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EIGENMODES OF A SLOTTED TUBE

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The resonant behaviour of slotted tubes, clamped at one end but free at the other, has been investigated to determine the influence of slot length upon their characteristic normal modes. FEM simulations have been performed for a series of steel tubes, identical apart from having slots of different lengths, but constant equal widths, running parallel to their symmetry axes. Three dimensional visualizations of the results allowed different modal classes to be identified. The lower frequency region of the spectrum was dominated by longitudinal, breathing and transverse modes. Other classes, such as oval modes, appeared progressively as the frequency increased. The results were checked experimentally using ESPI to view the tubes while exciting them using a magnetic transducer. This provided experimental confirmation of the mode classifications. It was found that the frequencies of the transverse modes and of the breathing modes were lowered as the slot was lengthened. The frequencies of the longitudinal modes were almost unaffected by slots of any size. Where modes occurred in degenerate pairs, always as predicted by group representation theory, the splitting increased with slot length. This was by as much as 100% of the lower frequency in the case of higher order transverse modes. From the FEM simulations a previously unknown type of modal form has been observed. These "modes" appeared in families but only in cases with longer slits. They have non-zero amplitude only around the edges of the slit. This discovery may provide a greater understanding of cylinders with cracks present and their failure under loading.

1. Introduction

It has been proposed that straight tubes, each with a single slot running parallel to its symmetry axis, could be used in arrays to form acoustic crystals or barriers[1]. As a preliminary it is desirable to understand the normal modes of individual slotted tubes clamped at one end and free at the other. However, the only tube-like structures to have been thoroughly investigated are complete ones and those in which the slot is either very short or runs along the tube's complete length[2]. Research into cylindrical shells with cut-outs has been concentrated on static stress distributions and loading[3,4]. A theoretical investigation with experimental confirmation has been reported by Brogan[5], who found that the cut-out had a significant influence only on breathing and transverse modes. We report an investigation into the normal modes of slotted tubes and, in particular, the influence on them of slot length (at fixed width). This has been done by means of finite-element (FEM) simulations using COMSOL Multiphysics and the results have been checked experimentally using Electronic Speckle Pattern Interferometry (ESPI) [6].

2. Symmetry Considerations

2.1 Basic Points

It is convenient to use cylindrical polar coordinates with z-axis along the tube's axis of symmetry. Displacements in the (r, θ, z) directions will be denoted by (u, v, w). The unslotted cylinder with clamped-free ends has symmetry group $C_{\infty v}$ with the same consequences for its normal modes as for any other system with this symmetry group[7]. Thus the normal modes must occur in degenerate pairs, the angular parts of whose modal functions are sin(mθ) and cos(mθ) where m = 0,1,2,… Consequently the nodal patterns must consist of 2m lines, parallel to the z-axis and equally spaced circumferentially, and n circles parallel to the end of the tube. Exceptions are the m = 0 cases, which are singlets. The group $C_{\infty v}$ has infinitely many classes and, because the modal functions are required to form a complete orthonormal set, every one of the possible symmetry types must occur in the spectrum. There are two different singlet classes Σ^{+} and Σ^{-} which differ only in being, respectively, symmetric and anti-symmetric under reflection in planes of fixed θ. The various doublet classes C_m, where m = 1, 2,.., can be identified just by quoting their m value.

2.2 Symmetry Breaking

If an unslotted tube is truly axisymmetric then the doublets will be exactly degenerate and the absolute locations of the m nodal lines will be indeterminate, being fixed in practice by the initial conditions. If the symmetry is broken by some azimuthally localized small perturbation then, according to Rayleigh's Principle[8], the doublets will split and the nodal lines become fixed. This happens in such a way as to give pair members maximum and minimum frequency shifts respectively from the original common value. If a point mass were to be the perturbation then one member would have a nodal line running through it (minimum shift) and the other an antinodal line (maximum shift). As the mass increased so would the splitting while the modal forms would become more remote from the original forms while retaining nodal/antinodal lines at the azimuthal location of the mass. If the perturbation were, instead, a narrow slit then the result would be similar although it is not immediately obvious whether the frequencies would fall or rise. The longer the slit, the greater the effect would be.

