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Numerical simulations of the flow of a continuously-stratified fluid, incorporating inertial effects

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Abstract

A high-resolution spectral numerical scheme is developed to solve the two-dimensional equations of motion for the flow of a density stratified, incompressible and inviscid fluid. It incorporates the inertial terms neglected in the Boussinesq approximation. Thus it aims, inter alia, to extend the numerical simulations of Rottman et. al. [12] and Aigner et. al. [1]. To test its validity, the code is used for two applications. One is the resonant flow over isolated bottom topography in a channel of finite depth, which has been studied extensively in the Boussinesq approximation. The inclusion of inertial effects, that is the influence of the stratification on the acceleration terms, discarded in the Boussinesq approximation, allows the comparison of the solution to the unsteady governing equations with the fully nonlinear, but weakly dispersive resonant theory of Grimshaw and Yi [7]. The focus is on topography of small to moderate amplitude and slope, and for conditions such that the flow is close to linear resonance for either of the first two internal wave modes. We also determine the vertical position where wave-breaking occurs. The other application is the propagation of large-amplitude internal solitary waves with vortex cores, again in a channel of finite depth. We aim to verify the existence and permanence of these types of waves derived by Derzho and Grimshaw [5]. Furthermore the time-dependent solution provides measurements of the structure of the vortex core and maximum adverse velocity at the top boundary.

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1 Introduction

The steady flow of a uniformly stratified, incompressible and inviscid fluid in a channel of finite depth was first studied extensively by Long [10], who derived a nonlinear equation commonly referred to as Long’s equation, although it was originally obtained by Dubreil-Jacotin[6]. It is usually employed when all streamlines originate upstream, and there is no upstream influence on the flow. It can be used to study the propagation of steady solitary waves, and is also applicable to the flow of a density stratified fluid over finite bottom topography.

In the Boussinesq approximation and for uniform stratification, Long’s equation is linear, without restriction in the wave amplitude. Solitary waves can then only be generated by a deviation from these conditions, as shown by Derzho and Grimshaw [5], although waves resembling solitary waves may sometimes be generated by topography. Grimshaw and Yi [7] derived an unsteady fully nonlinear, weakly dispersive equation encompassing small departures from the case of the Boussinesq approximation with uniform stratification, which also included the effects of localized topography. Rottman et. al. [12] report a detailed investigation, using numerical simulations of the fully nonlinear unsteady equations in the Boussinesq approximation for a uniform stratification, and compared their results to solutions of the finite-amplitude long wave equation (FALW) derived by Grimshaw and Yi [7]. Our aim here is to include the inertial terms neglected in the Boussinesq approximation (called non-Boussinesq effects in the sequel), and hence to verify, first, the asymptotic theory of Derzho and Grimshaw [5] for large-amplitude internal solitary waves with vortex cores and second, the FALW
equation for flow over topography. One can identify three sources of nonlinearity supporting the existence of solitary waves, or solitary-like waves. First, there is the nonlinearity induced by the deviation from uniform stratification in the Boussinesq approximation. A numerical study to verify the theory of Derzho and Grimshaw [5] for this case was undertaken by Aigner et. al. [1]. Second, there is the nonlinearity induced by the non-Boussinesq terms. Third, there is the role of the bottom topography.

The present study extends the aforementioned studies to the fully nonlinear equations, with the inclusion of the non-Boussinesq terms. For this purpose a novel numerical method is proposed to solve the discrete elliptic problem arising from the fully nonlinear inviscid equations, using a high-resolution pseudospectral method based on the scheme used by Rottman et. al. [12]. It involves the solution to a fixed-point problem using Liouville-Neumann iteration. The simulations allow comparison of the fully nonlinear equations with the FALW model of Grimshaw and Yi [7], and the large-amplitude solitary waves with vortex cores of Derzho and Grimshaw [5]. In addition, for the former case the vertical position of breaking will be of interest, since the FALW model assumes it to be at either the top or bottom of the domain. For the latter case, inter alia, the maximum adverse velocity at the top boundary will be measured. We will first formulate the governing equations and then proceed to describe the numerical method used. For completeness we also describe briefly the FALW model [7]. For a detailed review of the linear theory and the FALW model see Rottman et. al. [12]. Then we will present our numerical results for the two applications.
2 Governing equations

The governing equations for an incompressible, inviscid fluid are, in standard notation, 

\[ \rho \{ u_t + \bar{a} \cdot \nabla u \} + p_x = 0, \quad (1) \]
\[ \rho \{ w_t + \bar{a} \cdot \nabla w \} + p_z + \rho g = 0, \quad (2) \]
\[ \rho_t + \bar{a} \cdot \nabla \rho = 0, \quad (3) \]
\[ \nabla \cdot \bar{a} = 0. \quad (4) \]

Here we assume that \( \bar{u} \) is the perturbation velocity in the \( x \)-direction, relative to a uniform flow \( U \), so that the \( x \)-component of \( \bar{a} \) is \( U + u \). The continuity equation (4) is satisfied by introducing a perturbation streamfunction \( \psi \) such that \( u = -\psi_z, w = \psi_x \).

