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Models for instability in inviscid fluid flows, due to a resonance between two waves

Roger Grimshaw

Department of Mathematical Sciences
Loughborough University
Loughborough LE11 3TU, UK.
Email: R.H.J.Grimshaw@lboro.ac.uk

Abstract

In inviscid fluid flows instability arises generically due to a resonance between two wave modes. Here, it is shown that the structure of the weakly nonlinear régime depends crucially on whether the modal structure coincides, or remains distinct, at the resonance point where the wave phase speeds coincide. Then in the weakly nonlinear, long-wave limit the generic model consists either of a Boussinesq equation, or of two coupled Korteweg-de Vries equations, respectively. For short waves, the generic model is correspondingly either a nonlinear Klein-Gordon equation for the wave envelope, or a pair of coupled first-order envelope equations.

1 Introduction
In inviscid fluid flows it has long been recognised that instability generically arises due to a resonance between two wave modes. That is, as an appropriate parameter is varied, the phase speeds of the two waves coincide for some critical parameter value. A generic unfolding of this resonance yields either a stable “kissing” configuration, or “bubble” of instability, in the space of the external parameter. Many illustrations of this concept are reported in the monograph by Craik (1985) for shear flows, and recently Baines and Mitsudera (1994) have discussed the physical processes involved.

The origin of the concept lies in the Hamiltonian structure of inviscid fluid flows. Indeed in a finite-dimensional Hamiltonian dynamical system, it is well-known that linearisation about a steady state and a subsequent search for eigenfrequencies $\omega$ (i.e. a search for solutions proportional to $\exp(-i\omega t)$) will generically lead to sets of quartets $\{\omega, -\bar{\omega}; \bar{\omega}, -\omega\}$. Here for a given eigenfrequency $\omega$, $-\bar{\omega}$ is also an eigenfrequency due to the real-valued nature of the system, while the pair $\omega, -\omega$ follow from the Hamiltonian structure. Instability occurs if $\text{Im} \, \omega \neq 0$. Consider the situation as an appropriate parameter is varied. When stable, all the eigenfrequencies must lie on the real axis, with one eigenfrequency pair lying on the positive real axis, and the second pair being its mirror image on the negative real axis. For instability to occur as the external parameter is varied, the eigenfrequency pair on the positive real axis must come into coincidence, while the mirror-image pair on the negative real axis will do likewise. Further variation of the parameter which leads to instability will then cause the eigenfrequencies to split apart and move off the real axis, one member lying above the real axis, and the second member being its complex conjugate. The situation is sketched in Figure 1.

The inference from this generic situation for fluid flows leads to the concept sketched in Figure 2, where we plot the phase speed $c$ as a function of a parameter $\Delta$, while the unfolding parameter is $\delta$. The further exploration of the implication for fluid flows depends on whether one is considering long waves, or short waves. These two cases will be discussed in the next two sections respectively.

2 Weakly nonlinear models for long waves
Let us now explore the unfolding of this basic resonance for the case of long waves. We suppose that the long waves have a wavenumber $k$ and frequency $\omega$ (i.e. the linearised system supports solutions proportional to $\exp(ikx - i\omega t)$), so that the phase speed is $c = \omega / k$. Then, if the appropriate external parameter is $\Delta$, we may suppose without loss of generality that the dispersion relation which describes the situation sketched in Figure 2 is given by

$$c^2 = \Delta^2 + \delta$$

(1)

where $\delta$ is the unfolding parameter. Thus $\delta = 0$ corresponds to the resonance described in Figure 2a, while $\delta > 0$ ($< 0$) represents the stable or unstable configurations sketched in Figures 2b, c respectively. Note that the instability window in the unstable configuration is $\Delta^2 < -\delta$.

Next we convert the dispersion relation (1) into a partial differential equation (or equations) describing these linearised long waves. There are two cases to be considered. Let us first suppose that there is just a single mode at the resonance point where $\delta = 0$ and $\Delta = 0$. Then, noting that for a sinusoidal solution, $ik \leftrightarrow \partial / \partial x$ and $-i\omega \leftrightarrow \partial / \partial t$, we can infer that

$$\eta_{tt} - \Delta^2 \eta_{xx} = \delta \eta_{xx}$$

(2)