2.3 Inextensibility Condition

From experience with bells and rings one expects that most of the modes, at least until high frequencies are reached, will be inextensional in the sense that a neutral circle in each plane normal to the symmetry axis remains unstretched throughout the vibrational cycle. This means that the radial and transverse components of the motion are related by

$$u + \frac{\partial v}{\partial \theta} = 0 \qquad (1)$$

Thus, using the θ part of the modal functions, known from group theoretical considerations, we may write $u = mA\sin(m\theta)$ and $v = A\cos(m\theta)$ where A is an arbitrary constant. So, as m increases, the modes will have radial components whose amplitudes become increasingly larger than their transverse ones.

3. The Finite-element models

The tubes used in the present study are shown in Figure1. They were made of steel and were 33cm long with the first 3cm slotted into a heavy base so that they were effectively of 30cm length with one end rigidly clamped and the other free. They had external and internal diameters of 13mm and 9.7mm respectively. The slots, when present, were of width 4mm and were lengthwise symmetrical about the mid-point of the unclamped part of the tube. The shortest slot was 4mm long and the others had lengths increasing in steps up to a maximum of 280mm.

Figure 1. Photograph of steel tubes with increasing slot length; increasing from 4mm – 280mm.

The geometry of the tube was, in each case, fed into the COMSOL Multiphysics package and the automatic element selection and meshing facilities employed. Standard elastic constants for steel were taken from the package's library and the system treated as 30cm long with one end clamped and the other free. The Eigenfrequency Analysis facility of the package was employed and the frequencies and modal forms of a large number of modes were generated in each case. For our present purposes attention was restricted to modes with $n < 11$ and frequencies below 12 kHz. This was partly because of difficulty in interpreting the modal forms of higher orders as the slit became longer.

4. Mode types in the basic tube

Because of the completeness requirement one would expect to find modes of all possible symmetry types amongst the FEM predictions. In fact four main types were found, as listed below and shown in Figure 2(a-d):

(a) Breather modes (type Σ^+) where the motion is radial and the median surface is stretched circumferentially but not longitudinally. As expected they were all singlets.

(b) Longitudinal modes (also type Σ^+) where the tube is stretched in the axial direction only. Again they were all singlets.

(c) Transverse modes (type C_1) where the tube behaves like a cantilever. As expected these were in degenerate pairs with their directions of vibration at right angles to each other. For these m=1 modes *u* and *v* are equal in amplitude, in accordance with Eq. 1.

(d) Oval modes (type C_2) again occurred in degenerate pairs but were much higher in frequency than the modes of corresponding order of the above three types. From the example shown in Figure 2(d) it can be seen that, as in bells[9], these modes exhibit evanescent-type behaviour as one moves away from the open end. The oval appearance in planes of fixed z is due to the amplitude of the *u* component now being double that for *v*.
Modes of symmetry types C_m where m>2 were also found but were at frequencies well above the corresponding C_2 modes. Also the region of evanescence moved ever closer to the open end as m increased. No predictions were found for torsional modes (type Σ^-) but this was probably because we did not go high enough in frequency.