2.1 Stratified flow over topography

Next, eliminating the pressure \( p \), we get

\[ D_t \Delta \psi + \frac{\rho_z}{\rho} (g + D_t \psi_x) + \frac{\rho_z}{\rho} D_t \psi_z = 0, \quad (5) \]
\[ D_t \rho' + w \tilde{\rho}_z = 0, \quad (6) \]

where the total derivative is given by

\[ D_t = \frac{\partial}{\partial t} + U \frac{\partial}{\partial x} + J(\psi, \cdot), \quad (7) \]

\( J(\cdot, \cdot) \) is the Jacobian defined by \( J(a, b) = a_x b_z - a_z b_x \), and the total density \( \rho \) is defined by

\[ \rho(x, z, t) = \bar{\rho}(z) + \rho'(x, z, t). \quad (8) \]
The vorticity $\omega$ is given by $\omega = w_x - u_z = \Delta \psi$. The boundary conditions are

\[
\psi = U h(x) \quad \text{on} \quad z = h(x),
\]

\[
\psi = 0 \quad \text{on} \quad z = D,
\]

ensuring that there is no flow through the boundaries. When there is no topography present, we set $h = 0$. To identify the Boussinesq approximation, it is convenient to introduce a parameter $\kappa$, which is chosen such that the non-Boussinesq case is represented by $\kappa = 1$, and the equations in the Boussinesq approximation by $\kappa = 0$. Thus the vorticity equation (5) can be written as

\[
\omega_t = -U \omega_x - J(\psi, \omega) - \frac{\rho_x}{\rho_0 + \kappa(\rho - \rho_0)} g - \frac{\kappa}{\rho_0 + \kappa(\rho - \rho_0)} \left( \rho_x \frac{D\psi_x}{Dt} + \rho_z \frac{D\psi_z}{Dt} \right). \tag{11}
\]

The last term in equation (11) represents the inertial contribution to the vorticity equation. Note that it contains the variation of the velocities with time and space, and is neglected in the Boussinesq approximation. Furthermore, in the Boussinesq approximation, the total density $\rho = \rho_0 + \kappa(\rho - \rho_0)$ is set to a reference density $\rho = \rho_0$ except where it acts as the gravitational term, and is the case $\kappa = 0$ in equation (11).

The last term is retained in the non-Boussinesq case, which means that vorticity is generated by a nonzero horizontal density gradient, as is the case in the Boussinesq approximation, and also by a variation in the horizontal and vertical velocity components, that is, inertial terms in the horizontal and vertical directions. The variation in the perturbation density is given by equation (6),
or in full,

\[ \rho_i' = -U\rho_i - J(\psi, \rho') - \psi_{x}\rho_z. \] (12)

For the case of the flow over topography the initial conditions are that \( \psi = 0 \) and \( \rho' = 0 \) at \( t = 0 \). Together with the boundary conditions (9) and (10), these represent the case of an impulsively accelerated obstacle, and mimic the initial conditions in tow-tank experiments in the laboratory. The isopycnal surfaces are horizontal initially, and thus intersect the obstacle surface. This creates a small region of non-uniform distribution of density at the bottom boundary, which is swept rapidly downstream after the flow is initiated. The spectral filter that is being used filters out these modes and leaves the main flow behaviour unaltered. This small abnormality is seen in all our results, but gives no further reasons for concern, since the long-time behaviour near the obstacle is not affected (for a detailed description see Rottman et. al. [12]).

In the numerical simulations, a Gaussian curve was chosen as the obstacle shape,

\[ h(x) = ae^{-\frac{(x-x_0)^2}{2b^2}}. \] (13)

We introduce the dimensionless quantities based on the amplitude of the hill \( H \), the depth of the fluid \( D \), the horizontal dimension of the hill \( L \) and the flow speed \( U \). Thus, consider

\[ \frac{H}{D}, \quad \frac{L}{D}, \quad K = \frac{ND}{\pi U} \quad \text{and} \quad \beta = \frac{N^2D}{g}, \] (14)

where \( \beta \) is the Boussinesq parameter and \( K \) is the inverse Froude number. The resonant points are where \( K = n \), with \( n = 1, 2, ... \). We will consider small
values of $H/D$ and moderate values of $L/D$, with $0.95 \leq K \leq 1.2$. Equations (5,6) together with the boundary conditions (9,10) complete the formulation.

