Application of laser doppler velocimetry to rotor vibration measurement

This item was submitted to Loughborough University's Institutional Repository by the/an author.

Additional Information:

- A Doctoral Thesis. Submitted in partial fulfillment of the requirements for the award of Doctor of Philosophy of Loughborough University.

Metadata Record: [https://dspace.lboro.ac.uk/2134/7563](https://dspace.lboro.ac.uk/2134/7563)

Publisher: © John Robson Bell

Please cite the published version.
This item is held in Loughborough University’s Institutional Repository (https://dspace.lboro.ac.uk/) and was harvested from the British Library's EThOS service (http://www.ethos.bl.uk/). It is made available under the following Creative Commons Licence conditions.

For the full text of this licence, please go to: http://creativecommons.org/licenses/by-nc-nd/2.5/
Application of Laser Doppler Velocimetry to rotor vibration measurement

by

John Robson Bell

A Doctoral Thesis

Submitted in partial fulfilment of the requirements for the award of

Doctor of Philosophy of Loughborough University

May 2001

by John Robson Bell 2001
# Contents

<table>
<thead>
<tr>
<th>Contents</th>
<th>i</th>
</tr>
</thead>
<tbody>
<tr>
<td>Nomenclature</td>
<td>iv</td>
</tr>
<tr>
<td>Abstract</td>
<td>ix</td>
</tr>
<tr>
<td>Acknowledgements</td>
<td>x</td>
</tr>
</tbody>
</table>

## 1 Introduction

1.1 Vibration transducers
   1.1.1 Piezoelectric accelerometers 6
   1.1.2 Velocity transducers 7
   1.1.3 Proximity transducers 8
   1.1.4 The 'ideal' vibration transducer 9

1.2 Laser Doppler Velocimetry (LDV)
   1.2.1 Rotating diffraction gratings 12
   1.2.2 Acousto-optic cells (Bragg cells) 12
   1.2.3 Rotating scattering discs 13

1.3 Known problems with LDV for rotating machinery applications 13
   1.3.1 Laser speckle 14
   1.3.2 Cross-sensitivity 15

## 2 Theoretical analysis of measured velocity

2.1 Total velocity measured by a non-contact laser vibrometer 20
   2.1.1 Velocity at the point of incidence of the laser beam 20
   2.1.2 Velocity measured by a laser beam incident on a rotating shaft 22
   2.1.3 Isolating individual vibration sets 26
      2.1.3.1 Radial vibration measurement 26
      2.1.3.2 Axial vibration measurement 28

2.2 Total velocity measured at a fixed point by a contacting transducer 29

2.3 Comparison of non-contacting and contacting transducer outputs 32
   2.3.1 Non-rotating structures 32
   2.3.2 Rotating structures 33
2.4 Measurement of rotational vibration sets using multiple laser beams
   2.4.1 Concise theory for the difference velocity measured by a pair
       of parallel laser beams
   2.4.2 Isolating rotational vibration sets
   2.4.3 Measurements using multiple pairs of parallel laser beams
2.5 Resolution of individual motion components

3 Resolution of radial and angular vibration components
   3.1 Comparison of the theory of radial and angular vibration measurements
   3.2 Solution of the governing differential equations
       3.2.1 Analytical solution
       3.2.2 Frequency-by-frequency solution
       3.2.3 Synchronous vibration measurements
   3.3 Electronic implementation of frequency-by-frequency solution
   3.4 Experimental validation
   3.5 Diesel engine crankshaft vibration measurement

4 Angular vibration measurements on rotors
   4.1 Introduction
   4.2 Instrument design
   4.3 Experimental validation
   4.4 Diesel engine crankshaft angular vibration
   4.5 Torque fluctuation measurement

5 Errors in the resolution of radial and angular vibration measurements
   5.1 Errors in rotation speed measurement
   5.2 Effects of torsional vibrations
       5.2.1 Description of error terms
       5.2.2 Error terms in practical situations
   5.3 Correction of errors due to torsional vibrations in the resolution method
       5.3.1 Experimental implementation
6 Developments in the resolution of components:

Recommendations for further work 84

6.1 Introduction 84

6.2 Reduction of errors due to torsional vibrations 85
   6.2.1 Offset errors 85
   6.2.2 Repeated correction of errors due to torsional vibrations 86

6.3 Resolution of radial and angular vibration measurements using numerical integration 89

7 Conclusions 95

7.1 Introduction 95

7.2 Fundamentals of Laser Doppler Velocimetry measurements 96

7.3 Resolution of radial and angular vibration components 98

7.4 Developments in the use of LDV for vibration measurements on rotating components 101

Appendix 1 103

References 105

Figures 111
## Nomenclature

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\vec{A}$</td>
<td>Vector translation of shaft</td>
</tr>
<tr>
<td>$A_m$</td>
<td>Amplitude of $m^{th}$ component of vibration displacement</td>
</tr>
<tr>
<td>$A_x$</td>
<td>Amplitude of vibration displacement in $x$ direction</td>
</tr>
<tr>
<td>$a_v$</td>
<td>Vibration displacement amplitude</td>
</tr>
<tr>
<td>$a_x$</td>
<td>$x$ component of vibration displacement</td>
</tr>
<tr>
<td>$a_y$</td>
<td>$y$ component of vibration displacement</td>
</tr>
<tr>
<td>$a_z$</td>
<td>$z$ component of vibration displacement</td>
</tr>
<tr>
<td>$\hat{b}$</td>
<td>Unit vector defining beam direction</td>
</tr>
<tr>
<td>$C$</td>
<td>Abbreviation for Cosine</td>
</tr>
<tr>
<td>$C_1$</td>
<td>Calibration voltage</td>
</tr>
<tr>
<td>$C_2$</td>
<td>Calibration voltage</td>
</tr>
<tr>
<td>$d$</td>
<td>Magnitude of perpendicular beam separation</td>
</tr>
<tr>
<td>$\vec{d}$</td>
<td>Vector defining perpendicular beam separation</td>
</tr>
<tr>
<td>$EAF$</td>
<td>Error Amplification Factor</td>
</tr>
<tr>
<td>$\hat{e}$</td>
<td>Unit vector defining contacting transducers sensitivity axis</td>
</tr>
<tr>
<td>$\hat{e}_0$</td>
<td>Unit vector defining initial direction of contacting transducers sensitivity axis</td>
</tr>
<tr>
<td>$e_x$</td>
<td>$x$ component of initial transducer direction $\hat{e}_0$</td>
</tr>
<tr>
<td>$e_y$</td>
<td>$y$ component of initial transducer direction $\hat{e}_0$</td>
</tr>
<tr>
<td>$e_z$</td>
<td>$z$ component of initial transducer direction $\hat{e}_0$</td>
</tr>
<tr>
<td>$F_x$</td>
<td>$x$ component of position of point on rotating, non-vibrating structure</td>
</tr>
<tr>
<td>$f_D$</td>
<td>Doppler frequency shift</td>
</tr>
<tr>
<td>$f_s$</td>
<td>Reference beam frequency shift</td>
</tr>
<tr>
<td>$G_y$</td>
<td>$y$ component of position of point on rotating, non-vibrating structure</td>
</tr>
<tr>
<td>$G_z$</td>
<td>A group of vibration parameters</td>
</tr>
<tr>
<td>$G_y$</td>
<td>A group of vibration parameters</td>
</tr>
<tr>
<td>$I_{res}$</td>
<td>Effective resultant intensity</td>
</tr>
</tbody>
</table>
Photodetector current output

Overall circuit gain (x direction)

Overall circuit gain (y direction)

Integrator gain (x direction)

Integrator gain (y direction)

Vibrometer calibration constant (x direction)

Vibrometer calibration constant (y direction)

Torsional vibrometer calibration constant

Laser light wavenumber

Number of lines per unit length on diffraction grating

Ratio of vibration frequency: mean rotation frequency

Diffraction order

Unit vector defining rotation axis of transformation

Rotor radius

Position vector of arbitrary point.

Instantaneous position of point on a structure

Position vector of known point on line of beam

Position vector defining position of point on rotating, non-vibrating, structure

Position vector defining position of point P

Position vector defining position of point Q

Abbreviation for Sine

Radiant sensitivity of photodetector

Time

Magnitude of scattering particle velocity

Velocity measured by laser vibrometer

Velocity measured by contacting transducer

Measured x radial vibration set

a.c. coupled measured x radial vibration set

Measured y radial vibration set
\( \ddot{U}_y \)  a.c. coupled measured y radial vibration set

\( \dot{V}_0 \)  Velocity of origin O

\( \dot{V}_p \)  Velocity of point P

\( \dot{V}_Q \)  Velocity of point Q

\( V_x \)  Vibrometer output voltage (x direction)

\( \dot{V}_x \)  x component of vibration velocity of O

\( V_y \)  Vibrometer output voltage (y direction)

\( \dot{V}_y \)  y component of vibration velocity of O

\( \dot{V}_z \)  z component of vibration velocity of O

\( V_\Omega \)  Torsional vibrometer output voltage

\( v \)  Tangential velocity of point of incidence on diffraction grating

\( W(\omega_m) \)  Frequency dependent weighting factor

\( \dot{X} \)  Computed time signal (x direction)

\( \dot{X}_{out} \)  Analogue circuit output (x direction)

\( x \)  Direction of translating reference frame

\( x_m \)  \( m^{th} \) component of vibration displacement in the x direction

\( x_0 \)  x co-ordinate of known point on line of beam

\( \hat{x}_R \)  Unit vector defining direction of x axis of non-rotating reference frame

\( (x_m)_{est} \)  Estimate of \( m^{th} \) component of x direction vibration displacement

\( (\dot{x}_m)_{est} \)  Estimate of \( m^{th} \) component of x direction vibration velocity

\( \dot{Y} \)  Computed time signal (y direction)

\( \dot{Y}_{out} \)  Analogue circuit output (y direction)

\( y \)  Direction of translating reference frame

\( y_m \)  \( m^{th} \) component of vibration displacement in the x direction

\( y_0 \)  y co-ordinate of known point on line of beam

\( \hat{y} \)  Unit vector defining direction of y axis

\( \hat{y}_R \)  Unit vector defining direction of y axis of non-rotating reference frame
\((y^m_{\text{est}})\) Estimate of \(m\)\(^{\text{th}}\) component of \(y\) direction vibration displacement

\((\dot{y}^m_{\text{est}})\) Estimate of \(m\)\(^{\text{th}}\) component of \(y\) direction vibration velocity

\(z\) Direction of translating reference frame

\(z_0\) \(z\) co-ordinate of known point on line of beam

\(\hat{z}\) Unit vector defining direction of \(z\) axis

\(\hat{z}_R\) Unit vector defining direction of shaft rotation axis

\(\alpha\) Beam orientation in \(xy\) plane relative to \(x\) axis

\(\beta\) Beam orientation in \(xz\) plane relative to \(x\) axis

\(\Delta U_m\) Difference velocity measured by two beams

\(\Delta(\Delta U)\) Difference in velocities measured by two pairs of beams

\(\Delta\Omega\) Alternating component of rotation speed

\(\delta W(\omega_m)\) Error in weighting factor

\(\delta\Omega\) Error in measured mean velocity

\(\varepsilon\) Angle defining scattering direction

\(\Phi_{\text{res}}\) Effective resultant phase

\(\phi\) Phase

\(\phi_m\) Phase of \(m\)\(^{\text{th}}\) component of vibration displacement

\(\phi'\) Phase of vibrometer error velocity

\(\gamma\) Orientation of beam separator \(\vec{d}\) in \(yz\) plane relative to \(z\)

\(\lambda\) Laser wavelength

\(\mu\) Refractive index of air

\(\theta\) Magnitude of rotation transformation

\(\bar{\theta}\) Angular displacement of axial shaft element

\(\dot{\theta}_s\) Measured pitch vibration set

\(\theta_x\) \(x\) component of angular vibration displacement

\(\dot{\theta}_{x^m}\) \(m\)\(^{\text{th}}\) component of angular vibration velocity (Pitch)

\(\dot{\theta}_y\) Measured yaw vibration set

\(\theta_y\) \(y\) component of angular vibration displacement
\( \dot{\theta}_{y_m} \) \text{ } m^{th} \text{ component of angular vibration velocity (Yaw)}

\( \theta_z \) \text{ } z \text{ component of angular vibration displacement}

\( \rho \) \text{ } Angular rotation about } \hat{n}

\( \Sigma(\Delta U) \) \text{ } Sum of velocities measured by two pairs of beams

\( v \) \text{ Angle between velocity vector and bisector between scattering and source directions}

\( \Omega \) \text{ Rotation speed of axial shaft element}

\( \overline{\Omega} \) \text{ Mean rotation speed of axial shaft element}

\( \vec{\omega} \) \text{ Angular velocity vector}

\( \omega_c \) \text{ Angular frequency of test voltage}

\( \omega_B \) \text{ Photodetector beat frequency}

\( \omega_m \) \text{ Angular frequency of } m^{th} \text{ component of vibration}

\( \omega_v \) \text{ Vibration angular frequency}
Abstract

Vibration measurement is of fundamental importance in many machinery applications including for the development and monitoring of rotating machinery. In such applications, measurement of the vibration transmitted from the rotor into a non-rotating part of the structure is the most common arrangement but this cannot always be relied upon because vibration transmission may be low. In such cases, the use of a non-contacting vibration transducer capable of measuring vibration directly from the rotor itself is desirable.

Laser Doppler Velocimetry (LDV) is a non-contacting vibration technique capable of such measurements but vibration measurements on rotating structures using LDV have been shown to be ambiguous. The sensitivity of the measured velocity to other rotor vibration components can be significant enough to mask the intended vibration measurement entirely. This thesis examines the use of LDV for vibration measurements on rotating structures more comprehensively than in any previous study.

A new and completely general theory is developed to allow the velocity sensitivity of LDV measurements taken from rotating structures to be described for laser beam incidence in an arbitrary direction on a target element requiring 6 degrees of freedom to define its vibratory motion fully. Extension of the theory to optical configurations incorporating multiple laser beams is also included with a number of useful instrument configurations established.

The theory enables some fundamental questions regarding the use of LDV on rotating structures to be answered. Of particular importance is the confirmation that direct measurement of radial or pitch and yaw vibration is not possible because the measurements will always be unavoidably cross-sensitive to other motion components. Resolution of these components is possible, however and a new method of resolving steady state, non-synchronous radial, pitch and yaw vibrations is presented enabling a range of measurements to be made for the first time using LDV. Several of these measurements were made on a running IC engine and of special note are the angular vibration measurements made using a novel instrument incorporating 3 beams, the laser angular vibrometer, designed specifically for the task.

Errors within the resolution technique are considered in detail and, looking forward, a number of promising means by which to reduce error magnitudes are introduced and recommended for further investigation. LDV has great potential for rotating machinery diagnostics and such developments are key to achieving this potential.
Acknowledgements

My thanks go to the two sets of people who have made this thesis possible. Firstly to my supervisor Dr Steve Rothberg for his guidance, encouragement and endless enthusiasm which made him such a pleasure to work with. Secondly to my parents who have supported me in everything I have done. I dedicate this thesis to them.
To Mam and Dad
1 Introduction

Laser Doppler Velocimetry (LDV) is now accepted as a practical vibration measurement technique which can be used to complement traditional contacting vibration transducers, enabling vibration measurements to be made in some of the most challenging situations. This thesis describes the application of LDV to machinery diagnostics, concentrating primarily on vibration measurement directly from rotating structures.

There are few modes of transport or types of industrial machine in use today that do not include at least one rotating part, making rotating machinery one of the most important classes of machinery. Rotating machinery can often be found in situations where failures cannot be tolerated. This may be for financial reasons, such as in power generation where the revenue lost through unavailability can reach hundreds of thousands of pounds per day, or, in the case of aircraft engines, it may be because of the risk to human life. Together with these pressures, a general increase in quality expectations and safety requirements has led to widespread use of maintenance techniques, under the general heading of Condition Monitoring, aimed at the prevention and prediction of plant failure.

Several choices of measurement parameters exist which can give an indication of the condition of machinery including temperatures, loads, speeds, electrical currents, oil samples, debris etc., but vibration is generally accepted as the best indicator of machine health [1.1]. Vibration is a valuable parameter to measure because vibrations associated with the condition of a machine deteriorate in a predictable way and are generated in almost all dynamic systems. Individual components often generate distinctive vibration patterns or signatures which can be easily recorded and identified as the signature associated with a particular cause. Imbalance, mechanical looseness, misalignment, wear, oil whirl, cavitation, bearing element problems, eccentricity, cracks, rub are just some of the problems that can be identified [1.2].

In addition to identifying problems with machines in service, vibration data is used extensively in the design, development, manufacture and assembly of rotating machinery.
Balancing techniques rely on accurate vibration measurements and experimental data is used for performance verification and validation of computer models.

In this opening chapter the characteristics of the vibration transducers currently available for vibration measurements on rotating structures will be reviewed. The particular merits and failings of each transducer will be highlighted along with the characteristics that the 'ideal' vibration transducer for vibration measurements on rotating structures would have. The basic principles of Laser Doppler Velocimetry (LDV), which possesses many of the characteristics of the ideal vibration transducer, will then be introduced.

The non-contact nature of Laser Vibrometers offers significant advantages over contacting transducers and vibration measurements on hot, light and rotating structures are often cited as important applications. However, for vibration measurements on rotating structures the measurement can be ambiguous with the measurement sensitive to motion components other than the intended one. This "cross-sensitivity" to other motion components has been shown to be significant enough to entirely mask the intended vibration measurement and this presents the greatest problem in LDV vibration measurements from rotating structures. The chapter concludes with a demonstration of the measurement cross-sensitivity and a description of the effects of an additional source of ambiguity, laser speckle.

Chapter 2 begins with a comprehensive analysis of the velocity sensed by a single laser vibrometer beam incident in an arbitrary direction on a rotating structure. Previous studies have considered only special cases with selected vibration components but, for the first time, the vibration velocity sensitivity of a measurement will be described in terms of structure motion in all six degrees-of-freedom. Also, as a structure vibrates the position of the point of incidence changes on the structure. The effects of changes in the position of the point of incidence and the shape of the structure are included in the analysis. Comparison is made with previous theory and experimental validation of the new theory is carried out where appropriate.
The full expression for vibration velocity sensitivity shows that the measured velocity is made up of six separate "vibration sets", each a combination of motion parameters. Resolution of individual motion components within each set is shown not to be possible by geometric arrangement, or even manipulation, of the laser beam and direct measurements of radial, axial, pitch and yaw vibration, without cross-sensitivity to other motion components, is not possible. The theory also shows rotational vibration measurements cannot be made using a single laser beam and the chapter ends with the extension of the single laser beam theory to multiple laser beam arrangements.

Chapter 3 investigates the problem of cross-sensitivity found in radial and angular vibration measurements. Making two simultaneous vibration measurements, which individually show cross-sensitivity, for example, to motion in the y direction in an intended x direction measurement and vice-versa, along with a measurement of rotation speed enables the measurements to be described as a pair of linked differential equations. Resolution of the individual motion components requires the solution of the differential equations which can have time dependent coefficients if the rotation speed of the illuminated element fluctuates for example by torsional vibrations or during run up/down.

A practical resolution method is presented for situations where the rotation speed of the component is essentially constant. The method, however, is unable to resolve synchronous vibrations and the reasons behind this are discussed. An electronic implementation of the resolution method is described and the method is used to resolve radial vibrations on a test rotor. Comparison of the resolved measurements with measurements made using accelerometers fixed to the motor housing displays some differences which are shown to be due to genuine differences in the motion of the rotor and the housing. Resolved radial vibration measurements taken from a running diesel engine conclude the chapter. These enable, among other features, the first natural frequency of the crankshaft in bending to be identified.

Chapter 4 focuses on measurements of angular vibration and describes the design of a new instrument for angular vibration measurements. The new instrument, the Laser Angular Vibrometer (LAV), uses three parallel laser beams and enables simultaneous measurement
of two angular motion components. Using the LAV, a combined experimental validation of the theory presented in chapter 2 relating to angular vibration measurements and the resolution process described in chapter 3 is carried out on a small test rotor.

The remainder of the chapter concentrates on the measurement of crankshaft bending vibration in internal combustion engines. Crankshaft bending vibration is of particular importance in automotive NVH studies but investigations to date have been hindered by the difficulty in making direct measurements of bending vibration. Existing methods of deriving crankshaft bending vibration necessitate modification of either the engine or crankshaft. Successful identification of the first natural frequency of the crankshaft in bending, without modification to either the crankshaft or engine, is demonstrated using resolved angular vibration measurements made with the LAV. The chapter ends with a demonstration of a new application of LDV, to the measurement of torque fluctuations in rotating systems.

Chapter 5 concentrates on the effects of torsional vibrations and the accuracy of rotation speed measurement on the resolution process described in chapter 3. In the comparative tests made in chapters 3 and 4, where resolved vibration measurements made using laser vibrometers are compared with vibration measurements made using accelerometers fixed to a stationary structure, an increase in deviation between the two measurements sources is seen around the synchronous frequency. This deviation is not seen in comparative tests with the rotor non-rotating. The increase in sensitivity of the resolution process to an error in the measurement of rotation speed for vibration components close to synchronous is demonstrated and the “Error Amplification Factor” is defined to quantify the amount by which a error in the measurement of rotation speed is magnified to give an error in the calculated velocity at a particular frequency.

The method of resolving vibration measurements described in chapter 3 assumes that the rotation speed of the rotor is constant and that any torsional vibrations are negligible but torsional vibrations will be present to some extent in many practical situations. The error components in the resolved vibration velocity due to the presence of torsional vibrations are described and their effects simulated for a typical measurement. For measurements of
radial vibration the most significant error components are found to be those dependent on the initial alignment of the vibrometers. The chapter concludes with a demonstration of how the magnitudes of the remaining error components within the resolved vibration velocity can be reduced by subtracting an estimate of the error components, derived from knowledge of the torsional vibration spectrum, from the resolved vibration velocity.

Chapter 6 discusses the areas identified in this thesis which require further investigation work and suggests the ways in which progress could be made. The importance of accurately aligning the vibrometers has been seen throughout this work and an optical configuration which is insensitive to initial alignment is very desirable. Arrangements of parallel beams have proved to be useful in several measurement situations but the configuration required for translational measurements is shown to be equally sensitive to initial alignment as a single beam vibrometer.

In chapter 5 a promising method of reducing some of the errors was presented but routine application to all measurements requiring resolution is discussed. A relationship between the number of iterations of the method and the reduction in error obtained, for a specific torsional vibration amplitude, is demonstrated in chapter 6 along with the existence of a limit to the torsional vibration amplitude the method can tolerate, above which the method actually increases the errors present. Ultimately a resolution method insensitive to torsional vibrations and gross speed fluctuations is needed. The potential of one such resolution method, using a numerical integration method, is investigated in chapter 6. The method is simulated successfully and an initial attempt to implement the method in practice is made.

Concluding remarks are given in Chapter 7. The fundamentals of Laser Doppler Velocimetry are summarised including the “cross-sensitivity” of measurements to motion parameters other than the intended one. Two methods can be used to solve this problem, by employing different optical configurations and resolution using two simultaneous measurements, and the circumstances where these can be employed usefully are outlined. Finally the new areas where vibration measurement has been made possible as a result of this thesis are discussed.
1.1 Vibration Transducers

Three types of vibration transducer are commonly used for machine vibration measurements: piezoelectric accelerometers, velocity transducers and proximity probes. Each transducer has its own merits and the choice of which transducer to use is dependent on the specific application. The operating environment, dynamic range, robustness, permanency of the measurement and frequency range of interest are just a few of the parameters that need to be considered.

1.1.1 Piezoelectric accelerometers

A piezoelectric accelerometer is a seismic vibration transducer which generates a charge which is proportional to the acceleration to which it is subjected. At the heart of the accelerometer is its piezoelectric element. These elements produce an electrical charge proportional to the strain, and thus the applied force, when stressed in either tension, compression or shear. The piezoelectric elements form part of a mass-spring-damper system with the elements being loaded by a mass or masses. Figure 1.1 shows two different accelerometer configurations in which the piezoelectric elements are stressed in different ways.

When subjected to vibration the seismic mass exerts a force on the piezoelectric elements proportional to the relative displacement between the mass and the transducer base. For a system with little damping, which is the case for practical accelerometers, at frequencies well below the natural frequency, the relative motion between the seismic mass and the base, hence the accelerometer output, is directly proportional to the acceleration experienced by the transducer base with no phase difference.

The upper frequency limit of an accelerometer can be taken as approximately one third of its natural frequency, limiting the error to less than 1dB [1.3]. General use accelerometers have natural frequencies around 30 kHz but accelerometers with natural frequencies up to 180 kHz are available. The lower frequency limit of an accelerometer, typically 1Hz, is governed by the low frequency cut-off of the preamplifier which is used in conjunction with an accelerometer and by the effects of temperature transients [1.4].
As well as sensitivity to acceleration along its main axis, an accelerometer has a sensitivity to accelerations in a plane perpendicular to this, a transverse sensitivity. Figure 1.2 shows that the transverse sensitivity is direction dependent and is usually expressed as a percentage of the main axis sensitivity, typically it is below 1%. The effects of transverse vibrations can be reduced by positioning the accelerometer with its axis of minimum transverse sensitivity aligned with the axis of maximum transverse vibration.

The dynamic range of an accelerometer is determined by its structural strength. General purpose accelerometers can measure accelerations up to 100 km/s² while special shock accelerometers are available for accelerations up to 1000 km/s². The lowest acceleration that can be measured, typically below 0.01 m/s², is limited by electrical noise from the connecting cable and preamplifier circuitry.

The output from an accelerometer is fed into a preamplifier before connection to any measuring equipment. The preamplifier combines a very high input and low output impedance which prevent the loading and subsequent loss of sensitivity of the accelerometer that would occur if the accelerometer were directly connected to low input impedance measuring equipment. Usually the preamplifier also contains some variable amplification, integration and signal conditioning functions. The type of pre-amplifier is dependent on whether the accelerometer is a charge or voltage source.