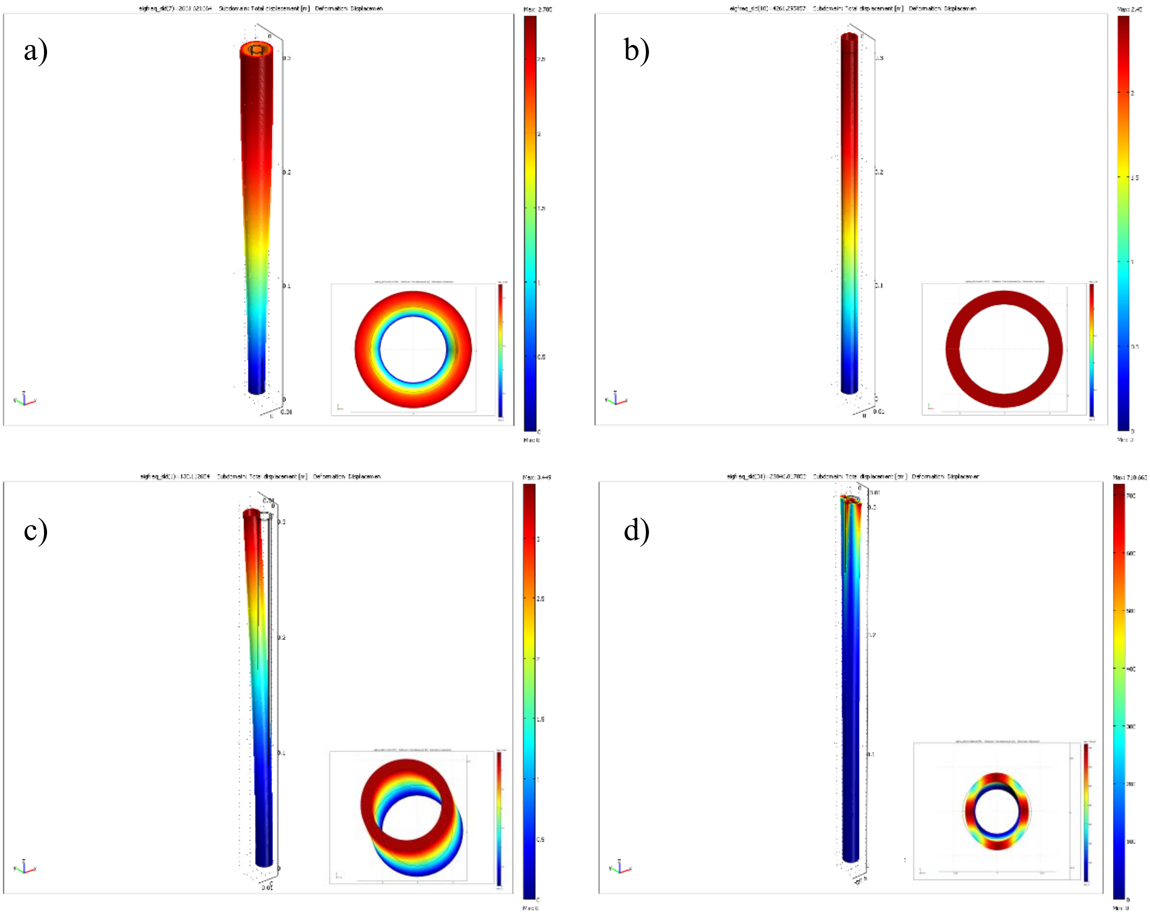

Figure 2. Modes of symmetry types for a cylindrical tube. a) Breather Modes, b) Longitudinal Modes, c) Transverse Modes and d) Oval Modes. Note: Deformation (Max displacement = Red, No displacement = blue) is exaggerated for clarity.

5. Experimental procedure

Initial testing involved finding the modes of the complete (unslotted) tube. This was achieved by exciting the steel with a magnetic transducer driven by an oscillator (B&K type 1022). The excitation frequency was slowly increased and the frequencies and nodal patterns observed at each resonance point. The procedure was then repeated in turn for tubes with slots of increasing length. For short slot lengths it was easy to follow the way in which modes mapped from one tube to the next but, as the slot lengths increased, some patterns became too complicated to identify with certainty. It was therefore decided to cut off the investigation at 12 kHz, corresponding to a maximum n value of about 10.

The nodal patterns were observed experimentally by using EPSI. We employed the subtraction method[10] of this non-contact real-time technique. This method reduces the effect of background noise and, although insufficient for making quantitative measurements of the modal forms, provides a very convenient way of visualizing them for the purpose of comparison with FEM results.

6. Results and Discussion

6.1 Longitudinal modes

The theory for longitudinal vibrations of bars of arbitrary but uniform cross-sectional shape is well known [11]. If the bar is fixed at one end and free at the other then the frequencies f_n of the normal modes are given by

$$f_n = \frac{2n-1}{4L}\sqrt{Y/\rho} \qquad n = 1, 2,\ldots \tag{2}$$

Here L is the length of the rod, while Y and ρ are respectively the Young's modulus and density of its material. No information about the cross-sectional shape appears in this equation.

Since our unslotted tube is just a bar of unusual cross-sectional shape, a graph of f_n vs. ($2n$-1) should be linear with slope as indicated in Eq. 2. Such a plot is included in Figure 3, where we show the results of both the FE calculation and experiment. The agreement between all three is very good.

As Eq. 2 applies to a bar of *any constant* cross-sectional shape, the results for a tube with the slot running along its *whole* length should be identical to those for the complete tube. Careful examination of the FEM results shows that, when the shortest slot is introduced, and the constant cross-sectional requirement is slightly broken, all the longitudinal mode frequencies rise slightly. As the slot length increases, and the constancy violation increases, the frequencies continue to rise to a maximum of about 2% (when the slot extends along about half the tube's length) and then fall back down again towards the original unslotted values. This is because Eq. 2 applies exactly only to the two extreme cases. In Figure 3, for clarity, we show only the unslotted and maximum shift cases but agreement between FEM and experiment was good in all cases.