**FALW model equation**

Grimshaw and Yi [7] derived the finite amplitude long wave equation (FALW) for the resonant interaction of flow with topography, and an improved numerical solution to the FALW equation was presented in Rottman et. al. [12]. The FALW equation is given by

$$\int_{-\infty}^{x} \frac{\partial G(A, A')}{\partial A} \frac{\partial A'}{\partial t} dx' + \{U - c_n \} A - \frac{c_n^2}{N^2} \beta m(A) - \frac{1}{2} \frac{c_n^2}{N^2} \frac{\partial^2 A}{\partial x'^2} - \frac{c_n^2}{ND} (1 - \frac{N}{c_n} A) h = 0 \quad (15)$$

The kernel $G(A, A')$ is a complicated function of $A$ and $A'$ given in [7, 12], where $A(x, t)$ is the amplitude of the $n^{th}$ linear long wave mode, and $c_n = n\pi/D$ is the linear long wave phase speed. Equation (15) holds for amplitudes less than a certain critical amplitude, that is, $|A| < A_\ast = 1/N\pi$. The non-Boussinesq term in the FALW equation is included in $m(A)$, and is asymptotically given by

$$m(A) = 3A^2,$$

as the Boussinesq limit is approached ($\beta \to 0$), see [7].

7
2.2 Large-amplitude internal solitary waves with vortex cores

For the second case of large-amplitude internal solitary waves, we introduce a reference frame moving with the wave in the positive x-direction at the phase speed $c$

$$s = x - c \, t,$$  \hspace{1cm} (17)

and a modified streamfunction $\phi(s,z)$ for steady flow,

$$\psi(s,z) = -c \, z + \phi(s,z).$$  \hspace{1cm} (18)

Then, for steady flow, equation (3) for conservation of density implies that

$$\rho = \rho(\phi),$$  \hspace{1cm} (19)

Elimination of the pressure between the momentum equations (1,2) and making use of the transformation (17,18,19) yields a single equation for the streamfunction $\phi$, (see Dubreil-Jacotin [6] and Long [10]),

$$\phi_{ss} + \phi_{zz} + \frac{1}{\rho} \frac{d\rho}{d\phi} \left( g \, z + \frac{1}{2} \left( \phi_s^2 + \phi_z^2 \right) \right) = G(\phi).$$  \hspace{1cm} (20)

The functions $\rho(\phi)$ and $G(\phi)$ can be obtained on those streamlines originating upstream, where we assume that $\psi \to 0$, so that

$$\phi \to c \, z,$$  \hspace{1cm} (21)

$$\rho \to \bar{\rho}(z),$$  \hspace{1cm} (22)

where $\bar{\rho}(z)$ is the basic density profile. It follows that

$$\rho(\phi) = \bar{\rho}(\phi/c),$$  \hspace{1cm} (23)

$$G(\phi) = \frac{1}{\rho} \frac{d\rho}{d\phi} \left( \frac{g \, \phi}{c} + \frac{1}{2} \phi^2 \right).$$  \hspace{1cm} (24)
Introducing dimensionless coordinates based on the height of undisturbed fluid $h$ and the phase speed $c$,

$$\phi' = \frac{\phi}{c h}, \quad s' = \frac{s}{h}, \quad z' = \frac{z}{h} \quad (25)$$

and considering a basic density field close to uniform stratification

$$\bar{\rho}(z') = \rho_0(1 - \sigma z' - \sigma^2 f(z')) \quad (26)$$

transforms equation (20) to the following equation, which, after omitting the prime superscripts, is given by

$$\phi_{ss} + \phi_{zz} + \lambda(\phi - z)(1 + \sigma f_\phi(\phi) + \sigma \phi) - \frac{1}{2} \sigma (\phi_x^2 + \phi_z^2 - 1) + O(\sigma^2) = 0, \quad (27)$$

where

$$\lambda = \frac{\sigma g h}{c^2} \quad (28)$$

Here $\lambda$ is an inverse Froude number, and scales with unity with respect to the small parameter $\sigma$, which characterizes the weak stratification. The boundary conditions are that,

$$\phi_x = 0 \quad \text{on} \quad z = 0,1 \quad (29)$$

$$\phi \sim z \quad \text{as} \quad x \to \pm \infty. \quad (30)$$

Equations (27,29,30) provide a complete formulation of the problem if all streamlines originate upstream, thus excluding the possible presence of a recirculation region.
Derivation of the steady solitary wave solutions

