Electrodynamics of fluxon and semifluxon in 2D T-shaped Josephson Nano-Junctions

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Electrodynamics of Fluxon and Semifluxon

in 2D T-shaped

Josephson Nano-Junctions

by

Hanaa S. Hassan

Doctoral Thesis
Submitted in partial fulfilment
of the requirements for the award of
Doctor of Philosophy of Loughborough University

November, 2010.

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Publications


Abstract

Dynamic properties of Josephson junctions are interesting due to the emission of high frequency radiation (up to THz range) from Josephson junctions, closely related to fluxon dynamics. A better understanding of this dynamics can help to improve the Josephson devices used for applications. Josephson junctions can also be of great use as T-shaped multiple Josephson junctions in Josephson electronic circuits. In general, T-junctions consist of two attached Josephson transmission lines: a main Josephson transmission line (MJTL) along the $x$-axis, and an additional Josephson transmission line (AJTL) along the $y$-axis. These junctions can use to create fluxons (solitons) in junctions without applied magnetic field, (called flux cloning phenomenon). This work is devoted to contributing to a clarification of the dynamic behaviour of solitons (fluxons) in 2D extended conventional T-shaped Josephson junctions (extended means an AJTL is larger than MJTL). A conventional T-junction is a MJTL along the $x$-axis which divides into two Josephson transmission lines along the $x$ and $y$-axes. In addition, we also attempt to elucidate further the concept of flux cloning in rotated T-junctions, which are $90^\circ$ anticlockwise rotation of conventional T-junction. In rotated T-junction, a MJTL along the $x$-axis divide into two Josephson transmission lines along along the $y$-axis. We find the first evidence of moving semifluxon and observe for the first time new phenomena of semifluxons and anti-semifluxons in both extended conventional and rotated T-junctions. We numerically study the electrodynamics behaviour of solitons in the standard T-shaped Josephson junction (conventional T-junction) in a magnetic field. Therefore, we describe theoretically how flux cloning circuits exist and give an opportunity for use as flux flow oscillators operating without applied magnetic field. The results that emerge give further support to the flux cloning mechanism.

Keywords: Josephson junctions, T-junction, flux cloning, fluxons, semifluxons.
Acknowledgements

During my graduate studies at Loughborough University, I have collaborated and worked with several professors and persons directly and indirectly for my research and when things became difficult and complicated, I was always being encouraged by different family members and friends. Without their support and help, it would definitely have been impossible for me to complete the research work and my thesis. I therefore want to dedicate this section to thank them and express my appreciation for their support.

Firstly, I would like to thank ALLAH for giving me the strength and health to carry on throughout the years to complete this work. Additionally, I would like to express my gratitude to my research supervisor Feo Kusmartsev for believing in me, giving me the opportunity to work with him, for guidance, advices, support, and encouragement. A further word of thanks goes out to Dima Gulevich, for very helpful comments and useful discussions. A special word of thanks should be directed to my senior colleague Olga Kusmartsev, who advised me and helped a lot to answer any questions. I am also grateful to Daniel Elford for proofreading the thesis.

A warm word of thanks is directed to my parents, for their love, for always being there for me and stimulating me in my studies and work. I would further like to express my sincere thanks to my sisters, brothers, nieces, and nephews who have always supported me from the beginning of my studies until the end and for all the fun we had. I would like to thank all the rest of my family and friends, who were always supportive during the years of my PhD, for just being who they are.