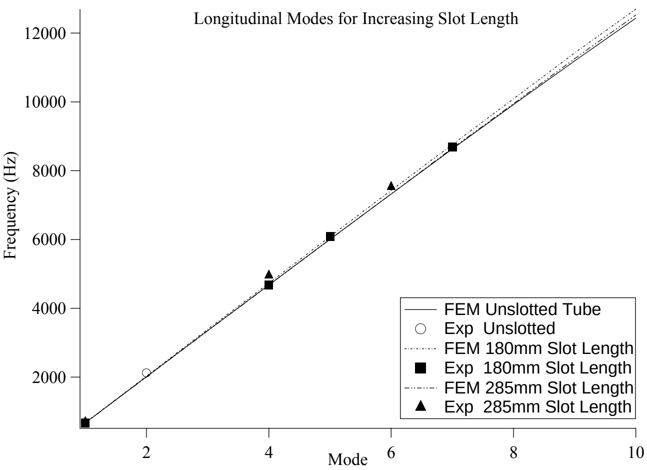

Figure 3. FE calculation and experimental results for longitudinal modes.

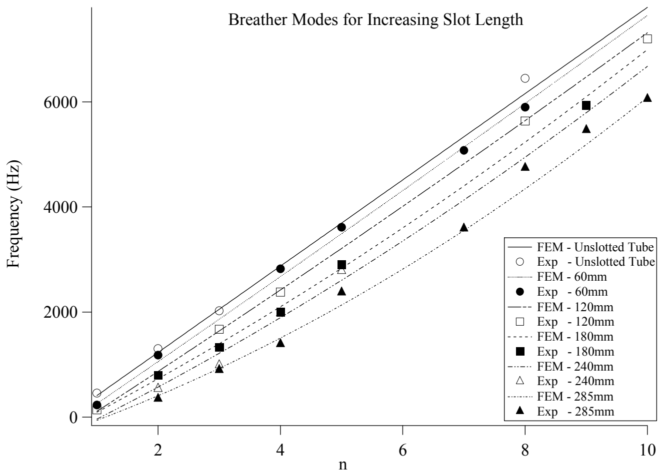

Figure 4. FE calculation and experimental results for breather modes with increasing slot length.

6.2 Breathing modes

The FEM predictions and experimental results for these modes for the complete tube are included in Figure 4 where they are seen to be in good agreement, with frequency increasing linearly with n, as one would expect. Since adding a small mass to the tube would certainly cause the frequencies to fall, due to increased inertia, one might anticipate that a slot would cause them to rise. However, according to thin shell theory[12], while the inertia is proportional to thickness h, the stiffness varies like h^3. Thus cutting metal away will have the net effect of causing all the frequencies to drop. The longer the slit the greater the drop will be. In Figure 4 we show the results for all the slot lengths investigated. Everything is more or less as expected, with the variation of frequency with n being more or less linear.

The modal functions for these modes are quite revealing. A consequence of the breaking of the axial symmetry is that torsional effects come into play. As the slot length increases so does the complexity of the modal forms.

6.3 Transverse modes

According to conventional cantilever theory for beams with arbitrary but *constant* cross-section the modal frequencies are given by[11]

$$f_n = \frac{\pi c \kappa}{8L^2}\left(1.194^2,\ 2.988^2,\ 5^2,\ 7^2,\ \ldots\ \right) \tag{3}$$

Here $c^2 = Y/\rho$ and κ is the area radius of gyration about the appropriate neutral plane direction. Thus a graph of f_n vs. $(2n-1)^2$ should be linear, except at low values of n. Figure 6 shows this to be reasonably well satisfied for both FEM predictions and experiment. For the complete tube, κ takes the same value for all possible azimuths so the frequencies are the same for any direction in which the initial conditions launch the vibrations. The degenerate pair arises because only two of these modes are linearly independent. Experiment shows that, as one would expect, imperfections in the tube have already produced small breakings of the symmetry so the pairs are already slightly split and the orientations of the nodal lines fixed.