To derive the steady solitary wave solution, an asymptotic expansion of $\phi$ and $\lambda$ in terms of $\epsilon^2$ is substituted into the governing equation (27)

$$
\phi(X,z) \sim \sum_{k=0}^{\infty} \epsilon^{2k} \phi^k(X,z),
$$

$$
\lambda \sim \sum_{k=0}^{\infty} \epsilon^{2k} \lambda^k,
$$

which yields the zeroth order solution, $\phi^{(0)} = A(X)\sin\pi z$ and $\lambda^0 = \pi^2$. Here we consider only the first mode, $n = 1$. A secularity condition applied at the next order then gives the amplitude equation,

$$
A_X^2 + \lambda A^2 + \kappa \frac{4}{3\pi} A^3 + 2 \int_0^A M(A')dA' = 0.
$$

(33)

The nonlinear function $M(A)$ is given by

$$
M(A) = 2\pi^2 \int_0^1 A \sin^2(\pi z) f(\phi(z + A \sin(\pi z)))dz.
$$

(34)

The parameter $\kappa$ is zero in the Boussinesq approximation and unity otherwise. In this study we choose $\kappa = 1$, while the case $\kappa = 0$ was studied by Aigner et al. (1999) [1]. Equation (33) holds in the whole $x-$domain where the amplitude $A < A_* = \frac{1}{\pi}$, and there are no flow reversals. However, if $A = A_*$ at some point $|x| = x_0$ on the wave profile there is an incipient flow reversal, and in the region $|x| < x_0$, an asymptotic solution containing a vortex core needs to be constructed. This is given by

$$
A(\xi) = A_* + \mu B(\xi) \quad \text{with} \quad 0 \leq B(\xi) \leq 1,
$$

(35)
where, again using a secularity condition, we get

\[
E^2_\xi = R(A_\ast) [1 - B] - \frac{8\nu}{15} \left[ 1 - B^2 \right], \tag{36}
\]

where

\[
R(A_\ast) = \kappa \frac{4}{3} \pi A_\ast^3 + 2M(A_\ast) - \frac{4}{A_\ast} \int_0^{A_\ast} M(A') dA' \tag{37}
\]

and

\[
\nu = \frac{(2\pi \mu)^{\frac{3}{2}}}{\epsilon^2}. \tag{38}
\]

For a solution to exist the right-hand side of (36) must be positive, yielding a bound on \(\nu\) and \(\mu\), which in turn places an upper bound on the maximum possible amplitude, that is,

\[
\nu < \nu_m = \frac{3}{4} R(A_\ast), \quad \mu < \frac{\epsilon^2}{2\pi} \nu_m^{\frac{3}{2}} \tag{39}
\]

\[
A_{max} = A_\ast + \mu. \tag{40}
\]

The eigenvalue \(\lambda^1\) is given by

\[
\lambda^1 = -\frac{1}{A_\ast^2} \left[ \kappa \frac{4}{3} \pi A_\ast^3 + 2 \int_0^{A_\ast} M(A') dA' + \mu \left( R(A_\ast) - \frac{8\nu}{15} \right) \right] \tag{41}
\]

Let us now suppose that the function \(f(z')\) is given by

\[
f(z') = \alpha_1 z'^2 + \alpha_2 z'^{\frac{3}{2}} + \alpha_3 z'^3, \tag{42}
\]

so that equations (34,37,41) yield

\[
M(A) = \frac{9\pi^2}{4} \alpha_3 A_\ast^3 + 8\pi \left( \alpha_3 + \frac{2}{3} \alpha_2 \right) A_\ast^2 \tag{43}
\]

\[
R(A_\ast) = \frac{9\pi^2}{4} \alpha_3 A_\ast^3 + \frac{4\pi}{3} \left( \delta + 4\alpha_3 + \frac{8}{3} \alpha_2 \right) A_\ast^2. \tag{44}
\]
and

$$\lambda^1 = - \left( \frac{4}{3} + \frac{32}{9} \alpha_2 + \frac{155}{24} \alpha_3 \right) \pi \alpha_{max} - \frac{9 \pi}{8} \alpha_3 \mu + \frac{2}{15} \left( \frac{2 \pi}{\epsilon} \right)^2 \lambda^2,$$ \hspace{0.5cm} (45)

where $\alpha_1$ is chosen conveniently to remove the linear term in $M(A)$ so that

$$\alpha_1 = -\alpha_2 - \left( 1 - \frac{3}{2 \pi^2} \right) \alpha_3.$$ \hspace{0.5cm} (46)