1.1.2 Velocity Transducers

Velocity transducers are seismic vibration transducers, consisting of a mass-spring-damper system. Their construction is similar to that of the voice coil in a loudspeaker, consisting of a moving coil surrounded by a permanent magnet fixed to the base of the transducer. When subjected to vibration the coil, the mass in the system, moves relative to the magnet. This motion of the coil within the constant magnetic field induces an emf proportional to the rate at which the lines of magnetic flux are cut. For vibrations above the natural frequency of the system the motion of the coil relative to the magnet is directly proportional to the velocity of the base with no phase difference. Therefore, above the natural frequency, the voltage induced in the coil is proportional to the vibration velocity.
of the transducer base. This voltage can be directly input into measuring equipment without conditioning.

Velocity transducers typically have a frequency range of 10-1000 Hz. In order to allow low frequency vibrations to be measured, the natural frequency of the transducer needs to be low. The transducers are therefore lightly sprung and contain a large seismic mass which means that they are generally delicate, physically large and heavy. This makes them unsuitable for the vibration measurement of low mass objects. For vibrations below 10 Hz the signal output decays exponentially but measurements can be obtained through correction tables.

1.1.3 Proximity Transducers

Proximity transducers are displacement transducers that operate using the eddy current principle. A transducer consists of an insulated probe containing a small coil within its tip. A driver unit, also called an oscillator-demodulator or proximeter, feeds the coil with a high frequency carrier signal, typically 1 MHz, which creates a magnetic field at the tip of the probe. When the tip of the probe is brought into proximity with the test object, which must be made of a conducting material, the eddy currents induced by the magnetic field modulate the amplitude of the carrier signal in proportion to the distance between the probe tip and the test object. The demodulator then converts this to a voltage output proportional to the gap between the tip of the probe and the test object. Measurements of vibration displacement 0.25-2.5mm from the test object with a frequency range from dc-10kHz are typical [1.5] with a resolution of approximately 1% of the measurement range [1.6].

A major application of proximity probes is to measure shaft vibrations in situations where the shaft is mounted in stiff bearings which prevents significant transmission of vibration to the bearing housing. However, because the probes measure the gap between the probe tip and the shaft they are sensitive to shaft run-out. Mechanical run-out is caused when the shaft is eccentric to the rotation axis, the cross-section of the shaft is not perfectly circular or the shaft is bowed. These cause a variation in the gap distance through each rotation of the shaft when no vibration is present. Proximity probes are also sensitive to electrical run-
out caused by variations in the shaft permeability or thickness of any plating on a shaft. These effects are indistinguishable from a genuine vibration.

Mechanical and electrical run-out can be overcome to some extent by digitally subtracting the run-out measured from a slow roll of the shaft (i.e. with negligible vibration) from the “live” probe measurement. However, this assumes that the probe senses the same cross-section of the shaft during normal running as it did during the slow roll which may be unreliable in the presence of vibration.

On small diameter shafts, where the curvature of the shaft surface is significant compared to the probe diameter, proximity probes become more sensitive to shaft vibration perpendicular to the intended measurement direction. This causes a variation in the gap distance when there is no motion in the intended measurement direction. In addition, orthogonally mounted probes need to be offset axially to prevent cross-talk as they are in close proximity when the shaft diameter is small.

1.1.4 The ‘ideal’ vibration transducer

The “ideal” vibration transducer for vibration measurements on rotating machinery would encompass the desirable features of traditional vibration transducers without the drawbacks described above. Firstly, the transducer would measure vibration velocity which is widely acknowledged as the best parameter to measure in the frequency range encountered in analysis of rotating machinery and is cited in the British Standards covering machinery vibration. Secondly, the transducer would be non-contact, like proximity probes, allowing measurement directly from the rotor because, in many situations, for example where a rotating structure is mounted into large or rigid bearings, measurement of the low vibration transmitted from the rotor into a non-rotating structure cannot be relied upon. Thirdly, unlike proximity probes, the transducer would be insensitive to the shape of the rotor preventing problems with run-out and allowing straightforward measurement from irregularly shaped parts such as gear wheels and cams. Finally, the transducer would be robust with large dynamic and frequency ranges comparable with piezoelectric accelerometers. LDV offers these features and commercial manufacturers often quote measurements on rotating machinery as a key application area. In the next section the basic principles of LDV will be introduced.
1.2 Laser Doppler Velocimetry (LDV)

Laser Doppler Velocimetry is an interferometric measurement technique initially developed for non-intrusive fluid flow measurements and latterly used to make velocity measurements on solid surfaces. The technique uses the frequency shift which occurs in coherent light when it is scattered from a moving object. By detecting this Doppler frequency shift, named after the Austrian physicist who first considered the phenomenon, information about the velocity of the object can be obtained. The first measurements made using the technique were made by Yeh and Cummins in 1964 [1.7] who measured the laminar flow of water within a pipe. This thesis will concentrate on solid surface vibration measurements.

Figure 1.3 describes the frequency shift, \( f_D \), of the incident laser beam by [1.8]:

\[
\frac{2\mu U}{\lambda} \cos \nu \sin \frac{\epsilon}{2}
\]  

where \( \mu \) is the refractive index of air, \( \lambda \) is the laser wavelength, \( U \) is the magnitude of the scattering particle velocity, \( \nu \) is the angle between the velocity vector and the bisector of the angle between source and scattering directions and \( \epsilon \) is the angle defining the scattering direction. The laser vibrometers available commercially and used throughout this study collect the reflected light in direct backscatter within air, i.e. \( \epsilon = \pi \) and \( \mu = 1 \), therefore, the frequency shift measured is:

\[
\frac{2U \cos \nu}{\lambda}
\]  

where \( U \cos \nu \) now represents the component of velocity in the direction of the incident laser beam. By measuring the change in \( f_D \) with time, the time-resolved solid surface velocity can be found. \( f_D \) cannot be measured directly, however, because, for the velocities encountered, the frequency shifts observed, typically \( 10^6 \) Hz, are too small in comparison to the frequency of the laser light, \( 10^{14} \) Hz, to be resolved. The frequency shift
is measured by mixing the backscattered light with a reference beam derived from the same coherent source. The light is then heterodyned, the addition of two signals through a non-linear element, on the surface of a photodetector with the detected signal modulated at the beat or difference frequency between the two beams. Demodulation of this signal then gives a time-resolved velocity measurement. The non-linear element in the process is the photodetector as its electrical output is proportional to the intensity of incident light which is proportional to the square of the optical electric field. Figure 1.4 shows a typical laser vibrometer arrangement.

The technique of heterodyning is only capable of giving the modulus of difference frequency between the two beams and cannot distinguish which of the two beams has the higher frequency. For this reason the arrangement described so far would have a directional ambiguity, as a change in the sign of the velocity gives no change in the difference frequency. The most popular method of solving this problem is by frequency shifting the reference beam by a constant amount. The difference frequency seen by the photodetector then has a non-zero value, equal to the frequency shift in the reference beam, \( f_s \), when the target is stationary. The Doppler shifts from a positive velocity, \( +f_D \), and a negative velocity, \( -f_D \), then give different difference frequencies, \(|f_s - f_D|\) and \(|f_s + f_D|\) respectively. The size of the frequency shift in the reference beam governs the range of Doppler frequency shifts that can be successfully demodulated without directional ambiguity, setting the maximum velocity which can be measured by the instrument. Three different methods of frequency shifting are employed in the commercial and research vibrometers used in this study and their operating principles are discussed in Sections 1.2.1-1.2.3.

An alternative to frequency shifting for discriminating direction is to use a two phase detection system. The scattered light mixed with the reference beam is split into two channels, identical except for a phase difference of 90°, and heterodyned onto two separate photodetectors. The directional ambiguity is resolved by considering the phase difference between the two photodetector outputs. This is the method of directional discrimination used in the vibrometers manufactured by Ometron.
1.2.1 Rotating diffraction gratings

A diffraction grating consists of a number of lines ruled onto a reflecting or transmitting medium. A laser beam incident on a moving diffraction grating will diffract into orders of differing frequency and so by ruling the lines radially onto a rotating disc a continuous frequency shift can be arranged. The frequency shift, $f_s$, of a diffracted beam is equal to the number of lines passing a point per second multiplied by the order of diffraction [1.8]:

$$f_s = n_o N v$$  \hspace{1cm} (1.3)

where $n_o$ is the diffraction order, $N$ is the number of lines per unit angle on the grating and $v$ is the tangential velocity of the point of incidence. The mechanical rotation speed of the grating limits the maximum frequency shift possible to $\approx 5$ MHz. A rotating diffraction grating is used in the new instrument for angular vibration measurement described in Section 4.1 and they have been employed previously in non-commercial laser vibrometers [1.9].

1.2.2 Acousto-optic cells (Bragg Cells)

Bragg cells are perhaps the most popular devices used in LDV to frequency shift the reference beam and consist of a transparent medium within which acoustic waves are forced to travel. The presence of the acoustic waves makes the medium act as a moving three dimensional diffraction grating by setting up alternate layers of slightly differing refractive index due to the compressions and rarefactions of the acoustic wave. The diffraction efficiency of the grating is low as the changes in refractive index are very small. At a specific angle of incidence, however, called the Bragg angle after W.L. Bragg who first applied the principle to the diffraction of X-rays, the waves can be made to interfere constructively reinforcing the diffracted beam.

The frequency shift in the diffracted light is equal to the frequency of the acoustic waves and the direction of the shift depends on the direction of acoustic wave propagation. Ideally, the acoustic wavelength should be small compared to the diameter of the laser beam so that the beam intercepts a number of acoustic waves to give a useful reinforcement and, hence, high efficiency. This means that as short an acoustic wavelength
as possible is used giving diffraction efficiencies over 95% for first order beams, very small deflection angles and frequency shifts in the order of 40 MHz. Frequency shifts of this magnitude are generally too large for the signal processing techniques used in LDV. This problem can be overcome either by frequency shifting the photodetector output signal downwards electronically by mixing it with an oscillator at around 30-35 MHz, as used in the first laser vibrometer to use a Bragg cell [1.10] and the vibrometers marketed by Polytec and Dantec, or by using two Bragg cells, each giving slightly different frequency shifts. The two Bragg cells can be used in series, with the shifts in opposite directions, or in parallel with the shifts in the same direction. Both arrangements reduce the mean frequency shift seen at the photodetector down to the difference in frequency shift between the two cells.

1.2.3 Rotating scattering disc
The scattering disc is a motor driven disc from which an incident laser beam is scattered. The light frequency is shifted by the constant tangential velocity of scatterers on the disc [1.11]. The frequency shift produced can be easily controlled by varying the disc speed and the angle of incidence and radial position on the disc providing a simple and robust means of frequency shifting a laser beam which can be made self-aligning with the use of retro-reflective tape or paint [1.12]. This method of frequency shifting was employed in a vibrometer marketed by Brüel and Kjær.

1.3 Known problems with LDV for rotating machinery applications
The laser vibrometer, being a non-contacting velocity transducer which is insensitive to target shape, matches the description of the ‘ideal’ vibration transducer in section 1.2.4. However, two sources of ambiguity exist in vibration measurements made using laser vibrometers. One, laser speckle, is present in all laser Doppler measurements and makes interpretation of low-level vibration signals more difficult. The second, cross-sensitivity, is specific to measurements on rotating targets using laser vibrometers and can, in certain circumstances, make the measurements entirely ambiguous. Cross-sensitivity presents the greatest problem to making accurate vibration measurements on rotating targets using laser vibrometers and the effects and their solutions will be the main focus of this thesis.
1.3.1 Laser speckle

Laser speckle was first seen by researchers pioneering laser light sources. The appearance of “random dark and light spots” [1.13] and light of a “granular or peppery nature” [1.14] were reported when the first lasers were incident on diffuse surfaces and transmitted through diffuse media. This random distribution of high and low intensity is referred to as a “speckle pattern”.

Speckle patterns occur when light from a coherent source is scattered from a surface which is rough on the scale of the optical wavelength, which includes most surfaces of engineering interest. The component wavelets of the incident laser beam become dephased as each wavelet has to travel a slightly different distance to reach the surface. The still coherent wavelets then interfere constructively and destructively to create a speckle pattern which is unique in detail to the scatterers creating it. Any motion of the surface that causes the laser beam to illuminate a different set of scatterers will also cause the characteristics of the speckle pattern to change.

In LDV, the reference beam, which has a frequency shift $f_s$, and the light backscattered from the target are heterodyned onto the surface of a photodetector. For a target surface vibrating sinusoidally with displacement amplitude $a_v$ and frequency $\omega_v$, the alternating component of the photodetector current output, $i(t)$, can be written as [1.15]:

$$i(t) = S_p I_{res} \cos \left(2\pi f_s t + 2k a_v \sin \omega_v t + \Phi_{res}\right)$$

(1.4)

where $S_p$ is the radiant sensitivity of the photodetector, $k$ is the light wavenumber and $I_{res}$ and $\Phi_{res}$ are the effective resultant intensity and phase respectively which are dependent on the characteristics of the speckle pattern incident on the photodetector. Frequency demodulation of the photodetector output reveals the angular beat frequency $\omega_B$, from which the target velocity is derived. This is equal to the time derivative of the argument of the cosine term in equation (1.4):
If the spatial characteristics of the speckle pattern on the photodetector do not change then 
\[ \frac{d\Phi_{\text{res}}}{dt} = 0 \] and \( \omega_B \) is equal to the required value of \( \omega_B = 2\pi f_s + 2k\alpha_x \omega_v \cos \omega_v t \). In this case the output spectrum of the instrument consists of a single peak at the target vibration frequency \( \omega_v / 2\pi \). However, any non-normal motion of the target will cause the speckle pattern to change and \( \frac{d\Phi_{\text{res}}}{dt} \) will become non-zero and time dependent introducing a spurious noise into the instrument output. In general any non-normal motion will be periodic with the same fundamental frequency as the normal to surface vibration and so \( \frac{d\Phi_{\text{res}}}{dt} \) will be a pseudo-random signal with a fundamental frequency equal to the normal to surface vibration frequency. The resulting spectrum of the pseudo-random noise is characterised by approximately equal amplitude peaks at the fundamental frequency and subsequent harmonics which are indistinguishable from genuine low level vibrations and they have been given the name "pseudo-vibrations" [1.15].

This thesis is primarily concerned with LDV measurements taken directly from rotating structures. Figure 1.5 shows the spectrum of the output from a laser vibrometer incident on a target rotating at a constant speed of 26Hz with nominally no vibration. Approximately equal amplitude peaks at the rotation speed and its harmonics can clearly be seen. While the amplitudes of the peaks are dependent on the amplitude and rate of fluctuations in \( \frac{d\Phi_{\text{res}}}{dt} \), experience has shown that amplitudes are typically in the order of 0.1 mm/s. A degree of judgement must, therefore, be used when interpreting low level vibration data which could be of this magnitude.

1.3.2 Cross-sensitivity

The velocity measured by a laser vibrometer is the velocity component of the point on the target on which the beam is incident, in the direction of the incident beam. In general, the intended measurement is of the translational velocity of the target in the direction of the
incident beam (or is derived from the translational velocity). For vibration measurements on non-rotating targets the measured velocity is as intended. However, for rotating structures the velocity component in the direction of the beam, i.e. the measured velocity, includes additional velocity components due to the rotation itself. This “cross-sensitivity” of the measurement to motion components other than the intended motion was first reported in radial vibration measurements [1.16]. The measured velocity was found to be cross-sensitive to rotation speed fluctuations, including torsional vibrations, and motion perpendicular to the intended measurement.

Figure 1.6 depicts a radial vibration measurement on a rotor of radius $R$ which is rotating with angular velocity $\Omega$ and is vibrating in the $x$ direction. The intended measurement is the radial vibration velocity of the rotor in the $x$ direction but it can be seen in figure 1.6a that the measured velocity also includes the component of the tangential velocity, $R\Omega$, in the direction of the laser beam. Simple trigonometry shows that this “error velocity” is equal to $\Omega y_0$, where $y_0$ is the distance the beam is offset from the rotor’s centre of rotation, and is independent of the rotor’s radius. This suggests that the measurement is insensitive to the shape of the rotor and section 2.1 will confirm that this is true for rotors of any shape vibrating in any arbitrary fashion. Laser users and instrument manufacturers have often recognised that the component of the rotor’s tangential velocity can be large enough to overload the instrument output and have stated that the sensitivity of the measurement to the rotor’s tangential velocity can be eliminated by aligning the beam through the centre of the rotor [1.17]. This is only true for the simple case considered above where the rotation speed and offset have constant values and the error velocity is a constant. In most practical situations the rotor will also be vibrating in the $y$ direction and/or the rotation speed may fluctuate. It can be seen that under these circumstances the error velocity will also include alternating components of similar magnitude which are indistinguishable from a genuine vibration in the $x$ direction. This means that the measured velocity can be large when there is, in fact, no vibration in the $x$ direction. Likewise, a situation can exist where the error velocity is of the same magnitude and frequency as the vibration in the $x$ direction but 180° out of phase which means that the measured velocity is actually zero.
Figure 1.7a shows measurements of the vibration of a small test rotor vibrating in both the $x$ and $y$ directions made using accelerometers fixed to the motor housing. Using two laser vibrometers, one aligned in the $x$ direction and the other in the $y$ direction, comparative measurements were made directly from the rotor under first non-rotating and then rotating conditions. The vibrometer outputs are shown in figures 1.7b-d while the accelerometer outputs remain as in figure 1.7a at all times. Figure 1.7b shows vibrometer measurements made with the rotor non-rotating. As expected the measurements are very similar to those made using the accelerometers. Figure 1.7c shows vibrometer measurements made with the rotor rotating at a frequency of twice the vibration frequency. These measurements include some cross-sensitivity and it can be seen that the amplitudes and phases of the measured velocities no longer match those of the accelerometers. Figure 1.7d shows vibrometer measurements made with the rotor rotating at a frequency equal to the vibration frequency i.e. a synchronous vibration. This shows the worst case situation where the measured velocities are nominally zero despite genuine vibrations of significant magnitude in both the $x$ and $y$ directions and only the speckle noise discussed in section 1.4.1 can be seen.

The comprehensive analysis of the velocity measured by a laser vibrometer presented in chapter 2 explains the effects seen in figure 1.7 and also shows how axial and angular vibration measurements on rotating targets also have cross-sensitivities. The chapter also shows that these cross-sensitivities cannot be eliminated by any geometric arrangement of the laser beam or by introduction of additional laser beams. An approximate method of resolving radial and bending vibrations using two simultaneous vibration measurements is presented in chapter 3.
Theoretical analysis of measured velocity

The non-contact nature of LDV makes direct measurement of rotor vibration an obvious application and this is often quoted in the literature supplied by commercial instrument manufacturers. Indeed, one of the first reported LDV applications was by Davis and Kulczyk who made radial vibration measurements directly from a rotating turbine blade [2.1] on a turbine rotating at 900rpm. The authors concluded that the method showed promise but the signal strength needed to be improved. They also acknowledged that changes in the rotation speed and vibration of the turbine in its mountings could contribute to the variation seen in the measured velocity. A similar investigation using a fibre-optic laser vibrometer successfully measured the radial vibration of a bladed disc [2.2] but the vibrometers working range limited disc rotation speeds to around 1000rpm. Axial vibration measurements have also been made on bladed discs [2.3] and mode shapes obtained successfully but only for a disc excited at a single frequency in the axial direction.

Another application, with the rapid development of personal computers, is the study of the rotating magnetic discs found in hard drives. An area of particular interest is the interaction between the disc surface and the stationary slider which reads the information from the disc. Two beam differential measurements between the stationary slider and rotating disc [2.4] as well as single beam measurements on a flexible disc [2.5] have been made. Both studies considered the shape of the disc to be important with the authors of the dual beam taking care to position the beams away from a scratch in the disc surface while the dual beam study concluded that the unexplained fluctuation in the measured velocity was due to permanent deformation of the disc.

Modal analysis on a rotating disc has been demonstrated successfully using a continuously scanning laser vibrometer [2.6]. For measurements made using a non-scanning laser vibrometer, peaks in the response spectrum can include contributions from many vibration
components but by scanning the beam around the rotating disc the contributions of other vibration components can be separated.

LDV measurements made on rotors, however, can be ambiguous, affected by vibration components perpendicular to the component it is intended to measure. A feature of much previous work has been prediction of acceptable performance in the presence of a single vibration component, neglecting the effects of other components present in the more complex motions likely to be encountered in practice. When previous studies have acknowledged cross-sensitivities, notably to radial vibration measurements [2.7] and torsional vibration measurements [2.8], these have merely been special cases of the totally general theory to be presented in this chapter. Interestingly, investigators have already reported designs for multi-dimensional vibrometers and a compact 3D vibrometer is now commercially available [2.9].

A three dimensional vibrometer [2.10] for measurement of underwater structures has been constructed which can measure one out-of-plane motion component and two in-plane motion components. The instrument incorporates five beams which are focused at a point on the surface of the structure and the scattered light from all five beams is collected through a single aperture. Discrimination of the source of the scattered light is made possible by using different coloured beams and frequency shifts of different magnitudes.

A novel six-degree-of-freedom vibrometer [2.11] has been devised which consists of three laser beams and three photodetectors. The six motion components are computed from simultaneous measurements of the position of three beams reflected from a tetrahedral target fixed to the structure but in static tests the absolute error reached values of up to 0.2 mm in the translational measurements and 0.5° in the rotational measurements.

Until now, however, there has been no analysis of the velocity sensed by a single laser beam incident in an arbitrary direction on a target that is of substantial interest in engineering – a rotating shaft requiring three translational and three rotational co-ordinates to describe its vibratory motion fully. This chapter presents a comprehensive theory describing the measured velocity, which is the total velocity of the measurement point in
the direction of the laser beam, in terms of the six motion components, taking account of
the changes in position of the measurement point. This new theory is placed in its proper
context by a similarly comprehensive description of the velocity that would be measured
by a contacting velocity transducer and is extended to include measurements made using
multiple beams.

The theory presented in this chapter, first for one beam, then two or more beams, will be a
useful tool for the engineer, allowing the sensitivity of any measurement to be predicted
easily for any combination of target motion components. In addition, remaining
fundamental questions about the use of laser vibrometers on rotating components, such as
shape insensitivity and resolution of individual motion components, will finally be
answered. The resulting theory, however, will be equally applicable to any non-rotating,
vibrating structure.

2.1 Total velocity measured by a non-contact laser vibrometer

2.1.1 Velocity at the point of incidence of the laser beam
As shown in figure 2.1, the case considered is that of a rotating shaft, of arbitrary shape
and cross-section, undergoing an arbitrary vibration requiring three translational and three
rotational co-ordinates for description.

A translating reference frame $xyz$, which maintains its direction at all times, has its origin
O fixed to a point along the spin axis within the shaft. Point P on the shaft is defined by
the position vector $\vec{r}_p$. The velocity of P, $\vec{V}_p$, is the sum of the translational velocity of
origin O, $\vec{V}_o$, and the velocity of P relative to O as a result of rotation about an
instantaneous rotation axis passing through O at angular velocity $\vec{\omega}$:

$$\vec{V}_p = \vec{V}_o + (\vec{\omega} \times \vec{r}_p) \quad (2.1)$$

where $\vec{V}_o = \dot{x}\hat{x} + \dot{y}\hat{y} + \dot{z}\hat{z}$ and $\vec{\omega} = \dot{\theta}_x\hat{x} + \dot{\theta}_y\hat{y} + \dot{\theta}_z\hat{z} + \Omega \hat{z}_R$. $\dot{x}$, $\dot{y}$ and $\dot{z}$ are the translational
vibration velocities of the origin O in the x, y and z directions, $\Omega$ is the total rotation speed
of the axial shaft element (combining shaft rotation speed and any torsional vibration of
the axial element) and \( \dot{\theta}_x, \dot{\theta}_y \) and \( \dot{\theta}_z \) are the angular vibration velocities of the shaft
around the \( x, y \) and \( z \) axes, referred to as pitch, yaw and roll respectively.

In the usual configuration, a laser vibrometer measures target velocity at the point of
incidence in the direction of the probe laser beam. The position vector \( \vec{r}_p \) can be used to
define the point in space where the line of the laser beam intersects the surface of the shaft.
However, as the shaft vibrates and rotates, the position of this point, not only on the target
but also in space, will change continuously, becoming a function of time.

Figure 2.2 shows three ways in which the position in space of the incident point can vary
from an initial location, shown by a 'X' in each figure, to a final location, shown by a '○';
firstly due to translation of the shaft, secondly due to variations in the shape of the shaft as
it rotates (or rolls) and finally due to pitch and yaw of the shaft. Since the shaft can have an
arbitrary shape and it would be inconvenient to have the velocity measured described in
terms of shape, this would appear to be a difficulty in progressing this analysis.

A reliable piece of information, however, is that, no matter how the shaft moves, the point
of incidence will always lie somewhere along the line of the beam. Any point on the line
of the beam can be described as the sum of the position vector of a known point \( \vec{r}_0 \) that
lies on the line of the beam and a multiple of the unit vector, \( \hat{b} \), defining the direction of
the beam. At some time, \( t \), the position of the point of incidence \( \vec{r}_p(t) \) will have changed
from the initial position due to shaft motion and/or shape variation, as outlined previously,
and the new point of incidence is depicted in figure 2.3. The shaft has undergone the
translation \( \vec{A}(t) \) and the cross-section on which the beam is now incident has also changed
due to shaft motion and/or shape variation. The new position of the incident point can be
written as:

\[
\vec{r}_p(t) = [\vec{r}_0 - \vec{A}(t)] + p(t)\hat{b}
\]  

(2.2)
\(p(t)\) is always unknown but it will prove to be a convenient quantity to account for changes in the point of incidence. This new approach simplifies the analysis required considerably compared to previous studies.

The velocity measured by the laser vibrometer, \(U_m\), is the component of the velocity of the changing incident point in the direction of the incident beam. If it is assumed that there is no change in motion across the illuminated shaft element, using equations 2.1 & 2.2, \(U_m\) can be written as:

\[
U_m = \hat{b} \cdot \vec{V}_p = \hat{b} \cdot \vec{V}_0 + \hat{b} \cdot (\vec{\omega} \times [\vec{r}_0 - \vec{A}]) + \hat{b} \cdot (\vec{\omega} \times p\hat{b})
\]  

(2.3)

The second scalar triple product in equation (2.3) is always zero as can be seen more clearly by a simple re-arrangement:

\[
\hat{b} \cdot (\vec{\omega} \times p\hat{b}) = p\vec{\omega} \cdot (\hat{b} \times \hat{b})
\]  

(2.4)

This important result means that the measured velocity, \(U_m\), is independent of the unknown parameter \(p\), the parameter used to account for changes in space of the point of incidence due to shaft motion and shape variation. This proves, more generally than in any previous study, that the velocity measured by a laser vibrometer incident on a vibrating shaft is insensitive to the shape of the shaft, despite the fact that the incident beam can change axial and radial position on the shaft in any arbitrary fashion. Such immunity to target shape gives this measurement technique a significant advantage over, for example, proximity probe measurements. Of course the same shape immunity is found for measurements on targets undergoing simpler motions and the analysis still holds for scanning applications where \(\hat{b}\) is a function of time.