Once a small slot is introduced the value of κ becomes θ-dependent in the slotted region. Its value is reduced for all directions, but is now largest about the slot and least at right angles to it. According to Rayleigh's principle it is the directions of maximum and minimum κ at which the normal modes will occur. Thus the frequencies of both doublet members will fall, but one more than the other. As the length of the slot is increased the new values of κ will apply to an increasing proportion of the tube's length so the frequencies will continue to fall and each doublet will become increasingly split. The FEM calculations confirmed these ideas and were in good agreement with experiment in all cases. In Figure 6, for clarity, we show only the cases of the unslotted tube and that with the greatest slot length. The other cases were intermediate between these two. Regarding the nodal patterns, it was found experimentally that, due to the manufacturing imperfections present in all the tubes; it was only when the slot became reasonably long that the patterns became fully aligned to the slot direction. For shorter cases the basic imperfections could swamp the orientation due to slot position.

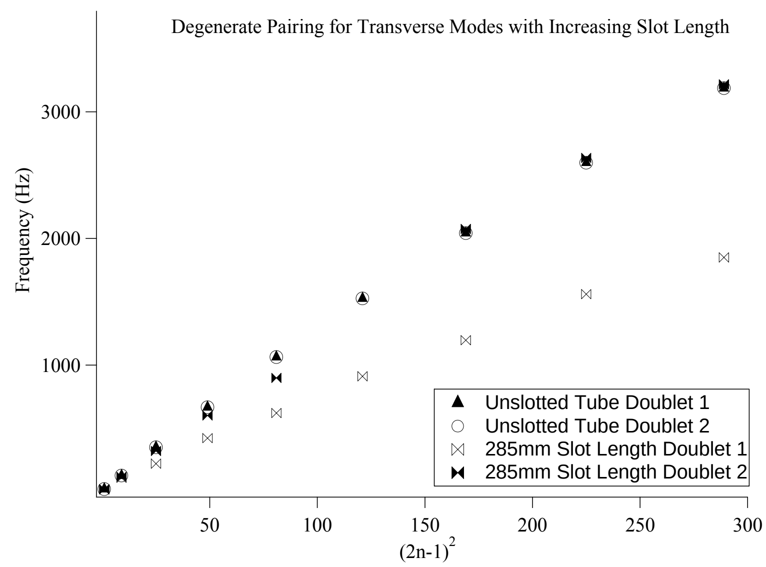

Figure 6. Degenerate pair splitting dependency on slot length for transverse modes.

6.4 Oval modes

These doublet m=2 modes are of great importance in bells but with relatively long, thin tubes such as those in the present study they do not appear in the spectrum until relatively high frequencies are reached. A plot, included in Figure 7, of f_n vs. n for the complete tube shows that frequency increases steadily with n, slowly at first but then more rapidly. Introducing the shortest slot causes all the pairs to split, with both members falling slightly. Increasing the length of the slot further then causes the frequencies to rise slowly but steadily for lower values of n. However, as n increases, the frequencies increase more and more rapidly as the slot length l goes up. Overall the splitting for a given n value increases with slot length. When the length becomes large the splitting then increases very rapidly with n. It would appear that the frequencies begin to increase rapidly when the wavelength of the z-variation of the modal function becomes comparable with the length of the slot. For wavelengths below this value the mode presumably "sees" the slot but does not do so above it. This point will occur when

$$\frac{l}{L} \approx \frac{4}{2n-1} \tag{4}$$

Where l is the slot length and L is the length of the unclamped part of the tube. This occurs at lower and lower values of l as n increases.