Introducing $\kappa^2 = -\lambda^1$ equation (33) becomes

$$A_x^2 - \kappa^2 A^2 + \tau_1 A^3 + \tau_2 A^4 = 0,$$ \hspace{0.5cm} (47)

where

$$\tau_1 = \frac{4 \pi}{3} (\delta + 4 \alpha_3 + 8 \alpha_2) \quad \text{and} \quad \tau_2 = \frac{9 \pi^2}{8} \alpha_3.$$ \hspace{0.5cm} (48)

Equation (47) can be readily solved. We consider here the two cases of a Korteweg-de Vries (KdV) ($\tau_2 = 0$) and modified KdV outer solution ($\tau_1 = 0$), which is then matched to the inner solution obtained by solving for $B$ in equation (36). The phase speed is given by

$$c = \sqrt{\frac{\sigma gh}{\lambda}}$$ \hspace{0.5cm} (49)

and is plotted in figure 15(a) for the case of a KdV outer solution (we also plot the phase speed of the conventional KdV solution for comparison).
3 Numerical Method

The governing equations for the vorticity can be written as

$$\nabla \cdot (\rho \nabla \Psi_t) = -\rho J(\Psi, \Delta \Psi) - \rho_x (g + J(\Psi, \Psi_x)) - \rho_z J(\Psi, \Psi_z),$$  \hfill (50)

where $\Psi$ is the total streamfunction $-Uz + \psi$. Using a Runge-Kutta Method in time amounts to solving the discrete elliptic problem in space at each iteration,

$$\nabla \cdot (\rho(x, z) \nabla \Psi(x, z)) = f(x, z).$$  \hfill (51)

In the Boussinesq approximation, $\rho(x, z)$ on the left-hand-side is replaced by $\rho_0$, and equation (51) reduces to a Poisson equation. In this special case the Poisson operator on the left-hand side of equation (51) is positive-definite and relaxation schemes succeed. The relaxation method was successfully used in the pseudo-spectral code of Rottman et. al. [12] and Aigner et. al. [1]. In the case where $\rho(x, z)$ is variable the operator on the left-hand side becomes

$$\rho(x, z) \nabla^2 + \nabla \rho(x, z) \cdot \nabla.$$

and for an equidistant discretisation, analogous problems have successfully been tackled, using multigrid methods and appropriate preconditioners, for the case of periodicity in two and three spatial dimensions, see [3, 13, 8].

For the present problem, we have an unequally spaced discretisation in conjunction with a transformation of the original differential operator to a topography-following differential operator, and also mixed periodic-nonperiodic boundary conditions. A multigrid technique due to Wesseling [15, 16] and McCarthy [11] was tried, but failed due to the operator not being sufficiently diago-
nally dominant. Also the multigrid technique requires a lot more computational effort to solve the elliptic problem (50). Considering our aim of examining the long-time behaviour of large-amplitude internal solitary waves with vortex cores in a non-Boussinesq fluid this is extremely undesirable.

For these reasons the multigrid technique was discarded in favour of a different iterative scheme that splits up the elliptic problem (50) into an easily solvable discrete elliptic problem and a fixed-point problem. Another method, also considered here, uses a relaxation scheme for the spatial integration, and a Runge-Kutta scheme combined with a Liouville-Neumann iteration to integrate forwards in time. Equation (50) can be regarded as a fixed-point problem

\[ \Delta \Phi = -J(\Psi, \Delta \Psi) - \frac{\rho_x}{\rho}(g + \Phi_x + J(\Psi, \Psi_x)) - \cdots - \frac{\rho_z}{\rho}(\Phi_z + J(\Psi, \Psi_z)), \]

(53)

where \( \Phi \) is the local temporal derivative of the streamfunction

\[ \Phi = \frac{\partial \Psi}{\partial t}. \]

(54)

The left-hand side of equation (53) is inverted at each time step using a relaxation scheme and a Runge-Kutta scheme to solve

\[ \Delta \frac{\partial \Psi}{\partial t}(x, z, t) = F(x, z, t, t^*) \]