2.1.2 Velocity measured by a laser beam incident on a rotating shaft

Using the general theory presented above, the velocity measured by a laser beam incident on an axial element of shaft, of arbitrary shape, rotating about its spin axis whilst undergoing an arbitrary vibration can be derived.
The origin of the translating reference frame xyz is fixed to a point on the centre line of the shaft with the undeflected shaft rotation axis defining the direction and position of the z-axis. The configuration is depicted in figure 2.1 with the time dependent unit vector $\hat{z}_R$ defining the changing direction of the shaft rotation axis, which deviates from the z-axis as the shaft tilts. The velocity measured by the laser vibrometer, $U_m$, is the component of the velocity of the point of incidence in the direction of the incident beam.

The velocity of the point of incidence P, given by equation (2.1), can be expanded as:

$$\vec{V}_p = (\ddot{x} + j \ddot{y} + \ddot{z}) + \hat{\dot{z}}_x (\hat{x} \times \vec{r}_p)$$

$$+ \hat{\dot{y}}_x (\hat{y} \times \vec{r}_p) + \hat{\dot{x}}_x (\hat{z} \times \vec{r}_p) + \Omega (\hat{z}_R \times \vec{r}_p)$$

(2.5)

The small angular deviation of $\hat{z}_R$ from $\hat{z}$ can be related to the shaft pitch and yaw with $\hat{z}_R$ written in terms of a constant component, $\hat{z}$, and a time dependent component:

$$\hat{z}_R = \hat{z} + (\hat{\theta} \times \hat{z}) = \hat{z} + \theta_x \hat{x} - \theta_y \hat{y}$$

(2.6)

where $\hat{\theta}$ is the angular vibration displacement of the shaft element due to pitch, yaw and roll. This eliminates the need for the time dependent unit vector $\hat{z}_R$ and the velocity of the point P can now be written only in terms of the three orthogonal vectors $\hat{x}$, $\hat{y}$ and $\hat{z}$:

$$\vec{V}_p = (\ddot{x} + j \ddot{y} + \ddot{z}) + \left(\hat{\dot{x}}_x + \Omega \theta_y\right) (\hat{x} \times \vec{r}_p)$$

$$+ \left(\hat{\dot{y}}_x - \Omega \theta_x\right) (\hat{y} \times \vec{r}_p) + \left(\hat{\dot{z}}_x + \Omega\right) (\hat{z} \times \vec{r}_p)$$

(2.7)

Substituting the position of the incident point, $\vec{r}_p$, given by equation (2.2), into equation (2.7) enables the velocity measured by a laser vibrometer incident on a rotating shaft, given by equation (2.3), to be written as:
\[ U_m = \hat{b} \cdot (\dot{x}\hat{x} + \dot{y}\hat{y} + \dot{z}\hat{z}) + \left( \dot{\theta}_x + \Omega \theta_y \right) \hat{b} \cdot \left( -(z_o - z)\hat{y} + (y_o - y)\hat{z} \right) + \left( \dot{\theta}_y - \Omega \theta_x \right) \hat{b} \cdot \left( (z_o - z)\hat{x} - (x_o - x)\hat{y} \right) + \left( \dot{\theta}_z + \Omega \right) \hat{b} \cdot \left( -(y_o - y)\hat{x} + (x_o - x)\hat{y} \right) \] (2.8)

where \( x, y \) and \( z \) are the translational vibration displacements of the origin \( O \) in the \( x, y \) and \( z \) directions and \( x_o, y_o \) and \( z_o \) are the co-ordinates of the known point on the line of the beam.

In order to make this equation of more direct practical use, \( \hat{b} \) needs to be described in terms of measurable parameters. Figure 2.4 shows how \( \hat{b} \) can be described as a combination of two angles; with \( \hat{b} = \hat{x} \) initially, rotating first by an angle \( \beta \) around \( \hat{y} \), then by an angle \( \alpha \) around \( \hat{z} \). These two rotations are finite and, therefore, this order of rotation must be maintained. In Cartesian form \( \hat{b} \) is given by:

\[ \hat{b} = \cos \beta \cos \alpha \hat{x} + \cos \beta \sin \alpha \hat{y} - \sin \beta \hat{z} \] (2.9)

Substituting equation (2.9) into equation (2.8) gives the velocity measured by a laser beam, orientated according to the angles \( \alpha \) and \( \beta \) (refer to figure 2.4) and incident on a rotating shaft as:

\[ U_m = \cos \beta \cos \alpha \left[ \dot{x} + \left( \dot{\theta}_z + \Omega \right) y - \left( \dot{\theta}_y - \Omega \theta_x \right) z \right] + \cos \beta \sin \alpha \left[ \dot{y} - \left( \dot{\theta}_z + \Omega \right) x + \left( \dot{\theta}_x + \Omega \theta_y \right) z \right] - \sin \beta \left[ \dot{z} - \left( \dot{\theta}_x + \Omega \theta_y \right) y + \left( \dot{\theta}_y - \Omega \theta_x \right) x \right] - (y_o \sin \beta + z_o \cos \beta \sin \alpha) \left[ \dot{\theta}_z + \Omega \theta_y \right] + (z_o \cos \beta \cos \alpha + x_o \sin \beta) \left[ \dot{\theta}_y - \Omega \theta_x \right] + (x_o \cos \beta \sin \alpha - y_o \cos \beta \cos \alpha) \left[ \dot{\theta}_z + \Omega \right] \] (2.10)
Derivation of equation (2.10) represents a significant progression from previous studies, allowing the vibration engineer to be sure of vibration component sensitivity for any laser beam arrangement on any target, including a rotating shaft. It shows that the measured velocity is the sum of six terms, each the product of a combination of geometric parameters and a combination of motion parameters - the "vibration sets". The six "vibration sets", shown in square brackets, are inseparable combinations of different motion parameters. This important result shows that, no matter how a laser beam is aligned, only the combinations of motion parameters within the square brackets can be measured directly. An intended measurement of the radial velocity in the $x$ direction, for example, also includes a sensitivity to displacements in the $y$ and $z$ directions combined with pitch, roll, yaw and rotation of the shaft about its spin axis.

The first description of the cross-sensitivity of radial vibration measurements using laser vibrometers [2.12] demonstrated how the "error terms" in the measured velocity, $(\text{principally } (\dot{\theta}_z + \Omega)y \text{ or } (\dot{\theta}_z + \Omega)x)$ in equation (2.10), could be of sufficient magnitude to mask the intended measurements of the radial velocity, $\dot{x}$ or $\dot{y}$. A particular problem in the measurement of synchronous radial vibrations was also highlighted. Since this first description, there has been a need to establish whether a particular arrangement of laser beams or a particular variation of the arrangement, for example by scanning the laser beams, might enable automatic resolution of individual motion components. Equation (2.10) shows that this is not possible.

The problem is simplified enormously on a non-rotating target as the remaining error terms in the translational vibration sets are of a significantly smaller magnitude. For a rotating shaft undergoing an arbitrary vibration, however, direct measurement of pure radial, axial or bending vibration is not possible because the measurement will always be sensitive to other motion components. It may be possible to assume the effects of additional shaft motions are negligible, enabling measurement. More reliably, these components may be estimated by post-processing, at least to some extent, and these techniques will be described in chapter 3. In contrast, unambiguous measurement of the axial element's time-resolved rotation speed is possible, accepting that the torsional
vibration and roll motion of the shaft are indistinguishable, although it will be seen in the following section that this requires more than one laser beam.

2.1.3 Isolating individual vibration sets

Throughout the remainder of the discussion, the six vibration sets in equation (2.10) will be referred to by the vibration parameter in each group that might be regarded as the intended measurement. In the order that they are written in equation (2.10), the translational vibration sets are the x radial, y radial and axial, while the rotational vibration sets are the pitch and yaw vibration sets and the rotation speed set (which includes torsional vibration). Equation (2.10) can be further simplified by setting \( z_0 = 0 \) so that the plane of the origin of the xyz axes and the “measurement plane” are coincident, since this is just a matter of definition.

Isolation of any one of the six sets requires appropriate choice of values for \( \alpha, \beta, x_0 \) and \( y_0 \). A radial and an axial vibration measurement are shown below as examples. While the pitch and yaw vibration sets and the rotation speed set can be eliminated from a measurement, no values exist that can isolate these sets. Isolation of any of these sets requires the geometric coefficients of the three translational vibration sets to equal zero i.e. \( \cos \beta \cos \alpha = \cos \beta \sin \alpha = \sin \beta = 0 \), to which there is no solution. Measurements made with a single laser beam will always be sensitive, therefore, to either radial or axial vibration or both. In section 2.4 it will be shown that the rotational vibration sets can be isolated using a pair of laser beams.

2.1.3.1 Radial vibration measurement

To measure the x radial vibration set requires alignment of the laser beam so that it passes through the centre of the shaft and along the x-axis making \( \alpha = \beta = 0^\circ \) and \( y_0 = 0 \). The measured velocity is then equal to:

\[
U_m = \left[ \dot{x} + (\dot{\theta}_z + \Omega) y - (\dot{\theta}_y - \Omega \theta_z) z \right]
\]

(2.11)

Similarly, values of \( \alpha, \beta, x_0 \) and \( y_0 \) can be found that enable the y radial vibration set to be isolated.
Equation (2.11) shows agreement with previous two dimensional theory for radial shaft vibration measurements [2.7] which was validated experimentally over a range of vibration amplitudes, frequencies and shaft rotation speeds. This equation, however, extends the theory to include motion of the shaft in all six degrees-of-freedom. In particular, equation (2.11) reveals a third and previously unreported term in the measured velocity, \((\dot{\theta}_y - \Omega \theta_x)z\). The technique presented in chapter 3 to resolve \(x\) and \(y\) radial motions by post-processing relies on the assumption that this third term is an order of magnitude smaller than the first two.

Radial measurements made during experimental studies on rotating turbine blades [2.1, 2.2] have shown satisfactory results because only single vibration components were present. In practice, rotating structures may have motion in all six degrees-of-freedom and the measurement can therefore be ambiguous with the cross-sensitivity significant enough to mask the intended measurement. In addition to the terms in equation (2.11), measurements on rotating blades generally have the laser beam offset from the centre of the shaft \((y_0 \neq 0)\) so the measured velocity will contain terms from the rotation speed set.

Experimental validation of the previously unreported terms in equation (2.11), \(\dot{\theta}_y z\) and \(\Omega \theta_x z\), was carried out using a test rig that allowed simultaneous axial and angular vibration of a small test rotor. The combination of axial and angular vibrations creates sum and difference frequency components in the measured velocity. In the validation, the motions were driven at different frequencies to distinguish the appropriate components, which are small in magnitude, from other vibrations produced by the mechanism used to generate the required motion of the rig. The driving frequencies were carefully chosen so that the sum and difference components were distinct and did not coincide with harmonics of either of the driving frequencies.

Figures 2.5a&b show the measured sum and difference frequency velocity amplitudes for a variety of vibration amplitudes along with the “theoretical” values derived from measurements using piezoelectric accelerometers fixed to the bearing housing of the shaft.
Measurements of angular vibration were obtained by subtracting the outputs of two accelerometers separated by a known distance with the axial measurements made in the usual way. Combinations of two rotation speeds (nominally 28Hz and 48Hz), three axial vibration amplitudes (nominally 50, 250 and 500µm) and three angular vibration amplitudes (nominally 30, 150 and 300mrad/s for \( \dot{\theta}_y \) and 1.5, 7.5 and 15mrad for \( \dot{\theta}_x \)) were used and the data sets numbered in ascending magnitude of the theoretical component for convenience. The measured data show reasonable agreement with the theoretical data with the differences attributed, at least in part, to genuine differences between the motion of the point probed by the laser and the points at which accelerometers were located. It is, of course, the very existence of these genuine differences that is the motivation behind the development of techniques for measurement directly from the rotor. The relatively small magnitude of the measured data compounded the difficulties encountered.

2.1.3.2 Axial vibration measurement

To measure the axial vibration set, aligning the laser beam so that it is parallel to the shaft rotation axis (\( \beta = 90^\circ \)), makes the measured velocity equal to:

\[
U_m = \left[ \dot{\theta}_x + \Omega \theta_y \right] y + \left[ \dot{\theta}_y - \Omega \theta_x \right] x + y_0 \left[ \dot{\theta}_x + \Omega \theta_y \right] - x_0 \left[ \dot{\theta}_y - \Omega \theta_x \right]
\]  

This shows that an axial vibration measurement is cross-sensitive to a combination of radial displacements, pitch and yaw. The pitch and yaw sets of vibration terms in equation (2.12) are dependent on the offsets \( x_0 \) and \( y_0 \), and are of significance, for example, in studies such as [2.3, 2.4, 2.5] where the measurement point was offset from the shaft rotation axis. The sensitivity to these terms can be eliminated if it is acceptable to align the laser beam so that it is collinear with the shaft rotation axis, in which case \( x_0 = y_0 = 0 \). In studies such as [2.6] where the laser beam is scanned, \( \alpha, \beta, x_0 \) and \( y_0 \) are all functions of time. Even if the variation in \( \alpha \) and \( \beta \) can be considered small, changes in \( x_0 \) and \( y_0 \) may still be significant. Alternatively \( z_0 \) may be reintroduced (\( z_0 \neq 0 \)) if the reference plane is
that containing the incident point on the scanning mirror making $x_0 = y_0 = 0$ potentially. Equation 2.10 is flexible enough to be used for such arrangements.

### 2.2 Total velocity measured at a fixed point by a contacting transducer

To place the description of the velocity sensed by a single beam laser vibrometer in its proper context, it is useful to compare equation (2.10) with its equivalent for a contacting transducer. The velocity measured by a contacting transducer fixed to a point on a vibrating structure can be derived in a similar way to the expression for the velocity sensed by a laser beam.

For the laser vibrometer, a combination of equations (2.1) and (2.3) gives the velocity of the measurement point $P$ as:

$$\vec{V}_p = \vec{V}_0 + (\vec{\omega} \times \vec{r}_0) - (\vec{\omega} \times \vec{A}) + (\vec{\omega} \times p \hat{b})$$ (2.13)

The derivation of the velocity measured by a contacting transducer differs in that, unlike the incident point of the laser beam, the position of its measurement point $Q$ remains fixed on the structure. However, as the structure tilts, the position of the measurement point deviates from its initial position in space by a small amount so that:

$$\vec{V}_q = \vec{V}_0 + (\vec{\omega} \times \left[ \vec{r}_0' + \left( \vec{\theta} \times \vec{r}_0 \right) \right]) = \vec{V}_0 + (\vec{\omega} \times \vec{r}_0') + \vec{\omega} \times (\vec{\theta} \times \vec{r}_0')$$ (2.14)

where $\vec{\theta}$ is the angular vibration displacement of the shaft element due to pitch, yaw and roll. $\vec{r}_0'$ is comparable with $\vec{r}_0$ but differs from $\vec{r}_0$ in that it is a function of time and is used to account for the gross changes of the measurement point in space due to rotation of the shaft.

It is interesting to compare equations (2.13) and (2.14) which show two common terms, "$\vec{V}_0 + (\vec{\omega} \times \vec{r}_0)$," and additional terms which account for the change in the measurement point as the structure vibrates and rotates.
In equation (2.13) the first of the additional terms, \( (\vec{\omega} \times \vec{a}) \), accounts for the effect of the change in the point of incidence of the laser beam on the target due to translation of the structure. The second term, \( (\vec{\omega} \times \vec{p} \vec{b}) \), accounts for changes in space of the point of incidence due to target motion and/or shape variation but this vector component is not present in the measured velocity because it is always perpendicular to the unit vector describing the incident laser beam. The additional term in equation (2.14), \( \vec{\omega} \times (\vec{\theta} \times \vec{r}_0') \), accounts for the effect of changes in the measurement point as the structure tilts.

For a non-vibrating structure, rotating about its spin axis, defined by the unit vector \( \hat{z} \), the instantaneous position of a point on the structure, \( \vec{r}_0' \), can be written as:

\[
\vec{r}_0' = (x_0 \cos \Omega t - y_0 \sin \Omega t)\hat{x} + (x_0 \sin \Omega t + y_0 \cos \Omega t)\hat{y} + z_0\hat{z} \quad (2.15a)
\]

As a result of the angular vibration of the structure the position vector becomes:

\[
\vec{r}_Q = (x_0 \cos \Omega t - y_0 \sin \Omega t)\hat{x}_R + (x_0 \sin \Omega t + y_0 \cos \Omega t)\hat{y}_R + z_0\hat{z}_R \quad (2.15b)
\]

where \( x_0, y_0 \) and \( z_0 \) define the initial position of the point in the translating reference frame \( xyz \) and \( \hat{x}_R, \hat{y}_R \) and \( \hat{z}_R \) are the unit vectors defining a translating and tilting, but non-rotating, reference frame fixed to the component.

As previously described by equation (2.6), the small angular deviations of \( \hat{x}_R, \hat{y}_R \) and \( \hat{z}_R \) from their undeflected positions, \( \hat{x}, \hat{y} \) and \( \hat{z} \), can be related to the pitch, yaw and roll of the component such that the instantaneous position of a point \( Q, \vec{r}_Q \), on the component can be written as:

\[
\vec{r}_Q = [F(t) + z_0 \theta_y - \theta_z G(t)]\hat{x} + [G(t) - z_0 \theta_x + \theta_z F(t)]\hat{y} + [z_0 + \theta_x G(t) - \theta_y F(t)]\hat{z} \quad (2.16)
\]
where \( F(t) = x_0 \cos \omega t - y_0 \sin \omega t \) and \( G(t) = x_0 \sin \omega t + y_0 \cos \omega t \).

By inserting this expression for the position of point \( Q \) into equation (2.7), which describes the velocity of any point on a rotating component, the velocity of the point \( Q \), \( \vec{V}_Q \), can be written as:

\[
\vec{V}_Q = \left( \hat{x} + \hat{y} + \hat{z} \right) + \left( \hat{\theta}_x + \Omega \hat{\theta}_y \right) \left[ \left( G(t) - z_0 \theta_x + \theta_z F(t) \right) \hat{z} - \left[ z_0 + \theta_x G(t) - \theta_y F(t) \right] \hat{y} \right] \\
+ \left( \hat{\theta}_y - \Omega \hat{\theta}_z \right) \left[ \left[ z_0 + \theta_y G(t) - \theta_z F(t) \right] \hat{z} - \left[ F(t) + z_0 \theta_y - \theta_z G(t) \right] \hat{x} \right] \\
+ \left( \hat{\theta}_z + \Omega \right) \left[ \left( F(t) + z_0 \theta_z - \theta_y G(t) \right) \hat{y} - \left[ G(t) - z_0 \theta_x + \theta_z F(t) \right] \hat{x} \right] \tag{2.17}
\]

A further effect in the measured velocity results from changes in the sensitivity vector for the contacting transducer. While the non-contacting transducer measures \( \hat{b} \cdot \vec{V}_p \), where \( \hat{b} \) is usually a constant, the contacting transducer measures \( \hat{e} \cdot \vec{V}_Q \), where \( \hat{e} \) is a unit vector defining the sensitivity axis of the transducer.

As the component tilts the direction of the sensitivity axis deviates a small amount from its undeflected direction, \( \hat{e}_0 \), such that:

\[
\hat{e} = \hat{e}_0 + \left( \hat{\theta} \times \hat{e}_0 \right) = \left( e_x + \theta_y e_z - \theta_z e_y \right) \hat{x} + \left( e_y + \theta_z e_x - \theta_x e_z \right) \hat{y} + \left( e_z + \theta_x e_y - \theta_y e_x \right) \hat{z} \tag{2.18}
\]

where \( \hat{e}_0 = e_x \hat{x} + e_y \hat{y} + e_z \hat{z} \). In addition \( \hat{e}_0 \) and, therefore, \( e_x \), \( e_y \) and \( e_z \) may be time dependent in the same way as \( \vec{r}_o' \) in equation (2.15a).

Such modification of \( \hat{e} \cdot \vec{V}_Q \) with equations (2.17) and (2.18) would reveal the velocity measured by the contacting transducer fixed to a rotating structure undergoing arbitrary motion. This is not done here because its complexity does not add significantly to the comparison being made.
2.3 Comparison of non-contacting and contacting transducer outputs

2.3.1 Non-rotating structures

Figure 2.6 shows a typical measurement situation. The $y$ vibration velocity of a cantilever beam is to be measured at some position along its length. The axial location of the measurement point and the co-ordinate axes are chosen to coincide, with the axis of the cantilever beam defining the direction of the $z$ axis.

For a contacting transducer with its base fixed to point $Q$ and its sensitivity axis orientated in the positive $y$ direction, from equations (2.17) and (2.18) with $\Omega = 0$, the velocity measured is:

$$U_Q = \dot{y} + \hat{\theta}_z (x_0 - \theta_z y_0) - \hat{\theta}_x (\theta_x y_0 - \theta_y x_0)$$

$$+ \theta_x [\dot{z} + \hat{\theta}_z (y_0 + \theta_z x_0) - \hat{\theta}_y (x_0 + \theta_y y_0)]$$

$$- \theta_z [\dot{x} + \hat{\theta}_y (\theta_x y_0 - \theta_y x_0) - \hat{\theta}_z (y_0 + \theta_z x_0)]$$

(2.19a)

To make the same intended measurement with a laser vibrometer requires the beam to be incident on a point $P$, where the points $P$ and $Q$ are initially coincident, and aligned parallel to the $y$ axis making $\alpha = 90^\circ$, $\beta = 0^\circ$, $z_0 = 0$. From equation (2.10) the measured velocity is:

$$U_m = \dot{y} + \hat{\theta}_z (x_0 - x) + \hat{\theta}_xz$$

(2.19b)

Equations (2.19a&b) show that, neglecting the higher order terms, the two measurements are both equal to $\left(\dot{y} + \hat{\theta}_z x_0\right)$, which is readily accepted as the $y$ velocity of the measurement point. The additional terms in equation (2.19a) are due to the change in direction of the transducer sensitivity axis as the structure tilts (those enclosed in square brackets) and to the change in position of the measurement point in space as the structure tilts. The additional terms in equation (2.19b) are due to the measurement point changing...
position on the target, while remaining essentially fixed in space because changes due to
target shape are unimportant.

2.3.2 Rotating structures
As discussed at the end of section 2.2, on a rotating structure the time dependence of \( \dot{e}_0 \) would make for a very complicated description of the measured velocity. If it could be achieved, a useful and interesting contacting transducer would be one whose sensitivity axis remained fixed in direction despite the target motion, especially the target rotation. For a measurement on a rotating structure it is interesting to compare the output of this 'ideal' contacting transducer with the fixed direction laser vibrometer measurement.

For a measurement of radial vibration in the y direction the velocity measured by the 'ideal' contacting transducer is the velocity component in the y direction. Using equation (2.17) and letting the axial location of the transducer define the location of the co-ordinate axes:

\[
\begin{align*}
\dot{y} = \dot{y} - (\dot{\theta}_z + \Omega \theta_y)[\theta_x G(t) - \theta_y F(t)] + (\dot{\theta}_z + \Omega)[F(t) - \theta_z G(t)] 
\end{align*}
\]  
(2.20a)

Making the same intended measurement using a laser vibrometer, equation (2.10) would give:

\[
U_m = \dot{y} + (\dot{\theta}_x + \Omega \theta_y)[z] + (\dot{\theta}_z + \Omega)[x_0 - x]
\]  
(2.20b)

Unlike equations (2.19&b), after neglecting the higher order terms, equations (2.20a&b) differ in their cross-sensitivities to rotation speed and roll. The terms enclosed in the square brackets in both equations result from the change in position of the measurement point. The differences between these sets of terms occur because the positions of the measurement points change in different ways. For the contacting transducer, displacement of the structure causes the measurement position to change position in space but remain in the same position relative to a rotating reference frame attached to the structure. In contrast, for the laser vibrometer, displacement of the structure causes the measurement point to change position on the target but it remains essentially fixed in space since
changes in position along the line of the laser beam have no effect on the measured velocity, as equation (2.4) showed.

The cross-sensitivity of the measurements can be compared for both transducers by studying the dominant ‘error velocity’ term, that of shaft rotation speed $\Omega$ multiplied either by the translation of the measurement point $F(t)$ for the contacting transducer or by the displacement $x$ for the laser vibrometer. Considering a harmonic vibration at frequency $\omega$, for the purposes of comparison, these terms can be written as:

\[
\begin{align*}
\Omega F(t) &= \Omega \sqrt{x_0^2 + y_0^2} \cos(\Omega t + \phi) \\
\Omega x &= \Omega A_x \cos(\omega t + \phi')
\end{align*}
\]

where $\phi$ and $\phi'$ are the phase terms and $A_x$ is the amplitude of the vibration displacement in the $x$ direction i.e. perpendicular to that of the intended measurement.

As $\sqrt{x_0^2 + y_0^2}$, which for a cylindrical shaft is equal to the shaft radius, is generally much larger than displacement amplitudes, the cross-sensitivity is much greater even for this ‘ideal’ contacting transducer. However, as $x_0, y_0, \phi$ and $\Omega$ are known or can be measured, the potential to subtract this component from the contacting transducer’s measured velocity does exist while this is not possible for the laser vibrometer as $A_x$ and $\phi'$ are unknown - indeed, they are part of the required measurement. Unfortunately, because it is likely that $\Omega F(t) \gg \dot{y}$, the resulting estimate of $\dot{y}$ would be very prone to error.

The cross-sensitivity in the laser vibrometer measurements, which can mask the intended measurement entirely, can be overcome to a reasonable extent for all but synchronous vibrations by post processing. The cross-sensitivity problems in the laser vibrometer measurement are dependent on the vibration frequencies whereas the cross-sensitivity in the contacting transducer measurement would principally only occur at synchronous frequencies. However, any torsional vibrations or speed fluctuations would cause time
dependence in $\Omega$ and introduce cross-sensitivity into the contacting transducer measurements at additional frequencies.