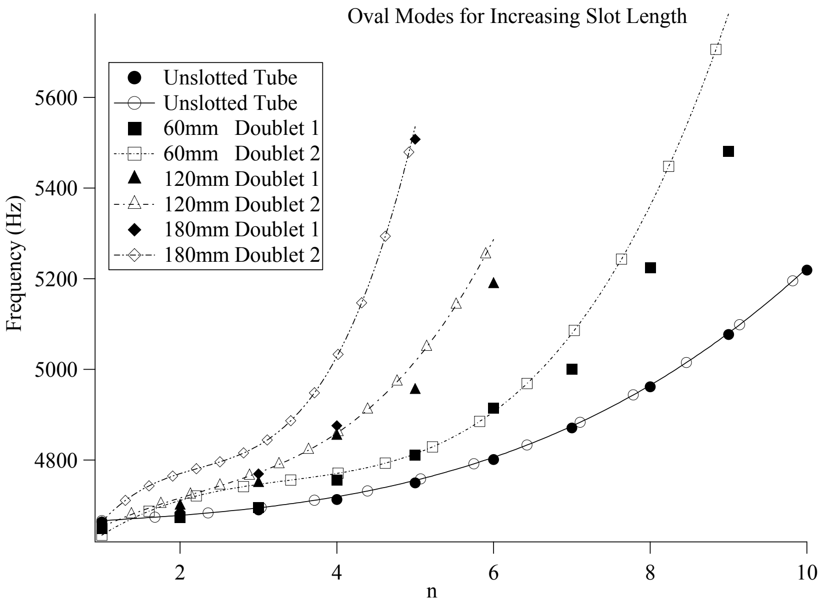

Figure 7. Degenerate pairing of oval modes with increasing slot length.

6.5 Flange modes

The FEM simulations predicted a previously unknown class of modes that we have coined "flange" modes. They appeared only in the presence of a slot of sufficient length. The deformation was localised to the region immediately adjacent to the slot, with no other movement being observed. Points of maximum and minimum deflection occurred along the straight edges of the slot. Examples are given in Figure 8 where we show the slot side and its obverse separately. For the tubes used in this study no flange modes were observed until the slot length reached 60mm. Thereafter, for a given slot length, there occurred a series of these modes with an increasing number of nodal "circles" whose frequencies increased with "n". This increase became roughly linear when n became large enough. The greater the slot length the larger this value of n needed to be. As the length increased so the frequency of the flange mode with a given value of n fell. The mechanism underlying these modes remains unclear but we hypothesise that Lamb waves may be involved. It is intended to present the results of a further study of these modes in a future paper.

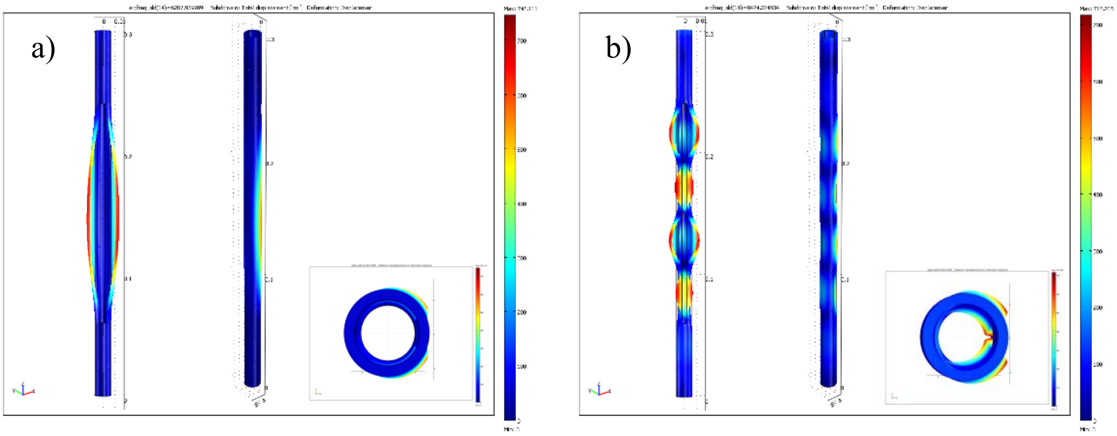

Figure 8. Observation of "flange" modes so non-zero amplitude deformation, localised to the region of the slot, a) first mode, b) fourth mode. (Max displacement = Red, No displacement = blue).

7. Conclusions

It has been shown that most of the normal modes of narrow slotted straight circular tubes can be understood as perturbations on the modes of the corresponding unslotted case. Although the spectrum is dominated over the first few kHz by longitudinal, breathing and transverse modes, some other types, expected from group theory, were also found at higher frequencies. Such modes as were still missing are expected to occur at frequencies higher than we investigated. The doublet modes all split and the nodal patterns of the members lined up, more or less as expected, with the slot's location. When the slot was short it was easy to follow the mapping of a given basic-case mode onto the slotted cases. However this became increasingly difficult as the slot length increased, especially at higher values of n. The variations in frequency with n and with slot length could be explained, at least qualitatively, in the cases of longitudinal, transverse and breathing modes. Perhaps the most interesting aspect of the study was the identification of a new type of "flange" mode, where motion is restricted to parts of the cylinder very close to the slot. These modes appear only when the slot is reasonably long. They merit further investigation.

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