(55)

for \( \Psi(x, z, t) \), where \( t^* \) is an intermediate time used in the iterative approximation of the local time derivatives, \( \Phi_x \) and \( \Phi_z \) on the right-hand side of equation (53). The time-differences on the right-hand side of (53) are approximated by
a second-order two-point stencil

\[ \Phi^N = \frac{\Psi^N - \Psi^0}{\Delta t} \]

where \( \Psi^N \) is the approximated streamfunction and \( \Psi^0 \) is the starting value for the iterated streamfunction \( \Psi \). The problem (53) can then be solved by iterating

\[ L\Phi^{N+1} = F\Phi^N \]

where \( L \) is the Laplace operator and \( F \) is the differential operator on the right hand side of equation (53). If \( \Psi^N \) lies in the space \( S \) and is complete, and the operator \( F \) on the right-hand side is Lipschitz-bounded in \( S \), then a converging Cauchy-series exists for all \( N \) (see Collatz [4] and [9, 14] cited therein).

The spatial integration uses a highly-accurate pseudospectral scheme with Fourier modes in the horizontal and Chebyshev modes in the vertical. The accuracy of the spatial integration is exponentially small (see Boyd [2]). The Runge-Kutta method used is a third-order scheme which is memory efficient and based on a method by Williamson [17]. The temporal approximation in the Liouville-Neumann iteration is second order.

An obvious restriction is imposed by the stability of the time-differencing on the right-hand side, and by the Runge-Kutta method itself. The former imposes the most severe restriction on the time-step. But once chosen sufficiently small (\( \Delta t = 0.46s \) as compared to \( \Delta t = 1.0s \) in the Boussinesq case) the velocity field does not change sufficiently fast to make more than a couple of iterations necessary. Conservation of mass and energy of this scheme is preserved to the required accuracy. The numerical model is valid until the disturbance starts
to overturn anywhere in the fluid, this is called wave breaking in the following. The condition for wave breaking is given by

$$\frac{\partial p}{\partial z} = 0$$  (58)

anywhere in the flow. Wave breaking generates small-scale disturbances that are not resolved by our model and leads to aliasing errors and eventual numerical blow-up. In Figure 2 at the non-dimensional time $Ut/D \approx 24$ the disturbance has reached wave breaking ($\rho_z \approx 0$) and the numerical model fails shortly after.

To compare the numerical results for two-dimensional flow over topography, with the one-dimensional model of the FALW equation by Grimshaw and Yi [7], the maximum amplitude of the disturbance and the drag $D$ on the obstacle are plotted, where

$$D = \int_{-\infty}^{\infty} p \frac{dh}{dx} dx$$  (39)

and $p$ is the pressure evaluated on the lower boundary.

4 Results

4.1 Stratified flow over topography

In this section we describe the numerical results of the non-Boussinesq model for resonant flow over topography. The basic density field is chosen to be

$$\bar{\rho}(z) = \rho_0 e^{\ln(1-\sigma)z}$$  (60)

where $\sigma$ is a measure of the strength of the stratification, typically chosen to be $\sigma = 0.01$. From equation (14) it follows that the Boussinesq parameter $\beta = 0.01$.
The Brunt-Väisälä frequency is given by,

\[ N^2 = -g \ln(1 - \sigma) \]  \hspace{1cm} (61)

and the flow speed by

\[ U_0 = \frac{ND}{\pi K} \]  \hspace{1cm} (62)

The region of stability for nonlinear hydrostatic flow in the Boussinesq limit is given by, (see [12])

\[ \pi H^* - |\sin(\pi(K^* - H^*))| \leq 0, \]  \hspace{1cm} (63)

where \( K^* \) and \( H^* \) are given by

\[ H^* = KH, \quad K^* = K - n + 1, \]  \hspace{1cm} (64)

where \( K \) is the inverse Froude number and \( n \) is the mode number. Equation (63) and the cases examined in detail in the following are plotted in Figure 1.

Four cases are of specific interest, when the inverse Froude number \( K \) equals 1.2, 1.1, 1.0 and 0.95, for a non-dimensional hill height \( H^* = 0.1 \), that is, the nondimensional parameter \( KH/D \) equals 0.1 and the length to depth ratio \( L/D = 20 \). Figure 2 shows a plot of the streamfunction computed by the spectral numerical model for \( K = 1.2 \), when it is essentially parallel to the density contours. Note the development of the deep trough downstream of the obstacle near the maximum slope of the hill and at the upper boundary at around \( Ut/D \approx 30 \).