These comparisons emphasise the difficulties encountered in attempting to measure vibrations directly from a rotating target not just for non-contacting transducers, as documented previously, but also for contacting transducers and even for an 'ideal' contacting transducer. The potential of laser vibrometry for such challenging applications is not so much diminished by the issues raised in this section as it is emphasised by the problems that would be encountered even if a suitable contacting transducer could be devised.

2.4 Measurement of rotational vibration sets using multiple laser beams

A variety of optical configurations have been proposed for angular vibration measurement on non-rotating structures. A dual beam laser vibrometer has been constructed which derives angular vibration velocity from differential velocity measurements made with two parallel laser beams separated by a known distance [2.13] and a commercial instrument with this arrangement of beams, but different optical configuration, is available [2.14].

Measurements of angular vibration have also been made using a single beam laser vibrometer which is continuously scanned across the component [2.15]. In this work, simultaneous measurement of one angular and one translational vibration component was possible by scanning the beam in a straight line while a circular scan enabled two angular vibration and one translational component to be derived. Although the time saving made by simultaneous acquisition of angular and translational data was considerable, the vibrometer outputs needed to be carefully interpreted and the technique was found to be best suited to modal analysis where the excitation frequency can be stepped.

Simultaneous measurements of two angular motion components and one translational component have also been proposed using a different type of dual beam vibrometer [2.16]. These measurements were derived from simultaneous measurements of the position of two beams reflected from a planar target using two-dimensional position sensing.
photodetectors. However, the frequency response and accuracy of the vibrometer was limited and the measurements needed to be processed off-line.

Section 2.1.3 confirmed that it is not possible to isolate any of the rotational vibration sets with a single laser beam and commercial LDV systems for measurement of rotational vibrations do use multiple laser beams. As far as the velocity sensitivity model presented in this thesis is concerned, the next logical step to take in attempting to isolate the rotational vibration sets is to include a second laser beam. In this section, the particular usefulness of a pair of parallel beams will be demonstrated. A parallel beam arrangement has been developed [2.17] and is available commercially for torsional vibration measurement on rotors. This parallel beam arrangement can also be used for measurement of other angular vibration components, either on rotating or non-rotating structures.

In an interferometer it is the difference velocity that can be obtained most conveniently by heterodyning and so it is the difference velocity that will be analysed in this section. A sum velocity could effectively be obtained by re-orientating one of the incident beams by 180° and so, by examining the difference velocity for arbitrarily orientated beams, all possible eventualities can be explored. Using equation (2.10) and assuming that the motions of the axial elements probed by each beam are the same, the difference in the velocity measured by two beams \( \Delta U_m = U_{m1} - U_{m2} \) can be written as:

\[
\Delta U_m = (\cos \beta_1 \cos \alpha_1 - \cos \beta_2 \cos \alpha_2) [\dot{x} + (\dot{\theta}_x + \Omega) y - (\dot{\theta}_y - \Omega \theta_y) z] \\
+ (\cos \beta_1 \sin \alpha_1 - \cos \beta_2 \sin \alpha_2) [\dot{y} - (\dot{\theta}_x + \Omega) x + (\dot{\theta}_z + \Omega \theta_y) z] \\
- (\sin \beta_1 - \sin \beta_2) [\dot{z} - (\dot{\theta}_x + \Omega \theta_y) y + (\dot{\theta}_y - \Omega \theta_x) x] \\
- (y_{01} \sin \beta_1 - y_{02} \sin \beta_2 + z_{01} \cos \beta_1 \sin \alpha_1 - z_{02} \cos \beta_2 \sin \alpha_2) [\dot{\theta}_z + \Omega \theta_y] \\
+ (x_{01} \sin \beta_1 - x_{02} \sin \beta_2 + z_{01} \cos \beta_1 \cos \alpha_1 - z_{02} \cos \beta_2 \cos \alpha_2) [\dot{\theta}_y - \Omega \theta_x] \\
+ (x_{01} \cos \beta_1 \sin \alpha_1 - y_{01} \cos \beta_1 \cos \alpha_1 \\
- x_{02} \cos \beta_2 \sin \alpha_2 + y_{02} \cos \beta_2 \cos \alpha_2) [\dot{\theta}_x + \Omega]
\]

where the numerical subscripts for the geometric parameters denote the particular beam.
At this stage it can be noted that, in measurement circumstances where it is not possible to assume that the motions of the axial element probed by each beam are the same, the approach to describe the difference in the velocity measured by the two beams would be to denote the vibration sets in equation (2.10) also according to the particular beam. Such a situation might be where one beam is incident on a vibrating turbine blade and the second is incident on the relatively rigid disc to which the blades are fixed.

Returning to the situation described by equation (2.22) and the issue of isolating one of the six vibration sets, it is necessary to choose geometric parameters that make the coefficients of the remaining vibration sets equal to zero while having a non-zero coefficient for the desired set. To measure one of the rotational vibration sets successfully, the coefficients of the radial and axial vibration sets must always be zero. Solving the following three simultaneous equations:

\[
\begin{align*}
\cos \beta_1 \cos \alpha_1 &= \cos \beta_2 \cos \alpha_2 \\
\cos \beta_1 \sin \alpha_1 &= \cos \beta_2 \sin \alpha_2 \\
\sin \beta_1 &= \sin \beta_2
\end{align*}
\]

(2.23)

gives solutions of \( \beta_1 = \beta_2 = \beta \) and \( \alpha_1 = \alpha_2 = \alpha \). Geometrically these solutions equate to the beams being parallel and orientated in the same direction for the differential measurement or in opposite directions for a sum velocity derivation. In this latter case, the beams would be incident on opposite sides of the shaft which would usually make for an inconvenient optical arrangement. In either case, it can be concluded that to make a measurement immune to both axial and radial vibrations, parallel beams are required. For this reason and after brief consideration of arrangements where the beams are at an angle to one another, the remainder of this chapter will concentrate on the use of parallel beams. In practice it is possible to use two separate interferometers and obtain the difference in outputs electronically [2.14] or to combine beams and find the difference velocity optically as in the arrangement proposed originally by Halliwell [2.17], the Laser Torsional Vibrometer (LTV).
Cross-beam arrangements have been used in the past for torsional vibration measurement [2.18] and they have also been used to determine the centre of rotation of rotating objects [2.19]. They differ from the arrangements described here in the method of light collection but the vibration sets to which measurements are sensitive can be investigated using the theory presented in this chapter by considering the difference in velocity measured by two single laser beams at an angle to one another, with the light scattered from each incident point collected in direct back-scatter. Figure 2.7 shows the comparable arrangement of the beams for a torsional vibration measurement with both beams lying in the xy plane. Using equation (2.22) the difference in the velocity measured by the two single beams is:

$$\Delta U_m = 2 \sin \alpha \left[ \dot{y} + (\dot{\theta}_z + \Omega)(x_0 - x) + (\dot{\theta}_z + \Omega \theta_y)z \right]$$

(2.24)

where \(\alpha\) is equal to the half-angle between the two beams and the intersection of the beams at \((x_0, y_0, z_0)\) is considered as the known point for both beams. As expected, the velocity measured includes radial, axial, pitch and yaw components in addition to the desired torsional vibration component. The recognised sensitivity to \(\dot{y}\) provided much of the motivation for the development of the LTV. In this configuration it is not necessary for the beams to cross on the target surface for equation (2.24) to hold but in the usual cross-beam configuration, which was developed to measure fluid flow, the method of light collection made this important. Light was collected in any direction but in the same direction for each beam allowing detection at any convenient location using only one photodetector. Compared to equation (2.24) the geometric coefficient was reduced by a factor of two but the velocity component sensitivity (within the square brackets) was the same. The reported use to determine the centre of rotation relied on minimising the mean value of \((\dot{\theta}_z + \Omega)x_0\) and experience has indicated that this is an acceptable approach. Error in this process is incurred, however, by the presence of products of oscillatory terms with the same frequency introducing signal components with non-zero means.

In the next section the difference velocity measured by a pair of parallel laser beams with arbitrary orientation and incident on a shaft undergoing arbitrary motion will be examined.
2.4.1 Concise theory for the difference velocity measured by a pair of parallel laser beams

The versatile approach developed in section 2.1 relied on the observation that although the point of incidence of the laser beam on the target might change position in space, it would always lie somewhere along the line of the beam itself. This simplified prediction of the vibrometer velocity sensitivity significantly compared to previous analyses and led to the derivation of equation (2.10) from which equation (2.22) followed. The same approach will be applied to this two beam arrangement, even though it is possible to manipulate equation (2.22), because the resulting derivation is more concise and comprehensive than any previous analysis.

The points $P_1$ and $P_2$, defined by the position vectors $\vec{r}_{p1}$ and $\vec{r}_{p2}$, are the points in space where the line of the laser beams intersect the surface of the shaft. The velocities of $P_1$ and $P_2$, $\vec{V}_{p1}$ and $\vec{V}_{p2}$, are the sum of the translational velocity of origin $O$, $\vec{V}_o$, and the velocity of each relative to $O$ as a result of rotation about an instantaneous rotation axis passing through $O$ at angular velocity $\vec{\omega}$:

$$\vec{V}_{p1} = \vec{V}_o + (\vec{\omega} \times \vec{r}_{p1})$$  \hspace{1cm} (2.25a)  

$$\vec{V}_{p2} = \vec{V}_o + (\vec{\omega} \times \vec{r}_{p2})$$  \hspace{1cm} (2.25b)  

As the shaft vibrates and rotates the positions of the points of incidence, not only on the target but also in space, will change continuously especially if the target has a non-circular cross-section. As figure 2.8 shows, the positions of the incident points at any arbitrary time are now written as:

$$\vec{r}_{p1} = [\vec{r}_{o1} - \vec{A}] + p_1 \vec{d}$$  \hspace{1cm} (2.26a)  

$$\vec{r}_{p2} = [\vec{r}_{o2} - \vec{A}] + p_2 \vec{d}$$  \hspace{1cm} (2.26b)  

where $\vec{r}_{o1}$ and $\vec{r}_{o2}$ are the (constant) position vectors for known points on the line of each beam, $\vec{A}$ is the (time-varying) translational vibration displacement of the origin and $p_1$ and
\( p_2 \) are (time-varying) unknown quantities that are used to account for changes in the points of incidence due to target shape and vibration. \( \hat{b} \) is the unit vector defining the direction of both beams since they are parallel.

The difference in the velocity measured by the two beams, \( \Delta U_m \), is the difference in the component of the velocity of the incident points in the direction of the incident beam. Assuming that \( \bar{\omega} \) is maintained across the probed region, by substituting equations (2.26a&b) into equations (2.25a&b), the difference velocity measured between the two beams is:

\[
\Delta U_m = \hat{b} \cdot (\vec{v}_{p1} - \vec{v}_{p2}) = \hat{b} \cdot \left[ \bar{\omega} \times (\vec{r}_{01} - \vec{r}_{02}) + \bar{\omega} \times (\vec{p}_1 - \vec{p}_2) \hat{b} \right]
\]

As for equation (2.3), inspection of equation (2.27) shows that the second scalar triple product can be re-arranged to \( (\vec{p}_1 - \vec{p}_2) \bar{\omega} \cdot (\hat{b} \times \hat{b}) \) which will always be zero and the measurement is, therefore, insensitive to the shape of the shaft. Equation (2.27) also demonstrates the immunity to translational motion offered by the parallel beam arrangement. The first scalar triple product in equation (2.27) is dependent on the difference in position of known points on the line of each beams. Since, the choice of known position is arbitrary, the difference in position can be chosen so that \( \vec{r}_{01} - \vec{r}_{02} = \vec{d} \) where \( \vec{d} \) is a vector in the plane of the beams perpendicular to \( \hat{b} \) and of magnitude equal to the perpendicular separation of the beams, \( d \). The measured velocity can therefore be written simply as:

\[
\Delta U_m = \bar{\omega} \cdot (\vec{d} \times \hat{b})
\]

In order to make equation (2.28) of more direct practical use, \( \hat{b} \) and \( \vec{d} \) need to be described in terms of measurable parameters. For \( \vec{d} \), consider an initial position in the \( yz \) plane inclined at an angle, \( \gamma \), to \( \hat{z} \), as shown in figure 2.9. After finite rotations first by \( \beta \) and then by \( \alpha \), \( \vec{d} \) becomes:
\[ \ddot{a} = d\left[ (\cos \gamma \sin \beta \cos \alpha + \sin \gamma \sin \alpha)\hat{x} + (\cos \gamma \sin \beta \sin \alpha - \sin \gamma \cos \alpha)\hat{y} + (\cos \gamma \cos \beta)\hat{z} \right] \]  

(2.29a)

This is easily obtained by repeated application of the transformation [2.20]:

\[ \vec{r}' = \vec{r} + (1 - \cos \rho)(\hat{n} \times (\hat{n} \times \vec{r})) + \sin \rho(\hat{n} \times \vec{r}) \]  

(2.29b)

which describes the new position, \( \vec{r}' \), of a vector, \( \vec{r} \), following rotation by an angle, \( \rho \), about an axis defined by the unit vector, \( \hat{n} \). The three rotations \( \gamma, \beta, \) and \( \alpha \) are finite and, therefore, this order of rotation must be maintained.

As shown in section 2.1.2, the angular velocity can also be expanded into its components:

\[ \vec{\omega} = (\hat{\theta}_x + \Omega \theta_x)\hat{x} + (\hat{\theta}_y - \Omega \theta_z)\hat{y} + (\hat{\theta}_z + \Omega)\hat{z} \]  

(2.29c)

which accounts for the change in direction of the shaft spin axis as the shaft tilts. Substituting equations (2.9 and 2.29a&c) into equation (2.28) enables the measured difference velocity to be written as:

\[ \Delta U_m = d\left[ (\sin \gamma \sin \beta \cos \alpha - \cos \gamma \sin \alpha)[\hat{\theta}_x + \Omega \theta_x] \right. \\
+ (\sin \gamma \sin \beta \sin \alpha + \cos \gamma \cos \alpha)[\hat{\theta}_y - \Omega \theta_z] + \sin \gamma \cos \beta[\hat{\theta}_z + \Omega] \]  

(2.30a)

which is consistent with equation (2.22), simplified according to the parallel beam arrangement under consideration. Equation (2.30a), however, has the advantage of also conveniently describing the difference velocity directly in terms of the perpendicular beam separation rather than the individual positions \( (x_{01}, y_{01}, z_{01}) \) and \( (x_{02}, y_{02}, z_{02}) \).

If the subtraction of the velocities is performed optically then it is, in fact, the modulus of the difference velocity that is measured. In addition, a practical optical configuration might
include frequency shifts in one or both beams and this is the case for the configuration in
the new instrument described in chapter 4. If shifts are included then the measured
difference velocity is:

$$\Delta U_m = \frac{\lambda}{2} \left[ f_{s1} - f_{s2} \right] + d \left[ (\sin \gamma \sin \beta \cos \alpha - \cos \gamma \sin \alpha) \left( \dot{\theta}_x + \Omega \theta_y \right) \right] + (\sin \gamma \sin \beta \sin \alpha + \cos \gamma \cos \alpha) \left( \dot{\theta}_y - \Omega \theta_x \right) + \sin \gamma \cos \beta \left( \dot{\theta}_z + \Omega \right) \right]$$  \hspace{1cm} (2.30b)

where $\lambda$ is the laser wavelength and $f_{s1}$ and $f_{s2}$ are the frequency shifts in beams one and
two respectively. The shifts enable direction discrimination in the measured difference
velocity. For brevity, the analysis will continue with acknowledgement of the ability to
discriminate direction but without explicit inclusion of the shifts.

### 2.4.2 Isolating rotational vibration sets

Isolation of the rotation speed set requires $\beta = 0$ and $\gamma = \frac{\pi}{2}$ so that:

$$\Delta U_m = d \left[ \dot{\theta}_z + \Omega \right]$$  \hspace{1cm} (2.31)

This shows that in order to make measurements of rotation speed, including torsional
vibration, with immunity to other angular vibrations, the beams must be orientated
perpendicular to the shaft rotation axis at any angle $\alpha$. A previous study highlighted the
sensitivity of measurements made with the Laser Torsional Vibrometer to angular
vibrations and showed that the error components due to pitch and yaw become of
measurable magnitude for values of $\beta$ in the range $-20^\circ < \beta < 20^\circ$ [2.8].

Isolation of the pitch and yaw vibration sets requires either $\gamma = \frac{\pi}{2}$ and $\beta = \pm \frac{\pi}{2}$, where the
beams are incident on the end faces of the shaft, or $\gamma = 0$, where the beams are incident on
the side of the shaft. Notably, the geometric arrangements that allow measurement of the
pitch and yaw vibration sets from the side of the shaft are independent of the angle $\beta$. 
With $\gamma = \frac{\pi}{2}$, $\beta = \pm \frac{\pi}{2}$ and $\alpha = 0$ or $\gamma = 0$ and $\alpha = \frac{\pi}{2}$ the pitch vibration set can be isolated:

$$\Delta U_{\eta} = d[\dot{\theta}_x + \Omega \theta_x] \quad (2.32a)$$

Similarly, with $\gamma = \frac{\pi}{2}$, $\beta = \pm \frac{\pi}{2}$ and $\alpha = \frac{\pi}{2}$ or $\gamma = 0$ and $\alpha = 0$ the yaw vibration set can be isolated:

$$\Delta U_{\eta} = d[\dot{\theta}_y - \Omega \theta_x] \quad (2.32b)$$

Note the sensitivity to both pitch and yaw in both of these sets. This cross-sensitivity is similar to that seen in section 2.1.3 for radial measurements and, again, can only be addressed by post-processing of the outputs. In the next chapter, section 3.2 looks at different methods of resolving individual radial and pitch and yaw motion components from simultaneous measurements of the radial vibration sets and pitch and yaw vibration sets respectively.

### 2.4.3 Measurements using multiple pairs of parallel laser beams

In many practical situations access to the shaft is restricted and it is not physically possible to align the beams at the angles set out in the previous section to isolate the required vibration set. The requirement for just one or two distinct values of $\alpha$ or $\beta$, rather than the possibility to measure at any orientation is very restrictive. For example, with this restriction, measurement of unambiguous rotation speed cannot be achieved from the end faces of the shaft which are often the only areas of access. An alternative optical arrangement capable of measuring the shaft rotation speed set in particular, but also the pitch and yaw vibration sets, from a variety of angles is, therefore, desirable.

Section 2.4 concluded that the most sensible optical arrangement to use was a pair of parallel beams so that the measurement is immune to the axial and radial vibration sets. Therefore, it seems natural to consider using more than one pair of parallel beams in the
search for a more versatile method of measuring the rotational vibration sets. In order to simplify the task, pairs of beams with equal separations will be considered.

Using the velocity measured by a single pair of beams, given by equation (2.30a), the measurements from two pairs of beams can be formulated. With the measured velocities from each pair of beams available as an electrical signal, the possibility to add, as well as subtract, the velocities exists equally conveniently. The sum, \( \Sigma(\Delta U) \), and difference, \( \Delta(\Delta U) \), in the measured velocities are:

\[
\Delta(\Delta U) = \Delta U_{m1} - \Delta U_{m2}
= d \left[ (S\gamma_1, S\beta_1, C\alpha_1 - C\gamma_1, S\alpha_1 - S\gamma_2, S\beta_2, C\alpha_2 + C\gamma_2, S\alpha_2) \left[ \dot{\theta}_x + \Omega \theta_y \right] 
+ (S\gamma_1, S\beta_1, S\alpha_1 + C\gamma_1, C\alpha_1 - S\gamma_2, S\beta_2, S\alpha_2 - C\gamma_2, C\alpha_2) \left[ \dot{\theta}_y - \Omega \theta_x \right] 
+ (S\gamma_1, C\beta_1 - S\gamma_2, C\beta_2) \left[ \dot{\theta}_z + \Omega \right] \right]
\]

\[
\Sigma(\Delta U) = \Delta U_{m1} + \Delta U_{m2}
= d \left[ (S\gamma_1, S\beta_1, C\alpha_1 - C\gamma_1, S\alpha_1 + S\gamma_2, S\beta_2, C\alpha_2 - C\gamma_2, S\alpha_2) \left[ \dot{\theta}_x + \Omega \theta_y \right] 
+ (S\gamma_1, S\beta_1, S\alpha_1 + C\gamma_1, C\alpha_1 + S\gamma_2, S\beta_2, S\alpha_2 + C\gamma_2, C\alpha_2) \left[ \dot{\theta}_y - \Omega \theta_x \right] 
+ (S\gamma_1, C\beta_1 + S\gamma_2, C\beta_2) \left[ \dot{\theta}_z + \Omega \right] \right]
\]

where the subscripts denote the particular pair of beams and \( S\alpha_1 = \sin \alpha_1, \ C\alpha_1 = \cos \alpha_1 \)

etc.

While there are numerous geometric arrangements of the pairs of beams that would theoretically enable the desired vibration set to be measured, in practice the number of useful geometric arrangements is limited to those that can be reliably set-up with acceptable accuracy in angle measurement. Experience suggests the most convenient way to arrange the pairs of beams is symmetrically, with a plane bisecting the angle between the planes of each laser beam pair, either perpendicular to, or parallel to, the shaft rotation axis and with equal values of \( \beta \). With such arrangements, measurements from both the side and end faces of the shaft are still possible. The geometric values required to measure
the vibration sets using two pairs of parallel beams are summarised in Table 2.1 and depicted in figures 2.10a-c.

<table>
<thead>
<tr>
<th>Desired Measurement</th>
<th>Signal</th>
<th>$\alpha_1$</th>
<th>$\alpha_2$</th>
<th>$\beta_1$</th>
<th>$\beta_2$</th>
<th>$\gamma_1=\gamma_2$</th>
<th>Side or end of shaft?</th>
<th>Figure</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\dot{\theta}_x + \Omega \dot{\theta}_y$</td>
<td>$\Delta(\Delta U)$</td>
<td>0</td>
<td>0</td>
<td>$\beta$</td>
<td>$-\beta$</td>
<td>$\pi/2$</td>
<td>Side</td>
<td>2.11a</td>
</tr>
<tr>
<td></td>
<td>$\Sigma(\Delta U)$</td>
<td>0</td>
<td>0</td>
<td>$\beta$</td>
<td>$\pi - \beta$</td>
<td>$\pi/2$</td>
<td>End</td>
<td>2.11a</td>
</tr>
<tr>
<td></td>
<td>$\Delta(\Delta U)$</td>
<td>$\alpha$</td>
<td>$-\alpha$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>Side</td>
<td>2.11c</td>
</tr>
<tr>
<td></td>
<td>$\Sigma(\Delta U)$</td>
<td>$\alpha$</td>
<td>$\pi - \alpha$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>Side</td>
<td>2.11c</td>
</tr>
<tr>
<td>$\dot{\theta}_y - \Omega \dot{\theta}_x$</td>
<td>$\Delta(\Delta U)$</td>
<td>$\pi/2$</td>
<td>$\pi/2$</td>
<td>$\beta$</td>
<td>$-\beta$</td>
<td>$\pi/2$</td>
<td>Side</td>
<td>2.11b</td>
</tr>
<tr>
<td></td>
<td>$\Sigma(\Delta U)$</td>
<td>$\pi/2$</td>
<td>$\pi/2$</td>
<td>$\beta$</td>
<td>$\pi - \beta$</td>
<td>$\pi/2$</td>
<td>End</td>
<td>2.11b</td>
</tr>
<tr>
<td></td>
<td>$\Sigma(\Delta U)$</td>
<td>$\alpha$</td>
<td>$-\alpha$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>Side</td>
<td>2.11c</td>
</tr>
<tr>
<td></td>
<td>$\Delta(\Delta U)$</td>
<td>$\alpha$</td>
<td>$\pi - \alpha$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>Side</td>
<td>2.11c</td>
</tr>
<tr>
<td>$\dot{\theta}_z + \Omega$</td>
<td>$\Sigma(\Delta U)$</td>
<td>$\alpha$</td>
<td>$\alpha$</td>
<td>$\beta$</td>
<td>$-\beta$</td>
<td>$\pi/2$</td>
<td>Side</td>
<td>2.11a &amp;b</td>
</tr>
<tr>
<td></td>
<td>$\Delta(\Delta U)$</td>
<td>$\alpha$</td>
<td>$\alpha$</td>
<td>$\beta$</td>
<td>$\pi - \beta$</td>
<td>$\pi/2$</td>
<td>End</td>
<td>2.11a &amp;b</td>
</tr>
</tbody>
</table>

Table 2.1: Summary of geometric arrangements for twin parallel beam measurements

Table 2.1 shows three basic beam configurations exist. In the first, $\left(\gamma_1 = \gamma_2 = \frac{\pi}{2}, \beta_1 = -\beta_2 = \beta\right)$, the beams are incident on the side of the shaft. In the second, $\left(\gamma_1 = \gamma_2 = \frac{\pi}{2}, \beta_1 = \pi - \beta_2 = \beta\right)$, the beams are incident on the end face of the shaft. As shown in figures 2.10a-c, the desired angular vibration set is selected by choosing the appropriate value for $\alpha$, while measurement of the rotation speed set can be
achieved with any value of $\alpha_1 = \alpha_2 = \alpha$. This means that, for the same physical arrangement, measurements of the rotation speed set and one of the angular vibration sets can be obtained simultaneously simply by choosing to take both the sum and the difference in the velocities measured by each individual pair of beams. In the third configuration, $(\gamma_1 = \gamma_2 = 0, \beta_1 = \beta_2 = 0)$, the beams are incident on the side of the shaft at different axial locations. This arrangement enables simultaneous measurement of the pitch and yaw vibration sets by taking both the sum and the difference in the velocities measured by each individual pair of beams.

The theory presented in this chapter has enabled all of the possible, useful geometric arrangements to be identified. These include the single arrangement used in a previous study of torsional and bending vibration measurement [2.8] in which two laser torsional vibrometers were used to assess bending vibration on a diesel engine crankshaft. In this previous study, simultaneous measurement of the yaw and rotation speed sets gave an indication of the bending motion of the crankshaft but resolution of the individual pitch and yaw motion was not possible.

