of a vehicle negotiating a large lateral kink in the track. The results showed that at such lateral irregularities impacts of the wheelset on the rails cause large dynamic loads to occur, an important feature of which is the lateral oscillation of the wheelset on the track resilience at a frequency of around 15 Hz for the conditions studied. The resilience of the track appears to come partly from lateral movement of the rail head relative to its foot, and partly due to lateral movement of the sleepers in the ballast.

The damping provided by the track to lateral vibration was quite high at the site studied, providing approximately 20% of critical damping levels in the predominant wheelset vibration mode. Damping of this mode is also apparently increased significantly if the forces in the plane of contact on the inner wheel are below the friction limit. The assumption that little damping is provided by lateral displacement of the rails relative to the sleeper seems to be a reasonable one as the theoretical predictions are certainly not under damped.

The predictions made by the theoretical model generally agree well with the experimental results showing the same main characteristics and reasonable accuracy. For the case studied a lumped spring, mass, damper model of the track system, with modelling of the rail deformation by a simple spring seems to be adequate.