The evolution in time of the amplitude function \( A(x,t) \) for the resonant mode as computed by the spectral model is plotted in Figure 4 and by the
\[
\begin{array}{|c|c|c|}
\hline
K^* & Ut/D (nB model) & Ut/D (FALW-model) \\
\hline
0.95 & * & * \\
1.0 & 88.23 & 77.9 \\
1.1 & 28.83 & 25.9 \\
1.2 & 23.74 & 21.1 \\
\hline
\end{array}
\]

*Table 4.1 Table of breaking times*

The FALW model in Figure 5, for the cases \( K = 1.2, 1.1 \) and \( K = 1.0 \) up to the breaking time \( t_{br} \) (refer to Table 4.1 for the exact breaking times of these cases). The amplitude function for the case \( K = 0.95 \) is plotted in Figure 6. The flow approaches a steady state and there is no breaking.

In Figure 6 the maximum amplitude for the cases considered is plotted up to the breaking time \( t_{br} \). The solid lines show the results for the spectral model and the dashed lines for the FALW model. For times \( Ut/D \approx 10 \) the amplitude in all these cases increases nearly linearly with time as predicted by linear resonant theory. For \( K = 0.95 \) the growth ceases altogether. For \( K = 1.0 \) the growth is quite slow but eventually reaches the breaking amplitude. For \( K > 1 \) the growth rate increases with \( K \). For \( K = 1.2 \) the growth is close to linear. Note that the growth as predicted by the FALW model is always bigger than the amplitude of the resonant mode calculated by the spectral model for \( K \geq 1 \).

For \( K = 1.0 \) and \( 1.1 \) the downstream trough grows to a breaking amplitude and the vertical position of breaking is located at the upper boundary. In Figure 7 the vertical position of breaking is plotted for several cases of the inverse Froude number in the range \( 1.0 \leq K \leq 1.2 \). Note that the vertical
position is always off centre and changes from close to the upper boundary to
close to the lower boundary, and simultaneously from downstream to upstream.
For \( K = 1.2 \) the development is similar, albeit more rapidly, but the crest that
develops over the obstacle does not start to propagate upstream as given in
the results of Grimshaw and Yi [7] (see in particular Figure 3). Overall, the
FALW model and the spectral model compare quite well in their time-dependent
behaviour.

Figure 8 is a plot of the drag over the hill for the cases considered. The
plot of the drag shows the agreement as well. Again the solid lines represent
the spectral model and the dashed lines the FALW model. The FALW model
slightly underpredicts the drag for \( K < 1.2 \). For \( K = 0.95 \) the drag approaches
zero, indicating that the flow eventually becomes symmetric about the obstacle.
For \( K \geq 1.0 \) the drag is non-zero when breaking occurs and the model indicates
that it remains non-zero even after breaking. This implies that the flow becomes
asymmetric about the obstacle with a high pressure on the upstream side and
low pressure on the downstream side.

4.2 Large-amplitude internal solitary waves with vortex cores

Next we describe the results of our numerical simulations for the long-time
behaviour of large-amplitude solitary waves with vortex cores. A maximum
time for the flow of \( t = 3 \cdot 10^4 \) was chosen in the following since the waves can
be considered to be of permanent form if their shape is preserved for more than
6 hours of physical time. Figure 9 shows the evolution of the streamfunction at
2/3 of the depth $D$ for $\mu = 0.99\mu_{max}$, and for the case of a KdV solution at the normalized time $t_n = tU/L$, where $L$ is the length scale of the solitary wave. Figure 10 is a sequence of contour plots of the streamfunction and density at the time computed. Figures 12 and 13 displays the results for the case of an mKdV solution for the same parameters as in the previous case. Noticeable from the contour plots are the transients propagating downstream and the essentially stagnant vortex core remaining intact. Figures 11 and 14 show the evolution of the maximum adverse velocity at the top boundary. Both show a nonvanishing velocity component in the upstream direction.

A series of results from 7 measurements, for $0 < \mu < 0.1$, shows a correction to the phase speed with an order of $O(10^{-4})$, see Figure 15(b). Note that the theoretical phase speed is known only to the order of $O(10^{-4})$, i.e. $O(\sigma^2)$ where here $\sigma = 0.01$ (see Figure 15(a)). The relative error maximum is 0.1%. In addition the numerical scheme is second order accurate. So the results suggest that the correction is within the error of the solution and numerical scheme.