where $\eta(x, t)$ is the modal amplitude. Let us now consider the generic weakly nonlinear, weakly dispersive unfolding of this equation. Let $\epsilon$ be a small parameter characterising weak dispersion, so that $\partial / \partial x$ is $O(\epsilon)$. The appropriate balance in (2) for the onset of instability is that $\delta$ is $O(\epsilon^{\frac{3}{2}})$, $\Delta$ is $O(\epsilon)$ and $\partial / \partial t$ is $O(\epsilon^2)$. Anticipating that the nonlinearity will be quadratic we can suppose that $\eta$ is $O(\epsilon^2)$, and infer that the canonical model equation will be the Boussinesq equation

$$\eta_{tt} - \Delta^2 \eta_{xx} + \frac{1}{2} \mu (\eta^2)_{xx} + \lambda \eta_{xxxx} = \delta \eta_{xx}.$$
Here $\mu$ and $\lambda$ are the nonlinear and dispersive coefficients respectively which depend on the particular physical system being considered. Equations of this form have been derived by Hickernell (1983a,b) for Kelvin-Helmholtz instability, by Helfrich and Pedlosky (1993) for baroclinic instability in a two-layer quasigeostrophic flow, and by Mitsudera (1994) in a study of Eady waves in the atmosphere. Remarkably the Boussinesq equation (3) is integrable for all combinations of the signs of $\lambda$ and $\delta$ (Hirota (1973) and Zakharov and Shabat (1974)). The nature of the solution set depends of course on the signs of $\lambda$ and $\delta$, but includes solitary waves, multiple-solitary waves, and interestingly, solutions which blow-up in finite time.

Next consider the alternative situation when there are two distinct modes at the resonance point where $\Delta = 0$ and $\delta = 0$. If we let $\eta(x,t)$ and $\zeta(x,t)$ be these two modes, then it is readily apparent that the generic set of equations describing the linear long wave case is given by

$$\begin{align*}
\eta_t + \Delta \eta_x + \kappa_1 \zeta_x &= 0, \\
\zeta_t - \Delta \zeta_x + \kappa_2 \eta_x &= 0.
\end{align*}$$

(4)

Indeed if we seek solutions for which $(\zeta, \eta)$ are given by $(A, B) \exp(ik(x-ct)) + c.c.$, then we get

$$\begin{align*}
-(c - \Delta)A + \kappa_1B &= 0, \\
-(c + \Delta)B + \kappa_2A &= 0.
\end{align*}$$

(5)

Elimination of $A$ or $B$ leads to (1) with $\delta = \kappa_1 \kappa_2$ so the system (5) is stable, or unstable, according as $\kappa_1 \kappa_2 > 0$, or $< 0$. Note that the system (5) includes the previous case, in that if we take the limit $\Delta \to 0$ and $\kappa_2 \to 0$ in (5), with $\kappa_1 \neq 0$, it follows that at this resonance point, $B = 0$ and there is just a single mode $(A \neq 0)$ with speed $c = 0$. However, the joint limit $\kappa_1, \kappa_2 \to 0$ with $\Delta \to 0$ leads to two independent modes $(A, B \neq 0)$ with speed $c = 0$. This is a fundamental distinction, and as we shall now see leads to a different canonical model for the weakly nonlinear, weakly dispersive unfolding.
of the system (4) even although the linear long-wave system (4) can be reduced to the form (2) for either of \( \eta \) or \( \varsigma \).

Let us then consider the generic weakly nonlinear, weakly dispersive unfolding of the system (4). We again let \( \varepsilon \) be a small parameter characterising weak dispersion so that \( \partial / \partial x \) is \( 0(\varepsilon) \). The appropriate balance in (4) for the onset of instability is that \( \kappa_1, \kappa_2 \) are \( 0(\varepsilon^2) \), \( \Delta \) is \( 0(\varepsilon^2) \), and \( \partial / \partial t \) is \( 0(\varepsilon^3) \). Again anticipating that the nonlinearity will be quadratic, we can suppose that \( \eta, \varsigma \) are \( 0(\varepsilon^2) \) and then infer that the canonical model equation will be the set of coupled Korteweg-de Vries (KdV) equations

\[
\begin{align*}
\eta_t + \Delta \eta_x + \mu_1 \eta x_x + \lambda_1 \eta_{xxx} + \kappa_1 \varsigma_x &= 0, \\
\varsigma_t - \Delta \varsigma_x + \mu_2 \varsigma x_x + \lambda_2 \varsigma_{xxx} + \kappa_2 \eta_x &= 0.
\end{align*}
\]

Here \( \mu_1, \mu_2 \) and \( \lambda_1, \lambda_2 \) are the nonlinear and dispersive coefficients respectively which depend on the particular physical system being considered. Note that we have not anticipated any nonlinear terms or dispersive terms which involve coupling of the two equations, since we are assuming that the coupling occurs predominantly at the linear order, and is a small effect which is in balance with nonlinearity and dispersion. Equations of this form have been derived by Mitsudera (1994) for the coupling of baroclinic Eady waves in an atmospheric model, by Gottwald and Grimshaw (1999a,b) for a two-layer quasigeostrophic flow, and by Grimshaw (1999) for a three-layered stratified shear flow.