2.5 Resolution of individual motion components

Section 2.1 showed conclusively that direct measurement of radial, axial and bending vibration is not possible on rotating components because no matter how the beam or beams are arranged geometrically, the measurement will always be sensitive to other motion components. Making two simultaneous vibration measurements which individually show cross-sensitivity, for example, to motion in the $y$ direction in an intended $x$ direction measurement and vice-versa along with a measurement of rotation speed enables the measurements to be described by a pair of linked differential equations. The next chapter presents a method of approximating the solutions of these linked differential equations to enable estimates of individual, non-synchronous, radial and bending vibration components to be resolved.
3 Resolution of radial and angular vibration components

The promise of non-contact vibration measurements directly from a rotating component of any shape makes LDV, in some ways, the "ideal" technique for vibration measurements on rotating machines. However, the detailed analysis of radial vibration measurements on rotating components using LDV carried out in chapter 2 revealed that these measurements are ambiguous, sensitive to both rotation speed (including speed fluctuations and torsional vibrations) and motion perpendicular to the intended measurement. These cross-sensitivities have been shown to be significant enough to mask the intended vibration measurement [3.1].

Since the first description of this cross-sensitivity, there has been discussion about whether a particular arrangement of laser beams or a particular manipulation of the arrangement, for example by scanning the laser beams, might enable automatic resolution of individual motion components. Section 2.1 considered the general case of the velocity sensed by a single laser beam incident in an arbitrary direction on a rotating target. Sensitivity to motion in all six degrees of freedom was considered for the first time and this revealed an additional cross-sensitivity in radial vibration measurements to axial and angular vibration. Also, a similar cross-sensitivity to that found in radial measurements was found in measurements of angular vibration where an intended measurement of pitch vibration is seen to be cross-sensitive to rotation speed and yaw motion and vice-versa. The section concluded that for a rotating component undergoing an arbitrary vibration, direct measurement of pure radial, axial or angular vibration is not possible because the measurement will always be sensitive to other motion components. Importantly, the theory showed isolation of the intended motion not to be possible by any geometric arrangement or manipulation of laser beams. A previous mathematical solution to cross-sensitivity in radial vibration measurements [3.2] used two simultaneous vibration measurements of radial vibration, one measurement in the x direction cross-sensitive to motion in the y direction and one in the y direction cross-sensitive to motion in the x direction, to resolve the genuine radial motion components. This idea of two simultaneous cross-sensitive measurements will be used in the new method described in this chapter.
3.1 Comparison of the theory of radial and angular vibration measurements

Section 2.1.2 showed that measurements of $x$ and $y$ radial vibration sets, $U_x$ and $U_y$, respectively, with the initial point of incidence defining the measurement plane, are given by:

$$U_x = x + \left( \dot{\theta}_z + \Omega \right) (y - y_0) - (\dot{\theta}_y - \Omega \theta_x) z$$

(3.1a)

$$U_y = y - \left( \dot{\theta}_z + \Omega \right) (x - x_0) + (\dot{\theta}_x + \Omega \theta_y) z$$

(3.1b)

where $(\dot{x}, \dot{y}, \dot{z})$ and $(x, y, z)$ are the translational vibration velocities and displacements of the origin $O$ in the $x$, $y$ and $z$ directions, $\Omega$ is the rotation speed of the axial shaft element (combining shaft rotation speed and any torsional vibration or speed fluctuation of the axial element) and $(\dot{\theta}_x, \dot{\theta}_y, \dot{\theta}_z)$ are the angular vibration velocities of the shaft around the $x$, $y$ and $z$ axes, referred to as pitch, yaw and roll respectively. $x_0$ and $y_0$ are the distances the beams are offset from the undeflected shaft rotation axis.

In order to reduce the number of unknown motions the contribution of the third set of terms in equations (3.1a&b), $(\dot{\theta}_y - \Omega \theta_x) z$ or $(\dot{\theta}_x + \Omega \theta_y) z$, will be neglected in the following analysis. These terms are the products of vibration parameters and are, therefore, usually an order of magnitude smaller than the first two terms. In practice, the shaft rotation speed and roll motion may be indistinguishable, for example, some transducer configurations attempting to measure the rotation speed of an internal combustion engine would also measure a contribution from the engine rocking in its mounts. In the remainder of this chapter the rotation speed and roll motion will be considered as one. Acknowledging these considerations, the measured velocity is equated to the motion parameters as follows:

$$U_x = \dot{x} + \Omega (y - y_0)$$

(3.2a)
\[ U_y = \dot{y} - \Omega(x - x_0) \] (3.2b)

Section 2.4 investigated the measurement of angular vibration. The measurements of the pitch and yaw vibration sets do not require the assumption made to reduce equations (3.1a&b) to (3.2a&b). From equations (2.32a&b), measurements of the pitch, \( \dot{\theta}_x \), and yaw, \( \dot{\theta}_y \), vibration sets yield:

\[ \dot{\theta}_x = \dot{\theta}_x + \Omega \theta_y \] (3.3a)

\[ \dot{\theta}_y = \dot{\theta}_y - \Omega \theta_x \] (3.3b)

Equations (3.3a&b) can be seen to be of the same form as equations (3.2a&b) and, therefore, the method of resolving cross-sensitivity in radial measurements that follows is equally applicable to measurements of pitch and yaw vibration. For brevity the discussion will proceed with the resolution of the radial vibrations.

### 3.2 Solution of the governing differential equations

All of the differential equations discussed in section 3.1 are classed as linear non-autonomous differential equations as their coefficients are time dependent. The theory of non-autonomous differential equations is complex and often a complete solution cannot be found. In most practical measurement situations the fluctuations in rotation speed or torsional vibrations are small compared to the mean rotation speed. A study of torsional vibrations in an internal combustion engine, a class of rotating machinery where torsional vibrations are usually significant, found a maximum ratio of torsional vibration velocity:mean rotation speed of 0.049 [3.3]. If the rotation speed can be considered constant, the problem of resolving the true vibration velocities can be greatly simplified and equations (3.2a&b) are reduced to a system of differential equations with constant coefficients which can be solved more easily:

\[ U_x = \dot{x} + \Omega(y - y_0) \] (3.4a)
where $\bar{\Omega}$ is the mean rotation speed, a constant.

In both equations (3.4a&b), only one term in the measured velocities is a constant, $\Omega x_0$ and $\Omega y_0$ respectively. By carefully aligning each vibrometer so that its mean output, equal to either $\Omega x_0$ or $\Omega y_0$ when $\Omega$ is constant, is zero, the perpendicular offsets $x_0$ and $y_0$ can be made approximately equal to zero. This would further simplify the equations to:

$$U_x = \dot{x} + \bar{\Omega} y$$  \hspace{1cm} (3.5a)

$$U_y = \dot{y} - \bar{\Omega} x$$  \hspace{1cm} (3.5b)

It is good practice to align the vibrometers in this way, as it maximises the vibrometers’ working range, but to make the effects of the offsets negligible they need to be reduced to values an order of magnitude smaller than displacement amplitudes encountered in practice. Experience of aligning vibrometers has shown this not to be achievable which means that the simplification made above is usually unrealistic and the offsets will be retained throughout the analysis.

### 3.2.1 Analytical solution

The first step in solving a system of linear differential equations is to transform the system into new equations containing only one dependent variable. Several systematic methods exist to transform systems of any size [3.4] but for a system with only two dependent variables the transformation can simply be achieved by elimination.

Differentiating and re-arranging equation (3.4a) enables $\dot{y}$ to be written as:

$$\dot{y} = \frac{\dot{U}_x - \dot{x}}{\bar{\Omega}}$$  \hspace{1cm} (3.6)
which can then be substituted into (3.4b) to give, in terms of $x$ and its differentials, the differential equation:

$$\ddot{x} + \Omega^2 x = \dot{U}_x - \Omega U_y + \Omega^2 x_0$$  \hspace{1cm} (3.7a)

Similarly, by first differentiating equation (3.4b):

$$\ddot{y} + \Omega^2 y = \dot{U}_y + \Omega U_x + \Omega^2 y_0$$ \hspace{1cm} (3.7b)

Equations (3.7a&b) are second order linear non-homogeneous differential equations. The situation they represent is analogous to the physical situation of a forced undamped vibration. The analytical solution of a non-homogeneous linear differential equation is the sum of the solution of the homogeneous equation and a particular solution to the non-homogenous equation. Using the method of variation of parameters [3.4] the solution to equation (3.7a) can be found to be:

$$x = x(0) \cos \Omega t + \frac{\dot{x}(0)}{\Omega} \sin \Omega t$$

$$+ \left( \cos \Omega t \int_0^t G_x \sin \Omega t \ dt - \sin \Omega t \int_0^t G_x \cos \Omega t \ dt \right) \frac{1}{\Omega}$$ \hspace{1cm} (3.8a)

where $G_x = \dot{U}_x - \Omega U_y + \Omega^2 x_0$.

Similarly

$$y = y(0) \cos \Omega t + \frac{\dot{y}(0)}{\Omega} \sin \Omega t$$

$$+ \left( \cos \Omega t \int_0^t G_x \sin \Omega t \ dt - \sin \Omega t \int_0^t G_x \cos \Omega t \ dt \right) \frac{1}{\Omega}$$ \hspace{1cm} (3.8b)
Each of these solutions introduces two new unknowns, the displacement and velocity at time $t = 0$. However, the only situation in which these quantities can be directly measured is when the rotor is not rotating. This means that, because the theory presented requires constant rotation speed, only the trivial case of a rotor rotating at zero rotation speed can satisfy the necessary conditions. Therefore, equations (3.8a&b), while perfectly valid, are of limited use because in practice the initial conditions cannot be found. An alternative type of solution which is not dependent on initial conditions is required.

### 3.2.2 Frequency-by-frequency solution

In practice the only motion components of interest are the alternating components. Representing the vibratory motion as a Fourier series of $M$ components, where the $m^{th}$ component $x_m$ (or $y_m$) has amplitude $A_m$, angular frequency $\omega_m$ and phase $\phi_m$, a relationship between vibration acceleration and vibration displacement at any one frequency $\omega_m$ can be found:

$$x_m = -A_m \omega_m^2 \sin(\omega_m t + \phi_m) = -\omega_m^2 x_m$$

(3.9)

Substituting equation (3.9) into equation (3.7a) and evaluating for the $m^{th}$ component i.e. at $\omega = \omega_m$ enables the magnitude of the vibration acceleration at frequency $\omega = \omega_m$ to be written in terms of the rotation speed and vibrometer outputs:

$$\ddot{x}_m = W(\omega_m)(\dot{U}_x - \Omega \dot{U}_y)\big|_{\omega = \omega_m}$$

(3.10a)

where

$$W(\omega_m) = \frac{\omega_m^2}{\omega_m^2 - \Omega^2}$$

(3.10b)

Similarly, considering a vibration in the $y$ direction:
\[ \ddot{y}_m = W(\omega_m)\left(\dot{U}_y + \Omega U_x\right) \bigg|_{\omega = \omega_m} \] (3.10c)

Therefore, the genuine vibration acceleration components can be resolved by evaluating the second bracketed terms of equations (3.10a&c) and then weighting by the frequency dependent function \( W(\omega_m) \). Alternatively, the acceleration time signal can be found (with some information missing, see Section 3.2.2) by filtering the time signal, given by the second bracketed term in either equation (3.10a) or (3.10c), by a filter with a frequency response equal to \( W(\omega_m) \).

Evaluating the second bracketed term in equation (3.10a) or (3.10c) requires the measured velocities to be differentiated. In general, numerical integration is preferred to numerical differentiation [3.4] because it is less sensitive to inaccuracies in the source data. Likewise, electronic differentiation circuits encounter more problems with stability and high frequency noise [3.5]. A solution based on integration of the vibrometer outputs would be preferred and as the solution that will be presented in this chapter is calculated on a frequency-by-frequency basis it can be differentiated or integrated as required in the frequency domain simply by complex multiplication or division by the frequency of vibration.

As an alternative to the more usual method of eliminating one of the dependent variables by differentiating one of the equations, a similar result can be achieved by integration. Integrating equation (3.4b) from an initial time \( t = 0 \) to the present time \( t \) gives:

\[ \int_0^t U_y(t) dt = \int_0^t \dot{y}(t) dt - \Omega \int_0^t (x(t) - x_0) dt \] .

Equation (3.11a) enables the vibration displacement at time \( t \), given by:
\[ y(t) = y(0) + \int_0^t \dot{y}(t) \, dt \]  
(3.11b)

to be written as:

\[ y(t) = y(0) + \int_0^t U_y(t) \, dt + \overline{\Omega} \int_0^t (x(t) - x_0) \, dt \]  
(3.11c)

This can then be substituted into equation (3.4a) to give the integro-differential equation:

\[ \ddot{x}(t) + \overline{\Omega}^2 \int_0^t x(t) \, dt = U_x(t) + \overline{\Omega} y_0 - \overline{\Omega} \int_0^t (U_y(t) - \overline{\Omega} x_0) \, dt - \overline{\Omega} y(0) \]  
(3.12a)

Similarly, by first integrating equation (3.4a):

\[ \dot{y}(t) + \overline{\Omega}^2 \int_0^t y(t) \, dt = U_y(t) - \overline{\Omega} x_0 + \overline{\Omega} \int_0^t (U_x(t) + \overline{\Omega} y_0) \, dt + \overline{\Omega} x(0) \]  
(3.12b)

Considering the pair of integro-differential equations (3.12a&b) and using the solution method presented above, again representing the vibratory motion as a Fourier series of \( M \) components, a relationship between vibration velocity and the integral of vibration displacement at any one frequency can be found:

\[ \int_0^t x_m \, dt \bigg|_{\omega = \omega_m} = \left[ -\frac{A_m}{\omega_m} \cos(\omega_m t + \phi_m) \right]' = \left[ -\frac{\dot{x}_m}{\omega_m^2} \right]' = \frac{\dot{x}_m(t) - \dot{x}_m(0)}{\omega_m^2} \]  
(3.13)

Substituting equation (3.13) into equation (3.12a) and evaluating for the \( m \text{th} \) component i.e. at \( \omega = \omega_m \) enables the magnitude of the vibration velocity at frequency \( \omega = \omega_m \) to be written in terms of the rotation speed and vibrometer outputs:
\[ \dot{x}_m = W(\omega_m) \left\{ U_x(t) + \Omega y_0 \right\} - \Omega \int_0^t \left\{ U_y(t) - \Omega x_0 \right\} dt \right|_{\omega = \omega_m} \] (3.14a)

Similarly, considering a vibration in the y direction:

\[ \dot{y}_m = W(\omega_m) \left\{ U_y(t) - \Omega x_0 \right\} + \Omega \int_0^t \left\{ U_x(t) + \Omega y_0 \right\} dt \right|_{\omega = \omega_m} \] (3.14b)

The genuine vibration motions can, therefore, be found on a frequency-by-frequency basis by weighting the forcing function of the differential equation, given by the second, bracketed term in equations (3.14a&b), by \( W(\omega_m) \).

On inspection it seems that it is impossible to calculate the quantities within the curved brackets in equations (3.14a&b) as \( x_0 \) and \( y_0 \) are unknown. The terms \( \Omega x_0 \) and \( \Omega y_0 \) are the only constant terms in the vibrometer outputs, however, and the quantities required are the alternating components of the vibrometer outputs. This means that they can be obtained by filtering the vibrometer outputs to leave only the alternating components, i.e. ac coupling the outputs. Although it is only the alternating components of motion that are of interest, in practice, the constants in the vibrometer outputs cannot be neglected and need to be filtered out to prevent the integration of the vibrometer outputs increasing linearly.

The initial condition dependent terms, \( \Omega x(0) \) and \( \Omega y(0) \), present within equations (3.12a&b), and the integration constant \( \left( \frac{\dot{x}_m(0)}{\omega_m^2} \right) \) in equation (3.13) are not included in equations (3.14a&b). These constants are associated with the values of the integrated quantities at the initial time \( t = 0 \), and take into account the fact that at the initial time, when measurement begins, the vibration can be at any point throughout its cycle. Selecting the amplitude of the component of the processed quantity at a non-zero frequency removes their influence.
The equivalent equations for angular vibration measurement are:

\[
\dot{\theta}_{x_m} = W(\omega_m) \left( \dot{\theta}_x(t) - \Gamma_0 \int_0^t \dot{\theta}_y(t) \, dt \right)_{\omega=\omega_m} \tag{3.15a}
\]

\[
\dot{\theta}_{y_m} = W(\omega_m) \left( \dot{\theta}_y(t) + \Gamma_0 \int_0^t \dot{\theta}_x(t) \, dt \right)_{\omega=\omega_m} \tag{3.15b}
\]

### 3.2.3 Synchronous vibration measurements

Section 3.2.1 showed that the solution of the homogeneous differential equation for a rotor rotating at a constant speed represents a synchronous vibration. Conversely, if a synchronous vibration is measured the situation is described by the homogeneous differential equation. In the frequency by frequency solutions presented above, the forcing function of the differential equation has been used to calculate vibration velocity but, by definition, the forcing function of a homogeneous equation is zero. It is, of course, a simple matter to substitute appropriate functions for the \(x\) and \(y\) vibrations and show that the forcing function is zero. This means that the solutions are unsuitable for synchronous vibration measurements as there is insufficient information, in the form of initial conditions, to base the solution on.

The consequences of this lack of information at synchronous frequency can be seen in both the analytical and frequency by frequency solutions. In equation (3.8a&b), the amplitude and phase of the synchronous vibration are solely dependent on the initial conditions which can only be found for the trivial case of a non-rotating rotor. In equations (3.10a&c), (3.14a&b) and (3.15a&b) the frequency dependent function \(W(\omega_m)\) becomes infinite, while the forcing function is zero because equations (3.4a&b) are no longer independent. This means that the synchronous content of the solution is missing, whether that be in the form of a spectrum or a time signal.
Measurements of synchronous rotor vibration form an important part of rotating machinery diagnostics as many defects are signified by a change in the synchronous vibration component. The inability to resolve the synchronous vibration component is, therefore, a serious limitation. It is, however, equally important to know the limitations if ambiguous measurements are to be avoided because, as was demonstrated in section 1.4.2, it is still possible for measurements of the radial vibration sets to be equal to zero when the actual magnitude of rotor vibration is large.

3.3 Electronic implementation of frequency by frequency solution

The frequency by frequency solution was implemented using an analogue electronic circuit to calculate the necessary time signals, according to equations (3.14a&b), which were then input into a spectrum analyser and the spectral amplitudes weighted accordingly. The signals required, $\dot{X}$ and $\dot{Y}$, are:

$$\dot{X} = \bar{U}_x - \Omega \int \bar{U}_y dt$$  \hspace{1cm} (3.16a)

$$\dot{Y} = \bar{U}_y + \Omega \int \bar{U}_x dt$$  \hspace{1cm} (3.16b)

where $\bar{U}_x$ and $\bar{U}_y$ are the ac coupled vibrometer outputs. $\dot{X}$ and $\dot{Y}$ are computed from the outputs of the two laser vibrometers, and a device capable of measuring rotation speed. In this investigation, a laser torsional vibrometer was used to provide real time rotation speed measurement but any suitable device could be used. Each instrument outputs a voltage proportional to the quantity it measures:

$$V_x = K_x \bar{U}_x$$  \hspace{1cm} (3.17a)

$$V_y = K_y \bar{U}_y$$  \hspace{1cm} (3.17b)

$$V_\Omega = K_\Omega \bar{\Omega}$$  \hspace{1cm} (3.17c)
where $V_x$ and $V_y$ are the output voltages from the $x$ and $y$ direction vibrometers, $V_\Omega$ is the output voltage from the torsional vibrometer, $K_x$ and $K_y$ are the known calibration constants of the vibrometers (V/m/s), and $K_\Omega$ is the calibration constant of the torsional vibrometer (V/rad/s).

A block diagram representation of the circuit used is shown in figure 3.1. The outputs of the circuit, $\dot{X}_{out}$ and $\dot{Y}_{out}$, are:

\[ \dot{X}_{out} = K_{Gx} \left( C_2 V_x - V_\Omega K_y \int V_y \, dt \right) \]  
\[ \dot{Y}_{out} = K_{Gy} \left( C_1 V_y + V_\Omega K_x \int V_x \, dt \right) \]

where $K_{Gx}$ and $K_{Gy}$ represent the overall gains of the circuit, $K_{ix}$ and $K_{iy}$ are the gains of the integrators.

In order to perform the correct calculations and give circuit outputs proportional to $\dot{X}$ and $\dot{Y}$, the gains on either side of the summing blocks must be equated, achieved using the calibration voltages $C_1$ and $C_2$. Therefore, inserting the vibrometer calibration constants the circuit outputs become:

\[ \dot{X}_{out} = C_2 K_x K_{Gx} \left( \bar{U}_x - \bar{\Omega} \int \bar{U}_y \, dt \right) \]  
\[ \dot{Y}_{out} = C_1 K_y K_{Gy} \left( \bar{U}_y + \bar{\Omega} \int \bar{U}_x \, dt \right) \]

where $C_1 = \frac{K_\Omega K_{ix} K_y}{K_y}$ and $C_2 = \frac{K_\Omega K_{iy} K_x}{K_x}$
The circuit needs to be calibrated experimentally to find the unknown gains $K_{x}, K_{y}, K_{Gx}$ and $K_{Gy}$. This can be done by applying known test voltages to the circuit and comparing the circuit outputs with the theoretical circuit outputs. The theoretical circuit outputs are detailed in Appendix 1.

3.4 Experimental validation

Using the electronic circuit, vibration measurements on a rotor were made. The test rig, shown in figure 3.2, consists of a rotating target and two electrodynamic shakers which allow vibration simultaneously in the $x$ and $y$ directions. The shakers were driven using a power amplifier supplied by a variable phase signal generator which controlled the frequency, phase difference and amplitudes of the $x$ and $y$ vibrations.

Two laser vibrometers were positioned orthogonally and incident on the target rotor with the laser torsional vibrometer positioned at a convenient location. For comparative purposes, two B&K Type 4374 sub-miniature accelerometers were fixed to the motor housing, also shown in figure 3.2, both of which had been cross-calibrated against the two laser vibrometers prior to testing. Measurements of vibration velocities were obtained using the vibrometer outputs, resolved using the electronic circuit described above, for a variety of vibration conditions along with accelerometer vibration measurements (integrated once for velocity).

Vibration frequencies were chosen above and below rotation speed with two values close to synchronous frequency to assess the technique's accuracy close to this condition. Multiples of rotation speed, where the speckle noise is concentrated, were avoided and values of vibration velocity amplitude were chosen to represent vibrations within the mid-range of severity for most classes of rotating machinery [3.6]. The nominal target vibration values used are shown in table 3.1 below:
Vibration Parameter | Values Used
--- | ---
Frequency $\omega_0 / 2\pi$ (Hz) | 10, 25, 55, 75, 180
x-direction velocity amplitude $\dot{x}$ (mm/s) | 7.5, 15
y-direction velocity amplitude $\dot{y}$ (mm/s) | 7.5, 15
Phase difference $\phi$ (°) | 0, ±45, ±90
Rotation frequency $\Omega / 2\pi$ (Hz) | 20, 40

Table 3.1: Nominal target vibration values

Figures 3.3a&b show the ratio of the resolved laser vibrometer data to the accelerometer data for the x and y directions respectively. At each value of $n$, where $n = \frac{\omega_m}{\Omega}$, the use of five values of $\phi$ and two values of $\Omega$ gives ten values for the ratio. This is presented as a plot of the mean and standard deviation at each value of $n$, the error bars corresponding to ±1 standard deviation.

A considerable, but not unacceptable, variation in the ratio can be seen in both x and y directions. Two different trends can be seen in the data. Firstly, the mean ratio values, which would be equal to one if the resolved laser vibrometer measurements matched the accelerometer measurements, are not randomly located and seem to follow a pattern. Secondly, a general trend of increasing standard deviation for ratios close to $n = 1$ can be seen. This phenomenon is investigated in the next chapter.

With the test rig and vibration amplitudes used, it is reasonable to assume that the vibrations produced by the electrodynamic shakers are well in excess of the vibration excited, for example, by imbalance or misalignment of the rotor and, therefore, the shaker driven vibrations are considered to be the only vibrations of the rotor.

In order to assess any genuine differences between the vibration response of the rotor (as measured by the laser vibrometers) and the motor housing (as measured by the accelerometers) to the shaker induced vibrations, a selection of the vibration conditions used during testing were repeated, this time with the target not rotating. Under these circumstances the vibrometers straightforwardly measure the genuine vibration of the
target. These measurements could then be compared directly with the accelerometer outputs (integrated once for velocity).

Figures 3.4a&b show the comparison between the rotating and non-rotating situations. The plots show the mean vibrometer:accelerometer ratios with error bars of ± 1 standard deviation from the rotating target tests for comparison with an error band set at the mean ± 1 standard deviation from the non-rotating tests. It can be seen from the plots that a genuine difference between the response of the rotor and the motor housing does exist. The vibrometer:accelerometer ratios for the rotating and non-rotating conditions show good agreement (for all but the 180Hz data in the y direction). It can therefore be concluded that the ratios plotted in figures 3.3a&b are not unity because of genuine differences in the vibration amplitudes between the target rotor and the motor housing. This emphasises the importance of being able to make measurements directly from the rotating component.

3.5 Diesel engine crankshaft radial vibration measurement

Measurements of radial vibration were made on a four cylinder two litre diesel engine running at a range of speeds, obtained by varying engine load while maintaining wide open throttle. Figure 3.5 shows the experimental arrangement with two laser vibrometers positioned orthogonally about the front of the engine and incident on the nut of the crankshaft pulley to provide simultaneous measurements of the x and y radial vibration sets. The beams were aligned through the centre of the crankshaft rotation axis by positioning the vibrometers so their mean outputs were minimised with the engine idling. A LTV was used to measure crankshaft rotation speed and was aligned as close to perpendicular (≈ 85°) to the rotation axis as access would allow to eliminate the sensitivity of the measurement to bending vibration. At this angle the sensitivity to bending vibration is negligible [3.3].