5 Conclusion

We have presented results for the flow in a channel of a uniformly stratified fluid over isolated bottom topography and for the long-time behaviour of large-amplitude internal solitary waves with vortex cores, introducing a novel numerical method to solve the time-dependent, fully nonlinear vorticity and density equation for a non-Boussinesq fluid. We used a spectral model in conjunction with an iterative scheme to solve the fully nonlinear inviscid two-dimensional
equations, with a flow impulsively started from rest and compared it to the finite-amplitude long-wave (FALW) evolution equation derived by Grimshaw and Yi [7]. The results concerning the FALW equation show that the theory developed by Grimshaw and Yi [7] agrees very well with the fully nonlinear spectral model proposed here. The amplitude growth and the drag on the obstacle are in very good agreement.

Furthermore we studied the large-amplitude solitary waves with vortex cores derived by Derzho and Grimshaw [5]. The results show that these latter waves are of permanent shape and that the vortex core stays stagnant to first order. In addition our results indicate that there is a non-vanishing adverse velocity at the top boundary.

The agreement between our spectral model and the FALW model and the DG theory allows us to conclude that to within the error of the computation this scheme models the fully nonlinear, inviscid, two-dimensional time-dependent governing equations very well.

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References


Figure 1: The $K^* - H^*$ parameter space diagram based on the hydrostatic Long's model solution for flow over two-dimensional obstacles. Crosses denote the cases plotted in figures 3 to 5.
Figure 2: Contour plot of the streamfunction for $K = 1.2$ and $H^* = 0.1$ at the times $Ut/D = 0.0, 7.66, 15.32, 22.98, 30.64$. Breaking occurs at $Ut/D = 23.74$. 

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Figure 3: Plot of the amplitude function $A(x, t)$ for the resonant mode of vertical displacement as computed by the spectral model for the case with $H^* = 0.1$, $L/D = 2.0$ and (a) $K = 1.2$, corresponding to the case shown in figure 2, (b) $K = 1.1$ and (c) $K = 1.0$. The obstacle is centered at $x/D = 20$. The corresponding breaking times are $Ut_{br}/D = 23.7, 28.8, 88.2$. 

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Figure 4: Plot of the amplitude function $A(x, t)$ for the resonant mode of vertical displacement as computed by the FALW model for the case with $H^* = 0.1$, $L/D = 2.0$ and (a) $K = 1.2$, (b) $K = 1.1$ and (c) $K = 1.0$. The obstacle is centered at $x/D = 20$. The corresponding breaking times are $Ut_{br}/D = 21.1, 25.9, 77.9$.
Figure 5: Plot of the amplitude function $A(x, t)$ for the resonant mode of vertical displacement as computed by a) the spectral model and b) the FALW model for the case with, $H^* = 0.1$, $L/D = 2.0$ and $K = 0.95$ up to the time $Ut/D = 115$ and 160 respectively. The obstacle is centered at $x/D = 20$. 

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Figure 6: Plot of the maximum amplitude $|\frac{A}{C}|_{\text{max}}$ as a function of time, corresponding to the calculations for $K = 0.95, 1.0, 1.1$ and 1.2: The solid line indicates the spectral model and the dotted line the FALW model.
Figure 7: Vertical position of wave breaking for several cases, $1.0 \leq K \leq 1.2$, note that the breaking location is either at the top or at the bottom and sets in downstream or upstream of the hill.
Figure 8: Plot of the drag as a function of time on the hill of height $H^* = 0.1$ corresponding to the calculations for $K = 0.95, 1.0, 1.1$ and 1.2: The solid line indicates the spectral model and the dotted line the FALW model.

Figure 9: Plot of the streamfunction at the depth $z = 2/3D$ for $\mu = 0.99\mu_{max}$ and a KdV outer solution.
Figure 10: Density (left) and streamfunction (right) contour plots for the times $t = n7500$ secs ($n = 0, 1, 2, 3, 4$) and a KdV outer solution for $\mu = 0.99\mu_{\text{max}}$. 
Figure 11: Maximum adverse velocity for a KdV outer solution.

Figure 12: Plot of the streamfunction at the depth $z = 2/3D$ for $\mu = 0.99\mu_{max}$ and an mKdV outer solution.
Figure 13: Density (left) and streamfunction (right) contour plots for the times $t = n7500$ secs ($n = 0, 1, 2, 3, 4$) and an mKdV outer solution for $\mu = 0.99\mu_{\text{max}}$.  

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Figure 14: Maximum adverse velocity for an mKdV outer solution.
Figure 15: Plot of (a) phase speed for the solitary wave with a vortex core, and the KdV solitary wave (solid), and (b) the absolute error of the phase speed of the theoretical phase speed to the phase speed in the numerical model for several simulations, with $0 < \mu < 0.1$. Notice that the error is constant over the time of integration, see, for example, Figure 9.