It is important to note that although these two cases of either a single mode, or two distinct modes, at the point of resonance are apparently equivalent in the linear long wave limit, they are very different in the weakly nonlinear, weakly dispersive unfolding of this limit. This is apparent first in the different time scales, and in the different scaling for the coupling parameters. Further, while the Boussinesq equation (3) can support solutions which blow-up in finite time (Hickernell, 1983a,b and Helfrich and Pedlosky, 1993), no such behaviour has yet been identified in the coupled KdV system (1.6). The coupled KdV system (6) is Hamiltonian, and can be written in the form
\[\kappa_2 A_x = -\partial_x \left( \frac{\delta H}{\delta A} \right), \quad \kappa_1 B_x = -\partial_x \left( \frac{\delta H}{\delta B} \right), \quad (7)\]

where

\[H = \int_{-\infty}^{\infty} J \, dx\]

\[J = \kappa_2 \left( \frac{1}{2} \Delta A^2 + \frac{1}{6} \mu_1 A^3 - \frac{1}{2} \lambda_2 A_x^2 \right) + \kappa_1 \left( -\frac{1}{2} \Delta B^2 + \frac{1}{6} \mu_2 B^3 - \frac{1}{2} \lambda_2 B_x^2 \right) + \kappa_1 \kappa_2 AB \quad (8)\]

The Hamiltonian \(H\) is a conserved quantity. Another conserved quantity is

\[P = \int_{-\infty}^{\infty} \left( \frac{1}{2} \kappa_2 A_x^2 + \frac{1}{2} \kappa_1 B_x^2 \right) dx. \quad (9)\]

It follows that \(\kappa_1 \kappa_2 > 0\) ensures nonlinear stability, and is just the linear stability criterion.

It is beyond the scope of this article to discuss the solutions of the coupled KdV equations (6). However, we note that the numerical simulations and analyses of Mitsudera (1994) and Gottwald and Grimshaw (1999a,b) give some indication of the range and diversity of the solutions. In essence, the solutions can be interpreted as interacting solitary waves. In the absence of the coupling parameters \(\kappa_1, \kappa_2\), each KdV equation can support the well-known solitary waves, which, in the presence of the coupling terms, then interact in various intriguing ways. The nature and strength of the interaction depends, inter alia, on the system parameters and in particular on the coupling parameters \(\kappa_1, \kappa_2\) and the phase speed detuning parameter \(\Delta\). In particular, from the numerical simulations of Gottwald and Grimshaw (1999a,b), we call attention to two common scenarios. In the first case, two solitary waves approach each other and then reflect, each undergoing significant amplitude modulation in the process. The variation of the relative position and amplitudes can be interpreted as analogous to orbits near a saddle point in an appropriate phase plane. In the second case, two solitary waves remain locked together, each
undergoing modulations such that their relative positions and amplitudes vary periodically. The phase plane interpretation is that of a centre.

3 Weakly nonlinear models for short waves

In this case, the mode resonance occurs for a finite wavenumber \( k_0 \), and a finite frequency \( \omega_0 \), for the phase speed \( c_0 = \omega_0 / k_0 \). Figure 2 is again relevant, but now the axis for \( c \) is replaced by that for frequency \( \omega \), and the axis for \( \Delta \) is replaced by that for wavenumber \( k \). The local dispersion relation, which now replaces (1) is

\[
(\omega - \omega_0)^+ = V^+(k - k_0)^+ + \delta, \tag{10}
\]

where \( \delta \) is again the unfolding parameter, and \( \pm V \) are the group velocities of the two modes at the resonance point. Thus \( \delta = 0 \) corresponds to the exact resonance (Figure 2a), while \( \delta > 0(< 0) \) represents the stable (unstable) configuration sketched in Figure 2b (2c). Note that the instability window in Figure 2c is \( V^+(k - k_0)^+ < -\delta \).