The vibration characteristics expected from a 4 cylinder 4 stroke engine are ½ order peaks, due to the 2 revolutions of the crankshaft required to complete a 4 stroke cycle, with larger 2\textsuperscript{nd}, 4\textsuperscript{th}, 6\textsuperscript{th}... orders due to the two combustion events that occur within every revolution.
of the crankshaft. Figure 3.6 shows a section of the x radial vibration set measured at 3000rpm. The repeating pattern of four combustion events within a cycle with a fundamental period equating to twice the rotation period can be seen. Figures 3.7a&b and 3.8a&b show the spectra of radial vibration sets and corresponding resolved radial vibration at 3000rpm in x and y directions respectively. The spectral peaks at half orders of rotation frequency can clearly be seen along with evidence of some cross-sensitivity. The effects of the post-processing can also be seen. The synchronous component of the spectra has been blanked but a broad peak centred at the synchronous frequency can be seen because the weighting factor, $W(\omega_m)$, which is very large close to synchronous, exaggerates the amplitude of those spectral components close to synchronous.

For vibration frequencies well above rotation frequency the terms that are the intended measurements, $\dot{x}$ and $\dot{y}$, dominate the cross-sensitivity terms, $\Omega y$ and $\Omega x$, in the measured vibration sets, assuming that vibration displacements in the x and y directions are of similar magnitude. Therefore, those regions of the spectra of the vibration sets and resolved vibration velocity are very similar. For frequencies below the rotation frequency the cross-sensitivity terms dominate the intended measurements and this can be clearly seen in the peaks at $\frac{1}{2}x$ rotation frequency. The resolved vibration spectra show a small magnitude vibration in the x direction and a larger vibration in the y direction whereas the x direction vibration set shows a large vibration component and the y direction vibration set a much smaller component.

Figures 3.7b and 3.8b also indicate an increasing vibration response in the region 300-500Hz. Figures 3.9a&b show waterfall plots of the resolved x and y vibration velocity respectively across the speed range of the engine. It is proposed that the broad vibration peak centred around 400Hz can be seen in both figures corresponds to the first natural frequency of the crankshaft in bending. Modal analysis of the stationary engine found this to be in the range 456-476Hz depending on the angular position of the crankshaft [3.3]. Bending vibration will be discussed in more detail in chapter 4 and is of particular interest in automotive NVH studies where a lack of suitable measurement techniques has hampered studies to date. The purpose of this work is to demonstrate practical application of post-processing technique rather than to identify the sources of individual vibration
components. Vibration responses centred around 225Hz in the x direction and 130Hz in the y direction had not, however, been anticipated and these peaks highlight how the technique might be applied for diagnostic work, among other applications.

The complete resolved time signal can not be shown because its synchronous component is missing but band-pass filtering the signal well above or below the synchronous frequency is valid. Figure 3.10 shows the resolved x vibration with the engine running at 3000rpm, band-pass filtered around the natural frequency of the crankshaft in bending (300-500Hz). The vibration due to the combustion events can clearly be seen within a cycle with a fundamental period equating to twice the rotation period. Previous studies of vibration on stationary crankshafts using modal analysis [3.7,3.8,3.9] have shown the 1st mode shape of the crankshaft in bending to have an anti-node at the crankshaft pulley. As a result the impulse created by the combustion of cylinder number 1, because it is closest to the anti-node, was found to be the most significant source of bending vibration. This can be seen in the repeating vibration signal which has one dominant impulse amongst the other combustion events that can be distinguished.

Successful analytical and computer models of vibration behaviour rely on accurate estimates of modal damping but this is particularly difficult to measure. The resolved vibration measurements enable the modal damping to be assessed and a modal damping factor in region of 2-4% of critical was estimated, using the logarithmic decrement, from the resolved data.

In the next chapter the measurement of angular vibrations is considered and the design of a new instrument capable of making two simultaneous measurements of angular vibration outlined. This instrument is then used to investigate crankshaft bending using measurements of the pitch and yaw vibration.
4 Angular vibration measurements on rotors

4.1 Introduction

Many problems in rotating machinery result in angular vibration of the rotor including unbalance, misalignment, bearing faults, rubs and the effects of gears, blades and vanes [4.1]. One area of particular interest is the bending vibrations in the crankshafts of internal combustion engines used in passenger vehicles where bending vibrations are responsible for engine "rumble" [4.2], a harsh and unpleasant noise heard during acceleration. Investigations have been hampered by the difficulty in making direct measurements of bending vibration and the studies to date have derived measurements of bending vibration using a variety of transducers and modifications to the engine and crankshaft.

A study comparing the accuracy of an analytical model for prediction of torsional and bending stress in engine crankshafts made measurements of bending vibration using strain gauges attached to the shaft [4.3]. A modified crankshaft enabled the strain gauge leads to pass from the crank web to a slip ring at the front of the crankshaft pulley. The study concluded that cylinder pressure, rather than the inertia forces, was the major contributor to the bending motion.

One approach to reducing engine rumble is to reduce the crankshaft bending vibration using a damper. The performance of these vibration dampers has been investigated using a variety of methods. One method used was to derive the crankshaft bending vibration from measurements of radial vibration made along the length of the crankshaft. The influence of ignition timing, crankshaft stiffness and counterbalance weights on crankshaft bending and engine rumble was analysed using measurements of the crankshaft journal locus at each of the main bearings using proximity probes fixed into the bearing caps [4.4]. Among other features, engine rumble was found to increase as ignition timing was advanced and that the radiated noise was most noticeable after the combustion in cylinder number 1 and number
4. The effectiveness of increasing crankshaft stiffness and the fitting of a crankshaft bending vibration damper on reducing engine rumble was also demonstrated. A similar study identified the natural frequency of the crankshaft in bending using two proximity probe measurements of the axial vibration on the crankshaft pulley, one above the centreline and one below it [4.54.5]. The bending motion was confirmed by the 180° phase difference in the two measurements.

Another technique that has been used to measure crankshaft bending vibration is to make measurements of the vibration transmitted through retro-fitted bearings to a non-rotating housing riding on the crankshaft [4.6,4.7]. Two measurements of radial vibration and one of axial vibration made from the housing using accelerometers enabled the bending motion of the crankshaft, before and after the fitting of a bending damper, to be assessed.

The studies of crankshaft bending vibration to date have all required some modification to the crankshaft or crankcase and great care needs to be taken when fitting vibration “transmitters” to the crankshaft to ensure that the natural frequency of the transmitter lies well above the frequencies of interest. A non-contact transducer capable of making bending vibration measurements directly from the rotating component without modification would be of significant value.

A commercial instrument, the Laser Torsional Vibrometer (LTV), was used in a study of torsional and bending vibration measurement [4.8] in which two LTV’s were used to assess bending vibration on a diesel engine crankshaft. In this study, simultaneous measurement of the yaw and rotation speed sets gave an indication of the bending motion but resolution of the individual pitch and yaw motion was not possible because, as chapter 2 demonstrated, angular vibration measurements on rotors using LDV have a cross-sensitivity similar to that seen in radial vibration measurements. Equations (2.32a&b) show that an intended yaw measurement is cross-sensitive to pitch and vice-versa. Resolution requires simultaneous measurement of the rotation speed set and the pitch and yaw vibration sets followed by post-processing. Chapter 3 described the first practical post-processing technique for resolution of radial or angular vibrations, with radial vibration measurements offered as a practical application of the procedure. The angular
measurements required in this chapter could have been performed in a similar manner using dual beam, rather than single beam, vibrometers but a novel optical configuration is to be used with sensitivity advantage over alternative, existing instruments.

Angular vibration measurements on non-rotating structures can simply be achieved using one pair of parallel beams but measurements on rotating structures requires simultaneous measurement of the pitch and yaw vibration sets. An arrangement used previously [4.8] would require four LTV's to be set-up in a complex configuration. Neglecting the considerable cost of such a set-up, the volume of equipment combined with the limited access usually found around an engine would make the measurement impractical. A single instrument capable of making simultaneous measurements of the pitch and yaw vibration sets that is simple in construction and easily set-up is desirable. The design of such an instrument, the Laser Angular Vibrometer (LAV), is described in the following section. For non-rotating structures the LAV gives straightforward measurement of two angular vibration components.

Another important area of interest in rotating machinery analysis is the transmission of torque through a system. In most systems steady transmission of torque is essential for successful operation with torque fluctuations resulting in increased wear and fatigue of shaft components, such as gears and bearings, while the operating life of flexible couplings, often included in the drive train to accommodate some misalignment, are severely reduced by large torque fluctuations. Accurate measurements of torque is needed to monitor the condition of shaft components but existing torque transducers usually require some modification to the drive system which results in costly downtime. Other disadvantages associated with some existing torque transducers include sensitivity to radial vibrations, difficulty in retrieving the signal from a rotating sensor and limited dynamic range. A robust torque transducer capable of accurate measurements without modification to the drive system is desirable. By making simultaneous measurements of torsional vibration, using the beam and geometric arrangements described in Section 2.4, at two axial locations along a shaft enables torque fluctuations to be derived without modification to the system. The final section of this chapter demonstrates one possible
application of these type of measurements by making a measurement of the torque transmitted through the alternator drive belt on a running diesel engine.

4.2 Instrument design

Chapter 2 demonstrated that the simplest beam arrangement for measurements of the pitch or yaw vibration sets is one incorporating two parallel beams aligned with the shaft spin axis and incident on the end face of the shaft. Measurement of the pitch and yaw vibration sets using two separate instruments would utilise two pairs of parallel beams but one beam in each pair shares a common function. Figure 4.1 shows the beam arrangement for a single instrument where the same measurements can be achieved using only three beams. The three parallel beams are aligned with the shaft spin axis and the angular vibration derived by subtracting optically the velocity measured by beams 1 and 2 for the pitch vibration set, and beams 1 and 3 for the yaw vibration set.

Figure 4.2 shows the optical configuration of the LAV which shares its basic features with the laser tiltmeter [4.9]. A rotating diffraction grating, which is used to provide the frequency shifts needed for directional discrimination, provides a convenient means of deriving three beams. The +1, 0, and -1 diffracted orders are used giving three beams frequency shifted +819.2, 0 and -819.2 kHz from the HeNe laser source. The lenses, L1 and L2, have their focal points coincident with the plane of the rotating diffraction grating so that the laser beam is focused onto the grating by L1 for optimum grating performance and the diffracted beams are made parallel by L2. At this stage the beams are all in one plane.

Two small mirrors, M1 and M2, are then used to steer the -819.2 kHz beam into the required arrangement shown previously in figure 4.1 where beam 1 is the zero order beam, beam 2 the -819.2 kHz beam and beam 3 the +819.2 kHz beam. A polarising beam splitter and quarter wave plate are used to increase the efficiency of the set-up by maximising the light sent to the target and ensuring that all the light collected from the target is directed towards the detectors and not back into the laser.
Non-polarising beam splitters, NPBS1, NPBS2 and NPBS3, and three mirrors, M3, M4 and M5, are then used to combine the 0 and -819.2 kHz beams onto detector 1 and the 0 and +819.2 kHz onto detector 2 where they beat. Demodulation is then performed by frequency trackers. Non-polarising beam splitters are used in the beam combining optics because, although some light is inevitably discarded, light both transmitted and reflected by the beams splitters needs to be combined.

To ensure the 819.2 kHz beam, which has a longer path length due to the steering mirrors, and the 0 kHz beam heterodyne strongly, the path length imbalance must be less than the coherence length of the laser so an additional path length is introduced into the 0 kHz beam controlled by the position of mirrors M3 and M4.

### 4.3 Experimental validation

Experimental validation of the theory presented in chapter 2 and the resolution process presented in chapter 3 was carried out using the test rig shown in figure 4.3 which allowed simultaneous angular vibration of a small test rotor in two orthogonal directions. Bearings with small clearances were employed to minimise the differences in vibration behaviour between the rotor and the bearing housing. Measurements of angular vibration were made from the rotor, using the LAV described in section 4.1, and from the bearing housing by subtracting the outputs of two accelerometers separated by a known distance.

Two sets of measurements were made; firstly, with the rotor rotating at a nominal speed of 20Hz, then repeated with the rotor non-rotating. From previous experience of such checking measurements (section 3.4) investigating genuine differences in vibration behaviour the two sets of data are compared differently here. Figure 4.4a shows the ratio of the resolved laser and accelerometer based measurements for the rotating measurements of pitch, normalised by the equivalent non-rotating measurements. Figure 4.4b shows the corresponding data for yaw. Measurements of one angular vibration amplitude (nominally $50 \times 10^{-3} \text{rad/s}$) were combined with four phase differences ($0^\circ$, $\pm 90^\circ$, $180^\circ$) and eight vibration frequencies (10, 25 30, 40, 60, 80, 100 and 120Hz) with the mean and standard deviation of the ratio at each vibration frequency presented in the figures.
In the non-rotating tests, the output from the LAV is straightforward to interpret and so these measurements give an indication of the genuine differences between the different points on the structure at which the LAV and accelerometer measurements were taken. In the rotating tests, the LAV outputs need to be post-processed with agreement between the measurements now acting as validation of the theory leading to equations (2.34a&b) and the post-processing method itself.

The resolved data from the rotating tests show good agreement with those from the non-rotating tests with the differences attributed to genuine differences between the vibration response of the rotor when rotating and not rotating. It is, of course, the very existence of these genuine differences that is the motivation behind the development of techniques for measurement directly from the rotor. A general increase in the deviation of values close to the rotation frequency (20Hz) can be seen which can be linked to an increase in the sensitivity of the resolution process to errors in the measurement of the shaft rotation speed and this will be discussed in section 5.1.

### 4.4 Diesel engine crankshaft angular vibration measurement

Figure 4.5 shows the LAV aligned parallel to the crankshaft rotation axis of a four cylinder two litre diesel engine and incident on the end face of the crankshaft pulley to provide simultaneous measurements of the pitch and yaw vibration sets. Unlike the radial measurements, the measurement of the pitch and yaw vibration sets is not dependent on the distances the beams are offset from the rotation axis. This means that the alignment of the LAV is relatively simple as the beams of the LAV can be positioned anywhere on the end face of the rotor. The LAV is a prototype constructed from bulk optics and, as figure 4.5 shows, is quite cumbersome but a commercial version of the instrument could be no larger than existing laser vibrometers.

The technique set out in section 3.2.2 requires simultaneous rotation speed measurement and a LTV was used for this purpose. The beams of the LTV were aligned as close to perpendicular (≈85°) to the rotation axis as access would allow to eliminate the
sensitivity of the measurement to bending vibration. At this angle the sensitivity to bending vibration is negligible [4.8].

Simultaneous measurements of the crankshaft pitch and yaw vibration sets were made across a range of engines speeds, obtained by varying engine load while maintaining wide open throttle. Figure 4.6 shows a section of the pitch vibration set measured at 3000rpm. The repeating pattern of the four combustion events that occur within each 4 stroke cycle, over a period of two crankshaft revolutions, can be clearly seen. Figures 4.7a&b and 4.8a&b show the spectra of the pitch and yaw vibration sets and corresponding resolved pitch and yaw vibrations at 3000rpm. The vibration characteristics expected from a 4 cylinder 4 stroke engine are \( \frac{1}{2} \) order peaks, due to the period of the 4 stroke cycle, with larger 2\(^{nd}\), 4\(^{th}\), 6\(^{th}\)... orders, due to the two combustion events that occur every crankshaft revolution, and these can be seen in both the pitch and yaw data. The effects of the cross-sensitivity in the measurements can also be seen by comparing the vibration sets with the resolved vibration. For vibration frequencies significantly above the rotation frequency the intended measurements, \( \dot{\theta}_x \) and \( \dot{\theta}_y \), dominate the cross-sensitivity terms, \( \Omega \dot{\theta}_x \) and \( \Omega \dot{\theta}_y \), in the measured vibration sets, assuming that the pitch and yaw are of similar magnitude. Therefore, those regions of the spectra of the vibration sets and resolved vibration velocities are very similar. For frequencies below the rotation frequency the cross-sensitivity terms dominate the intended measurements and this can be seen in the peaks at \( \frac{1}{2} \)X rotation frequency. The resolved vibration spectra show a smaller magnitude yaw vibration and a slightly larger pitch vibration which results in a larger vibration component in the yaw vibration set and a smaller component in the pitch vibration set.

Figures 4.9a&b show waterfall plots of the resolved crankshaft pitch and yaw vibration respectively across a range of engines speeds. Amongst other features, a broad vibration peak centred around 450Hz can be seen in both the pitch and yaw data which corresponds to the first natural frequency of the crankshaft in bending. Modal analysis has shown this to lie in the range 456-475Hz depending on the angular position of the crankshaft [4.8] and shows agreement with the radial vibration data presented in chapter 3. Similar values of natural frequency have also been reported on similar sized engines [4.7,4.10]. (Small peaks of relatively constant amplitude can also be seen at 50, 100, 150Hz... which are due
to the periodic noise signal created by the rotating diffraction grating within the vibrometer which rotates at 50Hz.)

As in chapter 3, the purpose of this work is to show practical application of the techniques developed. It is interesting to note, however, that while the radial measurements were of similar amplitudes in the x and y directions the pitch motion exceeds the yaw motion quite significantly. From data such as this an impression of the mode shape of this first bending mode can be constructed.

The information missing from the resolved spectra at the synchronous frequency prevents full reconstruction of the resolved pitch and yaw time signals but other regions of the spectra can be successfully reconstructed. Figure 4.10 shows the resolved pitch vibration with the engine running at 3000rpm, band-pass filtered around the natural frequency of the crankshaft in bending (350-550Hz). Modal analysis of the crankshafts of similar sized engines have shown the 1st mode shape of the crankshaft in bending to have a large anti-node at the crankshaft pulley, a smaller anti-node around the position of the 4th cylinder and a node around the 2nd cylinder [4.7,4.5]. A study on a 6 cylinder engine crankshaft also showed a large anti-node at the crankshaft pulley [4.2]. The studies concluded that the impulse created by the combustion of cylinder number 1, because it is closest to the anti-node, was the most significant source of crankshaft bending motion. This has been confirmed by measurements of engine noise where the peak noise was found to be synchronised with the firing of cylinder number 1 and a more subjective test where the ignition plug was removed from each cylinder in turn where the engine rumble "disappeared" when the plug was removed from cylinder number 1 [4.5]. The repeating pattern of combustion events can be seen in figure 4.10 where one impulse dominates the others. It is proposed that this impulse is due to the firing of cylinder number 1 and that firing of cylinder number 2 is not discernible (firing order 1, 3, 4, 2) because it is closest to a node in the mode shape of the crankshaft in bending. The resolved vibration measurements enable a modal damping factor in the region of 3-5% to be estimated, from the logarithmic decrement, which is consistent with the values obtained from the measurements of radial vibration made in chapter 3.
4.5 Torque fluctuation measurement

The operating principle of a number of torque transducers is to make use of the linear relationship between the torque applied to a shaft and the resulting twist. By measuring the twist in a shaft along a known axial length a measurement of torque can be derived from the material and section properties. Using this principle a real-time measurement of the torque fluctuations in a shaft can be achieved by making simultaneous measurements of torsional vibration at two axial locations along a shaft, the difference between the two torsional vibration measurements providing a measure of the time derivative of shaft twist.

In previous chapters of this thesis the advantages of vibration measurements using LDV have been highlighted, including the possibility to make non-contact torsional vibration measurements using the LTV. Using two LTVs to make two simultaneous measurements of torsional vibration provides a torque fluctuation measurement system which does not require any modification to the drive line and can be made with immunity to translational vibration on components of any shape.

As an example of the technique, figure 4.11 shows a measurement of twist, taken across a flexible rubber coupling fixed between an electric motor and the rotor on an experimental rig rotating nominally at 700rpm. The periodic fluctuations within each rotation corresponding to the pole-passing frequency of the motor can clearly be distinguished.

One important application where the ability to measure torque fluctuations would significantly improve understanding of component behaviour is the measurement of the fluctuations in the torque transmitted by timing chains and belts used in IC engines to synchronise the valvetrain with the crankshaft. Premature failure of these components results in costly engine damage as the valves collide with the pistons. Typically the fluctuations in the torque transmitted by these chains and belts will be large and measurement of the torque fluctuations should be possible using commercially available LTVs. In addition, for a chain or non-slipping belt, the ratio of the mean tangential velocities is simply the speed ratio of the system. This means that belt slippage can also be detected by monitoring changes in the ratio of mean tangential velocities.
To illustrate the ability to make this measurement, figures 4.12a&b show measurements of torsional vibration made on the crankshaft and alternator pulleys, respectively, of a diesel engine during idle using two LTVs. The difference in the form of the two time traces indicates that the torque transmitted to the alternator is fluctuating and the belt is experiencing an alternating stress. A measure of the rate of the strain in the belt can be obtained by calculating the difference in the tangential velocities of the belt leaving the crankshaft pulley and engaging the alternator pulley. Figure 4.13a&b show the spectra of the rate of strain in the belt at 1000rpm under no load and load respectively. An increase in the ½, 2, 4 and 6\textsuperscript{th} order components can be seen along with a 20\% increase in the rms level.

The two examples of torque fluctuation measurement presented in this section have demonstrated the potential of the measurement technique but both were chosen because, in each situation large differences in the two torsional vibration measurements existed. For a flexible rubber element the angle of twist obtained is large but in practice the twists found in metal shafts are small and a large axial separation between measurements is required to obtain a measurable twist. The smallest measurable twist that can be measured is dependent on the noise floor of the individual torsional vibration measurements. Hence, the levels of speckle noise in the LTV measurements govern the minimum twist and, therefore, torque that can be measured. Further investigation is required to determine the smallest torque that can be measured in practice with commercially available laser torsional vibrometers. It is likely that torque fluctuation measurement using LTV's will be limited to large amplitude torque fluctuations in relatively long flexible shafts until further progress is made in the reduction of speckle noise.
5 Errors in the resolution of radial and angular vibration measurements

5.1 Errors in rotation speed measurement

In section 3.4, resolved radial vibration measurements were compared with vibration measurements using accelerometers fixed to the motor housing. Figures 3.4a&b showed the ratio of the resolved laser vibrometer data to the accelerometer data for the x and y directions respectively. In addition to a trend in the data, found to be due to genuine differences in the vibration response of the rotor and motor housing, an increase in the standard deviation of vibrometer:accelerometer measurements for the ratios close to \( n = 1 \) was seen.

The unsuitability of the resolution technique for measurement of synchronous vibration, when the weighting function, \( W(\omega_m) \), becomes infinite, was discussed in section 3.2.1. Also, in regions where \( \omega_m \) is close to \( \Omega \), \( W(\omega_m) \) becomes very large and, therefore, in these regions, a small error in the measurement of \( \Omega \) will result in a large error in the value of the weighting function and, hence, in the calculated velocity. Considering a small error, \( \delta \Omega \), in \( \Omega \), the resulting error \( \delta W(\omega_m) \) in the weighting factor \( W(\omega_m) \) is given by:

\[
\delta W(\omega_m) = \frac{dW(\omega_m)}{d\delta \Omega} \delta \Omega
\]  
(5.1)

Defining the "Error Amplification Factor" (EAF) as the amount by which an error in the measurement of \( \Omega \) is magnified to give an error in the calculated velocity:

\[
EAF = \left( \frac{\delta W(\omega_m)}{W(\omega_m)} \right) = \frac{2}{(\Omega^2 - 1)}
\]  
(5.2)
Figure 5.1 shows the variation in \( EAF \) with \( n \). It can be seen that for sub-synchronous vibrations, \( n < 1 \), the percentage error in \( \overline{\Omega} \) is at least doubled. For \( n > 1 \), the amplification factor approaches zero as \( k \) increases with unity amplification at \( n = \sqrt{3} \).

Evidence of this form of error can be seen most clearly by comparing the variation of the standard deviation with \( n \) observed in the rotating tests shown in figures 3.3a&b with the non-rotating data shown in figures 3.4a&b. As expected the standard deviation of the non-rotating data shows no variation with \( n \). Figures 4.4a&b, which compare the angular vibration data of a rotating component with that of a non-rotating component, also show the same form of error.

### 5.2 Effect of torsional vibrations

In chapter 3 it was assumed that the rotation speed of the rotor was constant and that any torsional vibrations or speed fluctuations were negligible. This allowed the overall problem to be simplified greatly and allowed a practical resolution method to be generated. In many practical situations, torsional vibrations will be present to some extent. This section considers what effect any torsional vibrations will have on the accuracy of the resolution method described in chapter 3. For brevity the section will proceed with a description of the effects of torsional vibrations on the resolution of radial vibration measurements but, as was demonstrated in chapter 3, the resolution of pitch and yaw vibrations is very similar and the discussion is equally applicable with the exception of the errors due to the radial position offsets which are not present in the pitch and yaw vibration sets.

For a radial measurement in the \( x \) direction, vibration velocity components are resolved on a frequency-by-frequency basis by evaluating the function \( \hat{X} \) given by equation 3.16a from the vibrometer outputs and then weighting it by the frequency dependent function \( W'(\omega_m) \). For a rotor rotating at a constant speed, \( \overline{\Omega} \), the \( m^{th} \) vibration velocity component is equal to (see equations 3.14a and 3.16a):
\[ \dot{x}_m = W(\omega_m) \dot{x} \bigg|_{\omega=\omega_m} \]  

(5.3a)

where

\[ \dot{x} = \bar{U}_x - \bar{\Omega} \int_0^t \bar{U}_y dt \]  

(5.3b)

If the true rotation speed of the rotor is now considered to be:

\[ \Omega(t) = \bar{\Omega} + \Delta \Omega \]  

(5.4)

where \( \bar{\Omega} \) and \( \Delta \Omega \) are the mean and alternating components of rotation speed respectively, additional terms will be introduced into the calculated velocity due to the alternating component of rotation speed. By substituting \( \Omega(t) \) from equation (5.4) for \( \bar{\Omega} \) in equation (5.3b) and using \( \bar{U}_x \) and \( \bar{U}_y \) as described by equations (3.2a&b) rather than equations (3.4a&b), the actual velocity calculated is not simply equal to \( \dot{x}_m \) but to:

\[
W(\omega_m) \dot{x} = \dot{x}_m + W(\omega_m) \left( -\Delta \Omega \frac{\partial}{\partial x} \int_0^t \Delta \Omega x dt + \Delta \Omega \int_0^t \Delta \Omega x dt - \Delta \Omega x \frac{\partial}{\partial t} \int_0^t \Delta \Omega t dt + \Delta \Omega \int_0^t \Delta \Omega t dt \right)
\]  

(5.5a)

Similarly, for a measurement in the \( y \) direction, the actual calculated velocity is equal to:

\[
W(\omega_m) \dot{y} = \dot{y}_m + W(\omega_m) \left( \Delta \Omega \frac{\partial}{\partial y} \int_0^t \Delta \Omega y dt + \Delta \Omega \int_0^t \Delta \Omega y dt - \Delta \Omega y \frac{\partial}{\partial t} \int_0^t \Delta \Omega t dt + \Delta \Omega \int_0^t \Delta \Omega t dt \right)
\]  

(5.5b)
If the integral terms in equation (5.3b) and its y-direction equivalent are multiplied just by \( \bar{\Omega} \) rather than \( \Omega(t) \) a dependence on the in-plane motion multiplied by the alternating component of rotation speed, \( \Delta\Omega_x \) and \( \Delta\Omega_y \), can be found and with the solution no longer independent of the in-plane motion, which is the intended goal.

### 5.2.1 Description of error terms

The error terms in the calculated velocities, \( \hat{X} \) and \( \hat{Y} \), are described below with those found in \( \hat{Y} \) shown in brackets.

\( \Delta\Omega y_0 \ (\Delta\Omega x_0) \): A term equal to the speed fluctuations amplified by the offset. Its effect can be limited by careful alignment but it is problematic in situations where accurate alignment cannot be guaranteed, for example, when access to the laser vibrometer is limited, or the situation described later in section 5.2.2 where the radial and torsional vibration frequencies coincide.

\[
\bar{\Omega} \int_0^t \Delta\Omega x dt \left( \bar{\Omega} \int_0^t \Delta\Omega y dt \right) : \text{A cross-term amplified by the mean rotation speed that will introduce information at the sum and difference frequencies of the orthogonal motion and speed fluctuations. The integration will accentuate the lower frequency components.}
\]

\[
\Delta\Omega \int_0^t \Delta\Omega x dt \left( \Delta\Omega \int_0^t \Delta\Omega y dt \right) : \text{A cross-term that will introduce information at the vibration frequency of the orthogonal motion and the sum and difference frequencies of orthogonal motion and twice speed fluctuation.}
\]

\[
\bar{\Omega} x_0 \int_0^t \Delta\Omega dt \left( \bar{\Omega} y_0 \int_0^t \Delta\Omega dt \right) : \text{The integral of the speed fluctuations amplified by both the rotation speed and the offset. Its effect can be limited by careful alignment.}
\]
$\Delta \Omega x_0 \int_0^t \Delta \Omega dt + \Delta \Omega y_0 \int_0^t \Delta \Omega dt$: A cross-term amplified by the offset that will introduce information at dc and twice speed fluctuations. Its effect can be limited by careful alignment.

$\bar{\Omega} \Delta \Omega \int_0^t x dt + \bar{\Omega} \Delta \Omega \int_0^t y dt$: A cross-term amplified by the mean rotation speed that will introduce information at the sum and difference frequencies of the translational and speed fluctuations. The integration will accentuate the lower frequency components of translational vibration.

In the development of the frequency-by-frequency solution described in section 3.2.2, terms dependent on the integral of vibration displacement in equations (3.12a&b) were accounted for by linking them to the vibration velocity at any one frequency using $W(\omega_n)$. The frequency of the integral terms in equations (3.12a&b), however, are independent of the speed fluctuations of the rotor unlike the final two pairs of error terms described in this section which are the product of the alternating component $\Delta \Omega$ and the integral of displacement. This means that these two pairs of error terms cannot be accounted for by any weighting factor.

5.2.2 Error terms in practical situations

This section investigates the relative sizes of the error terms discussed above. To give appropriate context, a typical application where torsional vibrations are present is considered. The application chosen is the measurement of radial crankshaft vibration in an internal combustion engine as in section 3.5 where such measurements were made.

The spectra of both the radial and torsional vibrations of a 4 stroke 4 cylinder internal combustion engine's crankshaft are characterised by peaks at harmonics of half the fundamental rotation frequency, with larger amplitudes at 2x, 4x, 6x... rotation frequency due to the two combustion events that occur during every crankshaft rotation. Figure 5.2 shows the simulated radial and torsional vibration spectra with the amplitudes chosen to
correspond with a mid-severity radial vibration (10 mm/s at 2x, 4x, 6x... and 5mm/s at \( \frac{1}{2} \) orders), equal in both the x and y directions, and a very large torsional vibration amplitude \( \left( \frac{\Delta \Omega}{\Omega} = 0.05 \right) \) at 2x, 4x, 6x..., and \( \frac{\Delta \Omega}{\Omega} = 0.025 \) at \( \frac{1}{2} \) orders). Knowing both the radial and torsional vibrations enables \( \bar{U}_x \) and \( \bar{U}_y \) to be simulated and the individual error terms to be calculated.

Those error terms dependent on the offsets, \( x_o \) and \( y_o \), can, in theory, be eliminated by adjusting the vibrometer alignments to give a zero mean output. However, experience has shown that, in practice, the offsets can only be reliably reduced to 0.25-0.5mm using this process. The difficulty in alignment is compounded by the fact that a condition exists where alignment of the laser vibrometer to give a zero mean instrument output actually leaves a small offset. This occurs whenever the torsional and radial vibration frequencies coincide and the difference components within the measured velocity give a d.c. component.

Figures 5.3a-f show the spectra of the individual error terms present in the resolved x direction velocity in order of peak magnitude. In an effort to quantify the importance of each error term, the magnitudes have been normalised by the rms value of the true vibration velocity. Normalisation in this way enables error terms, some of which contain information at different frequencies to the true vibration due to sum and difference terms, to be compared usefully with the true vibration spectrum. The error terms dependent on the radial position offsets are shown for what would be a typically large offset of \( x_o = y_o = 0.5 \text{mm} \).

In the remainder of this section, the errors in an x-direction measurement will be discussed but these are obviously also applicable to a measurement in the y-direction. In addition, the effect of \( W(\omega_m) \) on the error terms is not included in the analysis. From figures 5.3a-f it can be seen that the most significant error terms are those that include the offsets \( x_o \) and \( y_o \). In particular, the error term \( \Delta \Omega x_o \) shown in figure 5.3a is notable because,
unlike the error term $\bar{\Omega}y_0 \int_0^t \Delta \Omega dt$ in figure 5.3b which includes an integral term, its magnitude does not tend to roll off with increasing frequency. However, $\bar{\Omega}y_0 \int_0^t \Delta \Omega dt$ might be the most significant error term in situations where there are large subsynchronous torsional vibrations such as in single cylinder internal combustion engines. Both of these error terms are independent of the radial vibration amplitudes which means that their significance can be assessed from the mean rotation speed, an estimate of the radial position offsets and a measurement/estimate of the torsional vibration spectrum. For example, measurements of crankshaft torsional vibration made during the radial engine measurements made in chapter 3 showed a peak 2\textsuperscript{nd} order vibration of 2.8rad/s at around 1200 rpm which means that for a 0.5mm radial offset the peak error velocity would be in the region of 1.5mm/s at 40Hz compared to the measured 2\textsuperscript{nd} order radial vibration velocity of approximately 8mm/s at this frequency. As can been seen from figures 5.3c-f, the magnitudes of the remaining error terms

Figure 5.4a shows the overall error due to the torsional vibrations. The rms value of the true vibration velocity is again used to normalise the difference between the true and calculated vibration velocities. For comparative purposes a plot of the normalised true vibration velocity is shown in figure 5.4b. It can be seen that the error velocity is of similar magnitude to the true vibration velocity in this situation, where the torsional vibration and offset are particularly severe. Figure 5.4c shows the overall error due to torsional vibration with the vibrometers perfectly aligned so $x_0 = y_0 = 0$. It can be seen that the error velocity is much reduced and is small compared to the true vibration velocity, showing the importance of aligning the vibrometers through the centre of the rotor.

5.3 Correction of errors due to torsional vibrations in the resolution method

Section 5.2 discussed the errors introduced into the resolution method presented in chapter 3 by the presence of torsional vibrations and concluded that three of the six error terms, those dependent on the offsets, $x_0$ and $y_0$, needed to be minimised by careful alignment of
the instruments. The remaining error terms were only dependent on the rotation speed, which is known, and the rotor displacement, which is unknown as it is the intended measurement. However, section 5.2 also showed that in most practical situations the errors due to torsional vibrations, neglecting those due to the offsets, are relatively small and that the resolved vibration velocity is a good estimate of the true vibration velocity. This means that the possibility to make an estimate of the error terms and subtract them from the estimate of vibration velocity and obtain an improved estimate exists.

Figures 5.6a&b show the simulated spectra for broadband radial vibrations of unit amplitude in both x and y directions (0-5x rotation frequency) and a broadband torsional vibration of unit amplitude (0-5x rotation frequency). By calculating the velocity measured by the laser vibrometers using equations (3.2a&b) and then using the resolution method described in chapter 3, an estimate of the vibration velocity can be obtained. It is assumed that the vibrometers are perfectly aligned so that \( x_0 = y_0 = 0 \). Figures 5.7a&b show the calculated velocity for the x radial direction and the total error in this estimate which is made up of the error terms described in section 5.2. The error in the calculated velocity can be seen to be approximately an order of magnitude larger than the true vibration velocity. An estimate of the total error can be made using the rotation speed and torsional vibration data measured and the calculated velocity and this process is studied in more detail in section 6.2.2. Figure 5.8a shows the estimated error velocity which can be seen to be a good approximation of the true error in the calculated velocity shown in figure 5.7b. Figure 5.8b shows the improved estimate of vibration velocity obtained by subtracting the estimate of the errors from the original calculated velocity. A dramatic reduction in the error can be seen.

Figures 5.9a&b show the original estimate, neglecting offset errors, and the improved estimate for the engine simulation used in section 5.2. Again, a useful improvement can be seen in the estimate of the radial vibration velocity.

5.3.1 Experimental implementation
The error correction process was applied to post-processed experimental data from radial vibration measurements on a test rig that allowed simultaneous radial vibration of a rotor.
in the $x$ and $y$ directions. Simultaneous measurements of the $x$ and $y$ radial vibration sets and the rotation speed set were made with the rotor undergoing nominal radial vibrations of 10mm/s at $\frac{1}{2}x$ rotation frequency in both the $x$ and $y$ directions. The radial vibration amplitudes were established from vibration measurements made using accelerometers attached to the motor housing. A torsional vibration at 2x rotation frequency of magnitude equating to $\frac{\Delta \Omega}{\Omega} = 0.05$ was induced. Radial and torsional vibration frequencies that were integer multiples of half rotation frequency were chosen because this typically occurs in practice.

In order to assess the effectiveness of the error correction process in practice the theoretical improvement in the resolved velocity were calculated. Figures 5.10a-d show simulated spectra of the actual and calculated vibration velocities in the $x$ direction along with the actual error and the estimate of the error. The largest peaks in the actual error spectrum occur at $1\frac{1}{2}$ and $2\frac{1}{2}$ times the rotation frequency. Figure 5.10e shows the spectrum of the improved estimate of the vibration velocity with the error components at $1\frac{1}{2}$ and $2\frac{1}{2}$ times rotation frequency reduced by a factor of 100.

Figures 5.11a&b show the calculated $x$ direction vibration velocity and improved estimate of the vibration velocity from the experimental measurements of the radial vibration sets. Table 5.1 shows a comparison of the theoretical and experimental improvements in the calculated vibration velocity measurements.
It can be seen that the error correction process is less effective in practice but still reduces the largest error velocity peaks at $1\frac{1}{2}$ and $2\frac{1}{2}$ times rotation frequency significantly. The likely cause of this discrepancy is that, while the simulation is not affected by the radial offset errors, the measured data is always sensitive to offset errors to some extent. Table 5.1 also shows that, in theory, the error correction process actually increases the error velocity components at $3\frac{1}{2}$ and $4\frac{1}{2}$ rotation frequency. In addition, figure 5.10e shows that a number of additional error components can appear. These phenomena and the effects of repeatedly subtracting the latest estimate of the error velocity to obtain a better estimate of the radial vibration velocity are discussed further in chapter 6.

Despite achieving a useful reduction in the error velocity components using the error correction process, figures 5.11a&b again highlight the importance of the offset error terms. Although great care was taken to align the laser beams through the centre of the shaft rotation axis the large error velocity component at 2x the rotation frequency, $\Delta\Omega x_0$, can be seen to be the largest error component. The magnitude of the error component, approximately 3mm/s, for the rotation frequency of 60Hz indicates a radial offset in the region of 0.16mm.
6 Developments in the resolution of components:

Recommendations for further work

6.1 Introduction

The work presented in this thesis has enabled significant progress to be made in the understanding and application of LDV to measurements of rotor vibration. In this chapter three areas where further investigation is required are discussed.

Section 5.2 described the errors introduced into the resolution method described in chapter 3 by torsional vibrations. The largest error terms in measurements of radial vibration are those dependent on the offsets $x_0$ and $y_0$ and an accurate method of aligning the vibrometer beams through the centre of the rotor would prove invaluable. A method of improving the estimate of the resolved vibration velocity by reducing those errors not dependent on the offsets $x_0$ and $y_0$ was also demonstrated in section 5.3. While the method reduced the effects of some of the error components, other smaller error components were actually increased. This phenomenon questions the validity of repeatedly applying the method to obtain progressively better estimates of the resolved vibration velocity.

Chapter 3 described a method that enables estimates of the resolved vibrations to be made by assuming that the rotation speed of the shaft was essentially constant. This made the method unsuitable for situations where there are significant torsional vibrations or gross speed fluctuations such as during run-up or run-down. A resolution method tolerant to torsional vibrations and gross speed fluctuations would be advantageous. Resolution of the measured vibration sets using numerical integration methods offers this possibility.
6.2 Reduction of errors due to torsional vibrations

6.2.1 Offset errors
Chapter 5 investigated the numerous errors introduced into the resolution process by torsional vibrations and concluded that the largest errors were the offset errors, those dependent on the offsets \(x_0\) and \(y_0\). Comparison of figures 5.4a, which shows a simulation of all the errors due to torsional vibrations, with figure 5.4c, which shows the same situation without the offset errors, highlights the significant improvement in the resolution process that would be obtained if a reliable method of eliminating the offset errors can be found.

Offset errors can be eliminated by aligning the beams exactly though the centre of the rotor but experience has shown that by aligning the vibrometers to give as close as possible to zero mean outputs is only capable of reducing the offset to 0.25-0.5mm. Further work is required to establish an alignment method that enables the offsets to be reduced to a lower value. In certain circumstances it may not be possible, for safety reasons, to gain access to the vibrometers once the rotor is rotating. In such cases accurate initial alignment of the vibrometers is very difficult. In practice, for circular shafts, the author has found reflective foil of use. Temporarily fixing the reflective foil to the surface of the rotor enables the alignment of the vibrometer to be adjusted until the reflected light is returned to the aperture of the vibrometer indicating that the vibrometer is aligned approximately with the centre of the rotor but this method can only achieve an alignment accuracy of approximately 0.5mm, depending on the diameter of the shaft.

Ideally an optical configuration that is insensitive to any offset could be found and in chapter 2 the advantages of parallel beam arrangements were demonstrated. The LTV makes measurements of torsional vibration with immunity to translational motion by subtracting the velocity measured by two parallel beams, knowing the perpendicular separation of the beams. One obvious optical configuration to investigate, with potential to give immunity to offsets, is an arrangement of parallel beams which adds the velocity measured by the two beams. Using equation (2.10) the sum of the velocity measured by
the two parallel beams incident on a rotating structure, both parallel to the x axis and lying in the xy plane is:

\[ U_{m1} + U_{m2} = 2\left[ \dot{x} + (\dot{\theta}_x + \Omega)y - (\dot{\theta}_y - \Omega \theta_x)z - (v_{01} + v_{02})(\dot{\theta}_x + \Omega) \right] \] (6.1)

Unfortunately it is not possible to eliminate the offset terms as \( y_{01} + y_{02} \), unlike \( y_{01} - y_{02} \) which is equal to the beam separation, is not known and the accuracy to which each individual beam can be aligned is the same as a single beam vibrometer.

### 6.2.2 Repeated correction of errors due to torsional vibrations

Section 5.3 outlined a method of improving the estimate of resolved vibration velocity which was able to produce a useful reduction in the largest error components due to torsional vibrations in both a simulated and experimental vibration measurement. However, the error correction process actually resulted in an increase in some of the smaller error velocity components, accompanied by new error components at frequencies different to those found in the original estimate of vibration velocity. The potential of the error correction process to increase certain error velocity components after only one iteration questions the validity of repeatedly applying the method in the hope of obtaining progressively better estimates of the resolved vibration velocity.

The increase in some of the error components can be seen by examining the error correction process in more detail. For an x-radial vibration measurement, the actual vibration velocity calculated, in the absence of offset errors, is given by equation (5.5a):

\[
(\dot{x}_m)_{est} = \dot{x}_m + W(\omega_m) \left[ (\Omega + \Delta \Omega) \int_0^t \Delta \Omega x dt + \Omega \Delta \Omega \int_0^t x dt \right]
\] (6.2)

The error correction process relies on the fact that in most practical situations the error components, shown in square brackets, are small compared to the true vibration velocity as demonstrated in Section 5.2. This means that a good estimate of the vibration
displacement, \((x_m)_{est}\), can be obtained by integrating the calculated vibration velocity given above:

\[
(x_m)_{est} = x_m + W(\omega_m) \int \left[ (\Omega + \Delta \Omega) \int \Delta \Omega x dt + \Omega \Delta \Omega \int x dt \right] dt
\]  

(6.3)

From this estimate of the vibration displacement the actual error components in equation (6.2) can themselves be estimated from the torsional vibration measurements made on the rotor:

\[
\begin{align*}
\left\{ (\Omega + \Delta \Omega) \int \Delta \Omega x dt \right\}_{est} &= (\Omega + \Delta \Omega) \int \Delta \Omega x dt + (\Omega + \Delta \Omega) \int \left\{ \Delta \Omega W(\omega_m) \int \left[ (\Omega + \Delta \Omega) \int \Delta \Omega x dt \right. \right. \\
& + \Omega \Delta \Omega \int x dt \left. \right] dt \right\} dt \\
& + \Omega \Delta \Omega \int x dt + \Omega \Delta \Omega W(\omega_m) \int \left[ (\Omega + \Delta \Omega) \int \Delta \Omega x dt + \Omega \Delta \Omega \int x dt \right] dt
\end{align*}
\]

(6.4a)

\[
\begin{align*}
\left\{ \Omega \Delta \Omega \int x dt \right\}_{est} &= \Omega \Delta \Omega \int x dt + \Omega \Delta \Omega W(\omega_m) \int \left[ (\Omega + \Delta \Omega) \int \Delta \Omega x dt + \Omega \Delta \Omega \int x dt \right] dt
\end{align*}
\]

(6.4b)

These are then weighted and subtracted from the original estimate of vibration velocity to give an improved estimate of vibration velocity, \((\dot{x}_m)_{est2}\):

\[
\begin{align*}
(\dot{x}_m)_{est2} &= \dot{x}_m - W(\omega_m) \left\{ (\Omega + \Delta \Omega) \int \left[ \Delta \Omega W(\omega_m) \int (\Omega + \Delta \Omega) \int \Delta \Omega x dt + \Omega \Delta \Omega \int x dt \right] dt \right\} dt \\
& + \Omega \Delta \Omega W(\omega_m) \int \left[ (\Omega + \Delta \Omega) \int \Delta \Omega x dt + \Omega \Delta \Omega \int x dt \right] dt
\end{align*}
\]

(6.5)

The simplest form of rotor vibration measurement where a torsional vibration is present is the case simulated in section 5.3.1, used to assess the effectiveness of the experimental measurements, where the rotor is undergoing single frequency orthogonal and torsional vibrations at angular frequencies \(\omega_v\) and \(\omega_s\) respectively. From equation (6.3) the
resulting error components in the initial estimate of vibration velocity due to the torsional vibration can be found to occur at \( \omega_r, (\omega_r \pm \omega_v) \) and \( (2\omega_r \pm \omega_v) \). Figure 5.10b shows these error components clearly which occur at \( n = 0.5, 1.5, 2.5, 3.5 \) and \( 4.5 \) for the frequencies simulated \( (\omega_r = 0.5\omega_R, \omega_v = 2\omega_R) \). Equation (6.5) gives the improved estimate of vibration velocity which now has, in addition to those in the initial estimate, error components at \( (3\omega_r \pm \omega_v) \) and \( (4\omega_r \pm \omega_v) \). Figure 5.10e shows the improved estimate of vibration velocity with additional error components at \( n = 5.5, 6.5, 7.5 \) and \( 8.5 \). Although the error correction process introduces error components in a predictable fashion, with the subsequent iterations adding components at \( (5\omega_r \pm \omega_v) \) and \( (6\omega_r \pm \omega_v) \), then at \( (7\omega_r \pm \omega_v) \) and \( (8\omega_r \pm \omega_v) \) etc., in practice it is likely that there will be more than one true vibration component and these components will have unknown frequencies. This makes it impossible to determine whether a change in magnitude of a particular vibration component in the estimated vibration spectrum is due to a reduction in the error present due to torsional vibrations or by the influence of a “new” error component, created by the correction process itself. It is, however, possible to identify those “new” error components created by the error correction process in situations, such as the situation simulated in Section 5.3.1, where new peaks occur in the estimated vibration spectrum at frequencies where previously no peaks were evident.

It can be seen that for the “improved” estimate of vibration to be a genuine improvement on the initial estimate of vibration, the complex set of error terms, shown in curved brackets, in equation (6.5) must be smaller in magnitude than those in equation (6.3) which may not be the case for all vibration amplitudes and frequencies. This can be demonstrated by taking the broadband radial vibration simulation used in Section 5.3 in which both the radial and torsional vibration amplitudes were originally of one arbitrary vibration unit and repeating the simulation for radial vibration amplitudes of 1 to 10 arbitrary units. Figure 6.1 shows error in the estimated vibration spectrum summed across the width of the spectrum, normalised to the actual vibration amplitude, against the number of iterations of the error correction process. For vibration amplitudes less than 5 arbitrary units, the error correction methods can be seen to reduce the error in the initial estimate. The rate of improvement can be seen to decay to a point where the error is then

88
unaffected by further iteration. The size of this residual error and number of iterations to achieve it can be seen to increase with vibration amplitude. For intermediate levels of vibration amplitude, approximately 6-7 units, the level of error can be seen to oscillate while decreasing gently and for amplitudes greater than 7 units it can be seen that the error correction process actually increases the error with every iteration.

Experience with a variety of different vibration simulations has demonstrated the error correction methods potential to reduce errors due to torsional vibration but has also shown circumstances where error correction can be disadvantageous. Use of simulated data, where the answer is known, has enabled this phenomenon to be identified along with the need to use the correct number of iterations. However, further work is required to establish the exact circumstances under which an improvement in the estimate of vibration velocity can be guaranteed and an indication of the optimum number of iterations for real data.

6.3 Resolution of radial and angular vibration measurements using numerical integration

The method of resolving radial and angular vibrations described in chapter 3 was unable to solve the ‘global’ equations governing measurements, (3.2a&b) and (3.3a&b) respectively, but was able to make an estimate of the resolved vibrations by assuming that the rotation speed of the shaft was approximately constant. A resolution method tolerant to torsional vibrations and gross speed fluctuations would enable significant progress to be made in the measurement of rotor vibrations using LDV as it would enable measurement in situations where there are large torsional vibrations or gross speed fluctuations such as during run-up or run-down. The ability to perform run-up and run-down tests is particularly useful as they enable rotor resonances to be identified.

Many engineering problems described by differential equations, such as equations (3.2a&b) and (3.3a&b), are too complex to be solved by analytical methods but can be solved using direct numerical integration methods. These methods generate solutions to the differential equations at discrete time intervals, starting from a set of initial conditions and proceeding in a step by step fashion. There are many different types of numerical
schemes and they differ in the number and form of the time interval used, the number of expansion orders used, and can be explicit or implicit. This section will use the relatively simple central difference scheme which is an explicit multi-step scheme calculating the solution at the next point based on the values at the previous two time intervals. The scheme is conditionally stable which means that a time step smaller than a critical value is needed for the method to remain stable. \( \pi \) time steps are required in the smallest period of the system for stability [6.1].

In order to solve the differential equations the initial conditions, usually displacement and velocity, need to be known. The very nature of the problem, the cross-sensitivity of the measurements, means that the vibration of the rotor cannot be measured directly using the laser vibrometers while the rotor is rotating. The only circumstances where the vibrometers measure the required vibration velocities directly are when the rotor is non-rotating. This means that this type of resolution method requires the measurements to begin before the rotor rotates, to provide the initial conditions, and then follow the rotor vibrations through run-up and into operating conditions. Similarly, post-processing measurements started during rotation and following them through run-down will provide the required ‘initial’ conditions when the rotor has stopped rotating.

This section will simulate the resolution of the genuine vibration velocities using numerical integration for a rotor undergoing a variety of vibrations. In each simulation, the rotor will run-up to a final rotation speed of 1000rpm (16.67Hz) and will undergo radial vibration displacements of unit amplitude in both the x and y directions. Using equations (3.2a&b) the velocity measured by the laser vibrometers can be calculated and then used as the inputs to resolution process. The accuracy of the method can then be assessed by comparing the outputs of the resolution process with the original vibrations. It is assumed that the vibrometers are perfectly aligned so that \( x_0 = y_0 = 0 \).

Figure 6.2 shows the rotor rotation speed, radial vibration and vibrometer outputs for a rotor undergoing a \( n=0.5 \) radial vibration. Figure 6.3 shows the corresponding plots of resolved radial vibration displacements and the difference between the resolved and actual values which can be seen to be very small in magnitude. The figures show that, unlike the
resolution method described in chapter 3, this resolution method can tolerate gross speed fluctuations. Figures 6.4 and 6.5 show the same situation but with the addition of a large torsional vibration, $\frac{\Delta \Omega}{\Omega} = 0.05$, at $n=2$. Again, the error is small and it can be seen that the torsional vibration seems to have little effect on the error in the resolution of the vibrations.

Another drawback of the resolution method described in chapter 3 was that it was unable to resolve synchronous vibrations. Section 1.4.2 also demonstrated that for synchronous radial vibrations of equal amplitude, a particular phase difference exists where the velocity measured by a laser vibrometer is zero and figures 6.6 and 6.7 show plots of the resolved vibrations and the error for just this situation. The errors in the resolved vibration displacement can be seen to be of the same order of magnitude as those seen previously in figures 6.3 and 6.5. This shows that this particular situation does not present a problem for this resolution method and that resolution of synchronous vibrations seems possible.