Next, we convert the dispersion relation (10) into a partial differential equation(s) for linear waves. There are again two cases to be considered. Let us first suppose that there is just a single mode at the resonance point where \( \delta = 0 \) and \( k = k_0 \). Then if \( u(x, t) \) is the modal variable, we introduce the envelope representation

\[
u \sim \alpha A(X, T) \exp(ik_0x - i\omega_0t) + c.c., \tag{11}
\]

where

\[
X = \varepsilon x, \; T = \varepsilon t, \tag{12}
\]

and c.c. denotes the complex conjugate. Here \( \alpha \) is the amplitude parameter to be determined below. Then we use the device that

\[
\frac{\partial}{\partial t} \sim -i\omega_0 + \varepsilon \frac{\partial}{\partial T}, \; \frac{\partial}{\partial x} \sim ik_0 + \varepsilon \frac{\partial}{\partial X}. \tag{13}
\]
where, as before, $ik \leftrightarrow \partial / \partial x$ and $-i\omega \leftrightarrow \partial / \partial t$. Assuming weak coupling, we let $\delta = \varepsilon \gamma$, so that

$$A_{tt} - V^2 A_{xx} + \gamma A = 0. \quad (14)$$

The generic weakly nonlinear unfolding of this linear equation requires the presence of cubic nonlinear terms, so that we set $\alpha = \varepsilon$, and then get

$$A_{tt} - V^2 A_{xx} + \gamma A + \mu |A|^2 A = 0. \quad (15)$$

This cubic nonlinear Klein-Gordon equation has been derived by Drazin (1970) and Weissman (1979) for Kelvin-Helmholtz instability, and by Pedlosky (1970, 1972) for baroclinic instability in a two-layer quasi-geostrophic flow. Recently (1998) has given a general Hamiltonian derivation of (15) for Kelvin-Helmholtz flows. The Klein-Gordon equation (12) supports solitary waves (for $\gamma \mu < 0$), kinks (for $\gamma \mu < 0$), solutions which blow-up in finite time (for $\gamma < 0$, $\mu < 0$), and linearly unstable solutions which can equilibrate at finite amplitude ($\gamma < 0$, $\mu > 0$). In general, equation (15) may be coupled to a mean flow equation. That is, an extra term $SA$, say, is inserted, where $S$ represents the mean flow, and can be expected to be governed by a first-order, or second-order, forced wave equation. A typical form in the first case might be

$$S_T + V_0 S_X = v |A|_X. \quad (16)$$

Next, we consider the alternative situation when there are two distinct modes at the resonance point where $\omega = \omega_0$, $k = k_0$, and $\delta = 0$. If we let $u(x, t)$ and $v(x, t)$ be these modes, then in place of the representation (6) we write

$$\begin{align*}
u &- \alpha A(X, T) \exp(i k_0 x - i \omega_0 t) + c.c. \\
v &- \alpha B(X, T) \exp(i k_0 x - i \omega_0 t) + c.c.
\end{align*}\quad (17)$$
Then, again using the device (10), we replace the linear equation (11) with the pair of first-order linear equations

\[
\begin{align*}
    i(A_T + VA_x) + \gamma_1 B &= 0, \\
    i(B_T - VB_x) + \gamma_2 A &= 0.
\end{align*}
\]

(18)

Here \( \delta = \varepsilon^2 \gamma \), where \( \gamma = \gamma_1 \gamma_2 \). Elimination of either \( B \) (or \( A \)) shows that \( A \) (or \( B \)) satisfies equation (14). If we seek solutions for \( (A, B) \) of the form \( (A_0, B_0) \exp\{i(k - k_0)x - i(\omega - \omega_0)t\} \), then we readily see that (18) collapses to the algebraic system

\[
\begin{align*}
    D_1 A_0 + \delta_1 B_0 &= 0, \\
    D_2 B_0 + \delta_2 A_0 &= 0,
\end{align*}
\]

(19)

where

\[
\begin{align*}
    D_1 &= (\omega - \omega_0) - V(k - k_0), \\
    D_2 &= (\omega - \omega_0) + V(k - k_0),
\end{align*}
\]

(20)

and \( \delta_{1,2} = \varepsilon \gamma_{1,2} \) so that \( \delta = \delta_1 \delta_2 \). Clearly the dispersion relation (7) is equivalent to \( D_1 D_2 - \delta_1 \delta_2 = 0 \).