In practice the vibrometer outputs will contain noise, both speckle noise originating from the target surface and random noise originating from other sources including the demodulating electronics. Speckle noise from a rotating target has a characteristic spectrum of approximately equal amplitude peaks at the fundamental rotation frequency and its harmonics. Figure 6.8 shows a repeat of the 0.5x radial vibration simulation, shown in figures 6.3, with a simulated speckle noise, consisting of four components, and random noise added to the vibrometer outputs. The speckle noise amplitude ramps up with the rotation speed to an apparent velocity in the vibrometer outputs of 0.1mm/s, typical for a rotating target [6.2], while the random noise has a nominally constant amplitude of 0.015mm/s. A considerable, but not intolerable, increase in the calculated displacement can be seen compared to that seen in figures 6.3, 6.5 and 6.7.

Although the errors seen in figure 6.3 and 6.8 have very different amplitudes and envelopes they both have the same fundamental period. Figures 6.9a&b show the spectra of these two error signals taken from the period in which the rotor rotation speed is constant. Figure 6.9a shows that the error signal from the no noise simulation is made up
of two distinct frequencies, one at approximately rotation frequency, and another at the Nyquist frequency minus this frequency. Figure 6.9b shows the same result with, as might be expected, additional frequency content at 1 to 4 integer multiples of the rotation frequency due to the simulated speckle noise, which is indistinguishable from a genuine low level vibration. The presence of noise, therefore, does not affect the frequencies at which the fundamental errors occur, only their magnitude.

The small difference between the lower fundamental error frequency and the rotation frequency is illustrated in figure 6.9b by the dual peak which can just be distinguished around the first order, one peak at exactly rotation frequency from the simulated speckle noise and the other slightly higher peak from the simulation error. Figures 6.10a&b show a repeat of the two simulations above using twice the sampling frequency. It can be seen that changing the sampling frequency does not affect the position that the fundamental error in the spectrum, again occurring at the approximately rotation frequency and Nyquist frequency minus rotation frequency. Unexpectedly, increasing the sampling frequency, for which a reduction in the magnitude of the errors is expected, actually increases the magnitude of the errors for the noise-added simulation.

All of the spectra above have a symmetrical appearance, with the fundamental errors symmetrical about half the Nyquist frequency, which is consistent with an error at rotation frequency and an error at either the Nyquist frequency minus the rotation frequency or at the Nyquist plus rotation frequency which is then aliased. Additional simulations show that as the sampling frequency is increased further, and the accuracy of the simulation is improved, the error fundamental frequency converges to the rotation frequency.

Figure 6.11 shows a plot of the error spectrum for the simulation including a 60Hz torsional vibration with no noise. Four additional frequency components are introduced in to the spectrum at the sum and difference of both the fundamental error peaks and the torsional vibration frequency. The highest and lowest frequencies of the four again appear to be aliased.
Simulation of the resolution method has shown that the errors encountered are generally small and the frequencies at which they occur are predictable. The practical implementation of the resolution method was investigated using a test rig that allowed radial vibration of a small test rotor in the x direction only. Simultaneous measurements of the x and y radial vibration sets, using single beam vibrometers, and the rotation speed set, using a LTV, were made with the rotor running up from standstill to a constant rotation frequency of nominally 20Hz while the vibration displacement amplitude in the x direction was increased from zero to 0.6mm rms at a constant frequency of 10Hz. The period of the run-up was approximately 15 seconds and the vibration displacement in the y direction was nominally zero throughout. Figures 6.12a&b show the calculated displacements in the x and y directions respectively. It can be seen that, despite having a greater number of time steps required for numerical stability, the calculation process has become unstable. The calculated displacements for both directions are also of approximately the same form and amplitude despite there being no genuine vibration in the y direction. One possible explanation for this behaviour is the assumption that the vibrometers are perfectly aligned. Experience with the simulation has shown that a relatively modest offset in one of the vibrometers, say in x₀ only, can result in an increase in the error in the calculated displacement in both the x and y directions. It may be that resolution using numerical integration methods requires a degree of vibrometer alignment that cannot be achieved currently with the existing set-up procedures. Figure 6.13 shows the spectrum of the calculated displacement in the x direction during the period when the rotor is rotating at a constant speed. The characteristic peaks at approximately rotation frequency and at Nyquist minus rotation frequency seen in the simulations are present and are of sufficient magnitude to obscure any spectral peaks relating to the genuine vibration displacement can be seen.

Simulation of the numerical integration method for a variety of measurement situations has demonstrated its potential to resolve the genuine vibration components in two important circumstances where resolution has not been possible to date, during gross speed fluctuations and in the presence of significant torsional vibration. The ability to resolve rotor vibrations successfully in these instances would represent a significant step forward in rotor vibration measurements using LDV but initial attempts to implement the
method have proved unsuccessful with the method becoming unstable. Further investigation is required into the effects of noise and initial vibrometer alignment if the potential of this resolution method is to be realised.
7 Conclusions

7.1 Introduction

The non-contact nature of the Laser Vibrometer offers significant advantages over traditional contacting vibration transducers. This thesis has looked at the application of Laser Doppler Velocimetry (LDV) to machinery diagnostics, concentrating primarily on vibration measurements directly from rotating structures.

A comprehensive new theory describing the velocity sensed by a single laser beam incident on a rotating structure was developed allowing the vibration engineer to determine the vibration component sensitivity of any measurement with any laser beam orientation. This theory was then extended to cover measurements made with multiple beams.

The first practical method of resolving steady-state, non-synchronous, radial and angular vibration components has been presented and was used to make the first successful non-contact measurements of crankshaft vibration on a running diesel engine. The development of a more comprehensive resolution method tolerant to severe torsional vibrations and capable of synchronous vibration measurements was also initiated.

A new instrument, with a novel three beam configuration, was developed to enable simultaneous measurements of two angular vibrations to be made and was used to study bending vibration in the crankshaft study referred to previously. For measurement of torque fluctuations in rotating structures, a new measurement system was demonstrated that enables measurement without modification to the system.
7.2 Fundamentals of Laser Doppler Velocimetry measurements

Previous studies of LDV vibration measurements have predicted acceptable performance because only single vibration components have been considered and the more complex motions likely to be encountered in practice have been neglected. In order to answer several questions regarding the use of LDV for measurements on rotating structures a fundamental basis for the study of all LDV vibration measurements was required. In this thesis a comprehensive new theory describing the velocity sensed by a single laser beam incident on a rotating structure requiring three translational and three rotational co-ordinates to describe its vibratory motion fully has been developed. The theory is equally applicable to measurements on targets with simpler motions, such as non-rotating targets.

Arbitrary motion of any arbitrarily shaped structure can cause the point of incidence of the laser beam on the target surface to change. This feature of such measurements has been incorporated in the new theory with the realisation that no matter how the incident point changes position on the target it is always located along the line of the laser beam in space. Although the insensitivity of LDV measurements to target shape has previously been demonstrated for planar target motion it was proved here in the most general fashion.

In 1992 a study of LDV measurements on rotating structures [7.1] showed that the measured radial velocity included components due to the rotation itself and motion perpendicular to the intended measurement. These “cross-sensitivities” were found to be significant enough to mask the intended measurement under certain circumstances. Since this first description, which was a special case of the totally general theory presented in this thesis, there has been discussion about whether a particular arrangement of laser beams or a particular variation of the arrangement, for example by scanning the laser beam, might enable resolution of individual motion components. The new theory shows that the full expression for vibration velocity sensitivity is made up of six separate “vibration sets”, each a combination of motion parameters [7.2]. Resolution of individual motion components within each set has been shown not to be possible by any geometric arrangement or manipulation of the laser beam or by introduction of additional laser beams. The task is simplified enormously on a non-rotating target ($\Omega = 0$) but, for a
rotating shaft undergoing arbitrary motion, direct measurement of radial, axial, pitch or yaw vibration is not possible because the measurement will always be sensitive to other motion components. It may be possible to assume the effects of additional shaft motions are negligible, enabling direct measurement. For example, if the amplitudes of the vibration components are known, somehow, to be similar then the intended measurement dominates at vibration frequencies much higher than rotation frequency. In contrast, unambiguous measurement of rotation speed is possible, accepting that the torsional vibration and roll motion are indistinguishable.

A variety of optical configurations have been used in LDV instruments, many using more than one laser beam. In order to measure any one of the vibration sets it is necessary to find a geometric arrangement that isolates the desired set. For measurements of translational vibration, the $x$ radial, $y$ radial and axial sets, an appropriate geometric set-up can be found using a single laser beam. In contrast, measurements of rotational vibration, the pitch and yaw vibration sets and the rotation speed set, cannot be isolated with a single laser beam. These measurements require at least two beams [7.3].

The new theory is easily extended to cover measurements made with multiple beams for rotational vibration measurements. Examples of how the theory can be applied have been presented and have shown agreement with the specific cases considered in previous work. In particular, the usefulness of arrangements of parallel beams was highlighted as it is the only two beam arrangement that enables the rotational vibration sets to be isolated [7.4]. Measurements with parallel beams have insensitivity to translational vibration unlike cross-beam arrangements used in previous instrument designs for rotational vibration measurement. In order to place the model of velocity sensitivity in a laser vibrometer measurement in its proper context it was compared with the velocity sensitivity of a contacting transducer under equivalent conditions. The comparison served to emphasise the challenging nature of measurement directly from rotating structures and showed that even for an ‘ideal’ contacting transducer, able to maintain its sensitivity axis during rotation, important cross-sensitivities would exist [7.3].
The importance of the parallel beam arrangement, employed in the Laser Torsional Vibrometer (LTV), led to the development of a new theory to describe the difference velocity sensitivity for two parallel laser beams using ideas developed in the derivation of the theory describing the velocity sensitivity of a single laser beam. This simplifies the description of the velocity sensitivity significantly compared to previous analyses. The theory shows that in order to make measurements of the rotation speed set, with immunity to pitch and yaw vibration, the beams must be orientated perpendicular to the shaft rotation axis.

In many practical situations it is not physically possible to align the beams at the required angles due to limited access. By using pairs of parallel beams the requirement for just one or two distinct arrangements can be eliminated. A full range of versatile arrangements using two pairs of parallel beams has been presented as an alternative means to measure the pitch, yaw and rotation speed sets [7.4].

7.3 Resolution of radial and angular vibration components

For measurements on rotating structures, significant cross-sensitivities are found when measurements of radial or angular vibration are tempted. Examination of the radial vibration sets within the measured vibration velocity revealed the cross-sensitivity to rotation speed and motion perpendicular to the intended measurement seen in a previous study of radial vibration measurements. The new theory shows agreement with the two dimensional analysis used in that study but the inclusion of shaft motion in all six degrees-of-freedom revealed an additional cross-sensitivity to angular and axial vibration. A similar cross-sensitivity to that found in radial measurements was also found in measurements of angular vibration where an intended measurement of pitch vibration was seen to be cross-sensitive to rotation speed and yaw motion and vice-versa.

In order to successfully measure radial and angular vibration on a rotating structure it is necessary to resolve the desired motion component from within the particular vibration set. By making two simultaneous vibration measurements, for example, one in the $x$ direction cross-sensitive to motion in the $y$ direction and the other in the $y$ direction cross-
sensitive to motion in the $x$ direction, along with a measurement of shaft rotation speed, the measured velocities can be described as a pair of linked differential equations. Solution of these differential equations reveals the desired vibration measurements.

The differential equations governing the most general measurement are linear non-autonomous differential equations, having time dependent coefficients due to speed fluctuations or torsional vibration of the rotor. The theory of this class of equation is complex and in general an analytical solution cannot be found. In most practical situations, and with the exception of ramp-up or run-down measurements, the fluctuations in rotation speed or torsional vibrations are small compared to the mean rotation speed. By assuming that the shaft rotation speed is essentially constant the governing equations are reduced to linear constant coefficient equations which are much easier to solve.

Analytical solution of the simplified governing equations is relatively easy but the solutions are initial condition dependent. The only time that the initial conditions can be measured directly, because of the cross-sensitivity, is when the shaft is not rotating but the equations are only valid if the shaft rotation speed is constant. This means that only the trivial case of a shaft rotating at zero rotation speed can satisfy the necessary conditions and that although the analytical solutions are perfectly valid, they are of limited practical use.

Until now there has been no practical means of resolving radial and angular vibration measurements. This thesis presented a new method of resolving steady state, non-synchronous radial and angular vibration measurements. The method calculates the resolved vibration velocities on a frequency by frequency basis [7.5] and it was validated experimentally for both radial and angular vibration measurements. The development of this method has been the key to enabling radial and angular vibration measurements on rotating structures to be made successfully for the first time using LDV. An electronic processor was designed specifically to derive the necessary functions for the resolution process. This enables the resolved vibration spectra to be generated in real time on a frequency analyser, after weighting of the spectral peaks, from inputs of shaft rotation
speed and the two vibrometer outputs. Although initially difficult to calibrate, this method proved successful in practice.

In tests to validate the resolution method, an increase in the deviation of the measurements of radial vibration using laser vibrometers from those measurements made using accelerometers fixed to the motor housing was seen for vibrations of frequencies close to synchronous. It was found that in this frequency region a small error in the measurement of rotation speed resulted in a large error in the value of the weighting function and, hence, in the calculated vibration velocity. The "Error Amplification Factor", the amount by which a percentage error in the measurement of rotation speed is magnified to give an error in the calculated velocity was defined, enabling the effects of an error in the measurement of rotation speed to be quantified [7.5].

The resolution method assumes that the rotation speed of the rotor is essentially constant and that any torsional vibrations are negligible but in most practical situations torsional vibrations will be present to some extent. The error terms introduced by torsional vibrations have been described which enables their effects to be assessed from knowledge of the torsional vibration spectrum. The effects of torsional vibrations on the resolution method were simulated for a typical measurement application where torsional vibrations are present and the error terms dependent on the offsets, $x_o$ and $y_o$, were shown to be the most significant. These terms can, in theory, be eliminated by aligning the vibrometers through the centre of rotation but experience has shown that, in practice, the offsets can only be reliably reduced to 0.25-0.5mm. The need for a method to reduce these offset errors has, therefore, been identified.

A method of reducing the remaining error terms, those only dependent on the rotation speed, which is known, and the rotor displacement, which is unknown as it is part of the intended measurement, has been demonstrated. The first estimate of the resolved vibration velocity is used to make an estimate of the error terms which are then subtracted from the estimated vibration velocity to obtain an improved estimate. In order to achieve this, the input signals were recorded and post-processed on a PC. The method was validated experimentally and a useful reduction in the largest of these error velocity terms was
demonstrated. Despite showing much promise, a limit to the methods applicability has been demonstrated and the circumstances where an improvement can be guaranteed are yet to be established.

Ideally a resolution method tolerant to torsional vibrations and gross speed fluctuations is needed which would enable significant progress to be made in the measurement of rotor vibrations using LDV as it would enable accurate measurements to be made in any situation, including during run-up or run-down. In chapter 6 the application of such a method was simulated and its insensitivity to torsional vibrations and speed fluctuations demonstrated. In addition, the method was able to resolve synchronous vibration components which offers a significant advantage over the resolution method presented in chapter 3. However, practical implementation of the method proved problematic and further investigation is required to discover the source of instability in the method.

7.4 Developments in the use of LDV for vibration measurements on rotating components

At the beginning of this thesis the Laser Vibrometer was cited as possibly the ideal transducer for vibration measurements on rotating structures. The development of a method to resolve the cross-sensitivity in both radial and angular vibration measurements has enabled these measurements to be made successfully in practical situations for the first time. The ability to measure the torque fluctuations in rotating structures has also been demonstrated.

The practical area chosen to study was the assessment of crankshaft bending in internal combustion engines. This is of particular interest for NVH purposes but previous investigations have been hampered by the difficulty in making direct measurements of bending vibration and have all required modification to the engine or crankshaft. The shape immunity of the laser vibrometer measurements enabled the crankshaft vibration to be assessed from measurements of both radial and angular vibration made with the beams incident on the bolt head of the crankshaft pulley of the engine. The radial measurements were made using two single point laser vibrometers and the angular vibration
measurements were made using a new instrument, the Laser Angular Vibrometer (LAV), designed specifically for angular vibration measurements. The LAV, which has a novel three beam configuration, allows simultaneous measurements of the pitch and yaw vibration sets [7.4]. Making the measurements with a single instrument simplifies the measurement set-up and guarantees correct arrangement of beams required for the angular vibration measurement.

The resolved measurements of radial and angular vibration enabled the first natural frequency of the crankshaft in bending to be identified, along with an estimate of modal damping [7.4]. In particular, the values for the natural frequency obtained from the angular vibration measurements agreed closely with the value obtained from modal analysis of the stationary engine. This highlights the advantage of being able to measure the bending motion of a structure directly. As the first such measurements to be taken by non-contact means, the potential of laser technology for machinery diagnostics under challenging conditions is, once again, emphasised.
Appendix 1 - Circuit calibration

The circuit outputs are described by equations (3.18a&b). By applying test voltages of a known amplitude and angular frequency, in the case of the alternating voltages, the gains of electronic components within the circuit can be found.

Considering the “X” side of the circuit and alternately grounding the test signals $V_x$ and $V_y$:

$$\dot{X}_{out} \bigg|_{V_x=0} = \frac{V_\Omega K_y V_y}{\omega_c} K_{Gx}$$  \hspace{1cm} (A1.1)

$$\dot{X}_{out} \bigg|_{V_y=0} = C_2 V_x K_{Gx}$$  \hspace{1cm} (A1.2)

where $\omega_c$ is the angular frequency of the test signals $V_x$ and $V_y$. Solving for the unknown constants gives:

$$K_{Gx} = \frac{\dot{X}_{out} \bigg|_{V_x=0}}{C_2 V_x}$$  \hspace{1cm} (A1.3)

$$K_{Gy} = \frac{\dot{Y}_{out} \bigg|_{V_y=0} V_x \omega_c C_2}{\dot{Y}_{out} \bigg|_{V_x=0} V_\Omega V_y}$$  \hspace{1cm} (A1.4)

Similarly for the “Y” side of the circuit:

$$K_{Gy} = \frac{\dot{Y}_{out} \bigg|_{V_y=0}}{C_2 V_y}$$  \hspace{1cm} (A1.5)
\[ K_{el} = \frac{\dot{Y}_{\text{out}}|_{\gamma=0}}{\dot{V}_{\text{out}}|_{\gamma=0}} \frac{V_x \omega_c C_1}{V_{\pi} V_x} \] (A1.6)
References

Chapter 1

1.5 Bentley Nevada, Product data booklet, "Transducer Systems."
1.6 Micro Epsilon, Product data booklet, "NCDT Non-contact displacement transducer system."

105

1.17 Brüll & Kjær, Instruction manual, "Laser velocity transducer set type 3544."

Chapter 2


2.9. Polytec, Product data booklet, "Compact 3D vibrometer".


Chapter 3


Chapter 4


Chapter 6


Chapter 7


Figures
Figure 1.1: Two different accelerometer configurations.
Figure 1.2: Transverse sensitivity of an accelerometer.
Figure 1.3: Doppler shift of scattered light.
Figure 1.4: Typical laser vibrometer arrangement.
Figure 1.5: Vibrometer output from a rotating target. (Nominally no vibration)
Figure 1.6: Cross-sensitivity in radial vibration measurements:
   a. Motion in $x$ direction
   b. Motion in $x$ and $y$ direction
Figure 1.7: Cross-sensitivity in radial vibration measurements:

a. Accelerometer measurements  
b. Vibrometer measurements (non-rotating)  
c. Vibrometer (2x vibration)  
d. Vibrometer (synchronous vibration)
Figure 2.1: General motion of a point on a vibrating structure.
Figure 2.2: Changes in the point of incidence:
a. Translation b. Rotation c. Pitch and Yaw (Tilt)
Figure 2.3: Change in position vector caused by target motion and shape variation.
Figure 2.4: Orientation of laser beam.
Figure 2.5a: Validation of radial vibration component $\Omega \theta_z \varepsilon$
Figure 2.5b: Validation of radial vibration component $\dot{\theta}_r z$
Figure 2.6: Vibration measurements on a cantilever beam.
Figure 2.7: Cross-beam velocimeter measurement.
Figure 2.8: Change in position vectors caused by target motion and shape variation.
Figure 2.9: Orientation of parallel beam separation.
Figure 2.10: Vibration measurements using pairs of parallel laser beams: c. Pitch and Yaw sets
Figure 3.1: Circuit block diagram.
Figure 3.2: Radial vibration test rig.
Figure 3.3: Ratio of resolved vibrometer:accelerometer data:
a. x direction b. y direction.
Figure 3.4: Comparison of non-rotating target and resolved vibrometer:accelerometer data: a. x direction b. y direction.
Figure 3.5: Diesel engine crankshaft radial vibration measurement.
Figure 3.6: x radial vibration set measurement at 3000rpm.
Figure 3.7: x direction radial vibration measurement at 3000rpm:
Figure 3.8: y direction radial vibration measurement at 3000rpm:
Figure 3.9a: Resolved $x$ direction radial vibration.
Figure 3.9b: Resolved y direction radial vibration.
Figure 3.10: Resolved $x$ radial vibration at 3000rpm filtered around 1st natural frequency of crankshaft in bending.
Figure 4.1: Novel beam arrangement for simultaneous measurement of pitch and yaw vibration sets.
Figure 4.2: Optical configuration of Laser Angular Vibrometer.
Figure 4.3: Test rig for rotor angular vibrations.
Figure 4.4a: Comparison of laser and accelerometer based pitch measurements.
Figure 4.4b: Comparison of laser and accelerometer based yaw measurements.
Figure 4.5: Diesel engine crankshaft vibration measurement.
Figure 4.6: Diesel engine pitch vibration set measurement at 3000rpm.
Figure 4.7: Pitch vibration measurement at 3000rpm:
Figure 4.8: Yaw vibration measurement at 3000rpm:
Figure 4.9a: Diesel engine crankshaft pitch vibration.
Figure 4.9b: Diesel engine crankshaft yaw vibration.
Figure 4.10: Resolved pitch vibration band-pass filtered (350-550Hz).
Figure 4.11: Twist measurement.
Figure 4.12: Engine torsional vibration measurement:
  a. Crankshaft pulley  b. Alternator pulley
Figure 4.13: Rate of strain measurement in alternator drive belt:
   a. No engine load  b. Engine loaded
Figure 5.1: Error Amplification Factor.
Figure 5.2: Simulated engine vibration spectra:
Figure 5.3: Individual error terms due to torsional vibration:

a. $\Delta \Omega x_0$  

b. $\Omega y_0 \int_{t_0}^{t} \Delta \Omega \, dt$  

c. $\Omega \Delta \Omega \int_{t_0}^{t} y \, dt$
Figure 5.3: Individual error terms due to torsional vibration:

- d. $\Delta \Omega y \int_{t_0}^{\tau} \Delta \Omega dt$
- e. $\Omega \int_{t_0}^{\tau} \Delta \Omega y dt$
- f. $\Delta \Omega \int_{t_0}^{\tau} \Delta \Omega y dt$
Figure 5.4: Overall error due to torsional vibration:
a. Total error in calculated velocity  b. True vibration velocity
Figure 5.5: Overall error due to torsional vibration (zero offsets):
   a. Total error in calculated velocity  b. True vibration velocity
Figure 5.6: Simulated broadband vibration:
(a) x and y radial vibration  (b) Torsional vibration
Figure 5.7: Simulated vibration:
(a) Calculated $x$ radial vibration velocity
(b) Actual error in calculated velocity
Figure 5.8: Simulated vibration:
(a) Estimated error in calculated velocity
(b) Improved estimate of radial vibration velocity
Figure 5.9: Simulated engine vibration:
(a) Actual vibration velocity  
(b) Calculated radial vibration velocity  
(c) Improved estimate of radial vibration velocity
Figure 5.10: Simulated vibration:
  a. Actual vibration velocity  b. Calculated vibration velocity
Figure 5.10: Simulated vibration:
c. Error in calculated vibration velocity  d. Estimated error in vibration velocity
Figure 5.10e: Simulated vibration: Improved estimate of vibration velocity
Figure 5.11: Experimental vibration measurements:
a. Calculated vibration velocity  b. Improved estimate of vibration velocity
Figure 6.1: Error reduction in broadband radial vibration simulation using error correction method
Figure 6.2: 0.5 order radial vibration simulation:
  a. Rotation speed  b. Radial vibration displacements
  c. Measured velocity (x direction)  d. Measured velocity (y direction)
Figure 6.3: 0.5 order radial vibration simulation:

a. Resolved displacement (x direction)  
b. Resolved displacement (y direction)  
c. Error in resolved displacement (x direction)  
d. Error in resolved displacement (y direction)
Figure 6.4: 0.5 order radial vibration simulation with torsional vibration:
  a. Rotation speed  b. Radial vibration displacements
  c. Measured velocity (x direction)  d. Measured velocity (y direction)
Figure 6.5: 0.5 order radial vibration simulation with torsional vibration:

a. Resolved displacement (x direction)

b. Resolved displacement (y direction)

c. Error in resolved displacement (x direction)

d. Error in resolved displacement (y direction)
Figure 6.6: Synchronous radial vibration simulation:
a. Rotation speed  b. Radial vibration displacements
c. Measured velocity (x direction)  d. Measured velocity (y direction)
Figure 6.7: Synchronous radial vibration simulation:
  a. Resolved displacement (x direction)  b. Resolved displacement (y direction)
  c. Error in resolved displacement (x direction)  d. Error in resolved displacement (y direction)
Figure 6.8: 0.5 order radial vibration simulation with speckle and random noise:
a. Resolved displacement (x direction)  b. Resolved displacement (y direction)
c. Error in resolved displacement (x direction)  d. Error in resolved displacement (y direction)
Figure 6.9: Error spectra for 0.5 order radial vibration simulation:
  a. Without speckle and random noise  b. With speckle and random noise
Figure 6.10: Error spectra for 0.5 order radial vibration simulation (Increased sampling frequency):
   a. Without speckle and random noise  b. With speckle and random noise
Figure 6.11: Error spectra for 0.5 order radial vibration simulation with torsional vibration
Figure 6.12: Calculated radial vibration:
a. x-direction  b. y-direction
Figure 6.13: Spectrum of calculated radial vibration displacement