The generic weakly nonlinear unfolding of the linear system (18) again requires the presence of cubic nonlinear terms, so that we set \( \alpha = \varepsilon^2 \), and then get

\[
\begin{align*}
    i(A_T + VA_x) + \gamma_1 B + \gamma_1 \sigma \left( |A|^2 + |B|^2 \right) A + \gamma_1 \beta B^2 \bar{A} + \ldots &= 0, \\
    i(B_T - VB_x) + \gamma_2 A + \gamma_2 \sigma \left( |B|^2 + |A|^2 \right) B + \gamma_2 \beta A^2 \bar{B} + \ldots &= 0.
\end{align*}
\]

(21)

Here the omitted terms contain, in general, the full suite of allowed cubic nonlinear terms (i.e. cubic combinations of \( A, A, B \) and \( \bar{B} \)). Equations of this form have been discussed by Grimshaw and Malomed (1994, 1995), Grimshaw et. al (1995, 1998), and Gottwald et. al (1997). Recently, (1998) has derived a system of this form for marginal instability in a three-layer stratified fluid flow. The system is Hamiltonian, and can be written in the form
\[ i\gamma_2 A_t = -\frac{\delta H}{\delta A}, \quad i\gamma_1 B_t = -\frac{\delta H}{\delta B}, \quad (22) \]

where \[ H = \int_{-\infty}^{\infty} J dX. \quad (23) \]

and
\[
J = \frac{1}{\gamma_1} i\gamma_2 V(\bar{A}A_x - A\bar{A}_x) - \frac{1}{\gamma_1} i\gamma_1 V(\bar{B}B_x - B\bar{B}_x) + \gamma_1\gamma_2(\bar{A}B + \bar{B}A) \\
\quad + \frac{1}{2} \sigma\gamma_1\gamma_2(\|A\|^2 + 2|A|^2|B|^2 + |B|^4) + \frac{1}{2} \beta\gamma_1\gamma_2(\bar{A}^2B^2 + \bar{B}^2A^2) + ... \quad (24)
\]

The Hamiltonian \( H \) is a conserved quantity. Note that the assumed symmetry in the nonlinear terms in (21) is to ensure that there is a Hamiltonian structure. Another conserved quantity is
\[
P = \int_{-\infty}^{\infty} (\gamma_2 \|A\|^2 + \gamma_1 \|B\|^2) dX. \quad (25)
\]

It follows that \( \gamma_1\gamma_2 > 0 \) ensures nonlinear stability, and is just the linear stability criterion. In general, the system (21) may also contain mean flow terms, of the form \( S_1A \) or \( S_2B \), where \( S_{1,2} \) satisfy forced wave equations of the form (16) (see, for instance, Grimshaw and Malomed (1994), or Grimshaw (2000) who derived a model of this form for a certain three-layered stratified shear flow). However, we shall not consider the implication of such terms here.

It is beyond the scope of this article to discuss the solutions of the coupled envelope system (21). However, we note here that the theoretical and numerical analyses of Grimshaw and Malomed (1994), and Grimshaw et. al (1995, 1998) have delineated some aspects of the expected behaviour. In the stable case \( (\gamma_1\gamma_2 > 0) \), Grimshaw and Malomed (1994) demonstrated that the system (21) can support gap solitary waves; that is, envelope solitary waves, whose speeds and frequencies are such that they lie in the gap in the linear spectrum (see Figure 2b). In the unstable case \( (\gamma_1\gamma_2 < 0) \) Grimshaw et. al (1998) showed that two basic scenarios could be expected, depending on the relative coefficients of the nonlinear terms. In one case, solutions develop a singularity in finite time, while in the other case, the solutions evolve into successively finer temporal and spatial structures,
due to modulational instability. It is significant to note here that the envelope system (21) only contains low-order frequency dispersion (i.e. the terms $\gamma_1 B, \gamma_2 A$ respectively), and the “usual” NLS-type of higher-order dispersion (i.e. $A_{xx}, B_{xx}$ terms) are absent here, since they are clearly of higher order in the parameter $\epsilon$.

References


Figure Captions

Figure 1: A sketch of the typical configuration of eigenfrequencies (X) in a Hamiltonian dynamical system.

Figure 2: A schematic sketch of the dispersion relation for mode resonance, in which the phase speed c (frequency ω is plotted as a function of an external parameter Δ (wavenumber k), for various values of an unfolding parameter δ (a) the uncoupled case, δ = 0, (b) the stable case, δ > 0, (c) the unstable case, δ < 0 (only Real (c) is shown).
Figure 1: A sketch of the typical configuration of eigenfrequencies ($\omega$) in a Hamiltonian dynamical system.
Figure 2: A schematic sketch of the dispersion relation for mode resonance, in which the phase speed $c$ (frequency $\omega$) is plotted as a function of an external parameter $\Delta$ (wave number $k_{\alpha}$) for various values of the coupling parameter $\delta$. (a) the uncoupled case, $\delta = 0$, (b) the stable case $\delta > 0$, (c) the unstable case $\delta < 0$ (only $k_{\alpha}c/\omega > 0$ is shown).