A last, but certainly not least, word of thanks is reserved for the Physics departments’ members especially Maureen McKenzie, Victoria Webster and Martin Stenlake for their help in everything.
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<tbody>
<tr>
<td>JTL</td>
<td>Josephson transmission line</td>
</tr>
<tr>
<td>AJTL</td>
<td>Additional Josephson transmission line</td>
</tr>
<tr>
<td>MJTL</td>
<td>Main Josephson transmission line</td>
</tr>
<tr>
<td>$\lambda_J$</td>
<td>Josephson penetration depth</td>
</tr>
<tr>
<td>$\phi$</td>
<td>The superconducting phase difference</td>
</tr>
<tr>
<td>PSGE</td>
<td>Perturbed Sine-Gordon Equation</td>
</tr>
<tr>
<td>$I_c$</td>
<td>The critical current</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>Damping coefficient</td>
</tr>
<tr>
<td>$u$</td>
<td>Normalized velocity in junction</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>The normalized external current</td>
</tr>
<tr>
<td>$\omega$</td>
<td>Frequency of oscillations in units of $\omega_p$</td>
</tr>
<tr>
<td>$h$</td>
<td>The normalized external magnetic field</td>
</tr>
<tr>
<td>$\phi_x$</td>
<td>Magnetic field component in 0 LJJ</td>
</tr>
<tr>
<td>$\mu(x)$</td>
<td>Magnetic field component in 0- $\pi$ LJJ</td>
</tr>
<tr>
<td>$H_x$</td>
<td>Magnetic field component in $x$-direction</td>
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<tr>
<td>$H_y$</td>
<td>Magnetic field component in $y$-direction</td>
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<tr>
<td>$</td>
<td>\nabla \phi</td>
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<table>
<thead>
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<tbody>
<tr>
<td>$LJJ$</td>
<td>Long Josephson junction</td>
</tr>
<tr>
<td>$W$</td>
<td>Width of AJTL</td>
</tr>
<tr>
<td>$W_0$</td>
<td>Width of MJTL</td>
</tr>
<tr>
<td>$\lambda_L$</td>
<td>London penetration depth</td>
</tr>
<tr>
<td>$\phi_0$</td>
<td>Magnetic flux quantum</td>
</tr>
<tr>
<td>SGE</td>
<td>Sine-Gorden Equation</td>
</tr>
<tr>
<td>$V$</td>
<td>ac-voltage in junction</td>
</tr>
<tr>
<td>$\bar{c}$</td>
<td>Swihart velocity</td>
</tr>
<tr>
<td>$u_c$</td>
<td>The critical velocity</td>
</tr>
<tr>
<td>$\gamma_c$</td>
<td>The critical driving current</td>
</tr>
<tr>
<td>$\omega_p$</td>
<td>Plasma frequency</td>
</tr>
<tr>
<td>$h$</td>
<td>The reduced Planck constant</td>
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<tr>
<td>ZFSs</td>
<td>Zero field steps</td>
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<tr>
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<td>Fiske steps</td>
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<tr>
<td>FFS</td>
<td>Flux flow steps</td>
</tr>
<tr>
<td>FFO</td>
<td>Flux flow oscillator</td>
</tr>
<tr>
<td>CC</td>
<td>Cloning circuit</td>
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Chapter 1

Introduction

1.1 Research Aim

Solitons have attracted a great attention for many years, because they can be excited and observed in diverse nonlinear systems. Nowadays, there is growing interest in the usage of fluxons and semifluxons in a variety of Josephson nano-junctions, in particular for quantum computing. A fluxon, which is a Josephson vortex carrying a magnetic flux quantum $\phi_0$, is created in a junction by applying magnetic field. A semifluxon, is a spontaneous Josephson vortex carrying a half of the magnetic flux quantum $\phi_0/2$, appears in 0- $\pi$ junction. Therefore, the properties of semifluxons are very different from the properties of fluxons. Recently, there has been the appearance of a new important phenomenon, which is called flux cloning. Flux cloning can be generated in junction without applying magnetic field. Flux cloning can be achieved by means of, for example, T-shaped Josephson nano-junction. In general, however, there is still a rare clarifications about this phenomenon. At the start of my PhD research, there were several physical questions for effecting of size of geometry T-junction on fluxon dynamics, which no adequate answer had been found. For instance, the natural question is: Is there any constriction for the widths particularly if the width AJTL becomes extensive? Also if the conventional T-junction is rotated counter clockwise through $90^0$, does flux cloning still occur? In addition, how the processing of cloning is going in the present external magnetic field when there is dissipation of energy in system. The work, presented in this thesis, is the result of a few years of research on the vortex dynamics in T-shaped Josephson junctions.
1.2 Thesis Structure

In this thesis the computer simulations of the vortex in T-shaped long Josephson junctions are carried out by using the finite-element method to solve the sine-Gordon equation (SGE) with boundary conditions and with the initial condition associated with the soliton located at the coordinate $x_0$ or soliton entering into junction by applying magnetic field. The structure of this thesis can be summarized as follows, see Fig. 1.1:

**Chapter 2** contains a short introduction to the Josephson junction dynamics (conventional Josephson junction), the wave equation model describing the dynamics of Josephson junctions and static and dynamic properties of Josephson junctions. In addition, it provides a review of physics of the $\pi$ Josephson junction dynamics, semifluxon and their properties. The theory of semifluxon relevant for the topics discussed in this thesis is given. Moreover, this chapter includes theoretical description for electrodynamics of soliton in conventional T-shaped Josephson junction. It contains conditions of vortices for cloning.

In the **Chapter 3** we describe briefly then the technical details of the numerical simulation techniques. Some simplified junctions allowing for a better understanding of the junction dynamics are modelled.

The following **Chapter 4** is the first results chapter. The purpose of the work is to demonstrate the existence of a new novel phenomenon for semifluxons carrying half-integer magnetic flux quantum in extended both kinds of T-shaped Josephson junction. We present the first evidence and observation of moving semifluxon which may arise in a manner similar to normal fluxon. In addition, we show here that semifluxon can be created and moved in extended Josephson junctions by means of flux cloning phenomenon arising in both kinds of T-shaped Josephson junctions. For this proposal, the effects of widths are studied by a set of 2D geometries with varying JTL widths. In the first part, we study the effects of widths in conventional T-junctions in case $W > W_0$. In the second part, we propose a new kind of T-junction, which is a rotated conventional T-junction counter clockwise through $90^\circ$ and attempted to estimate theoretically and numerically the critical velocities. The differences and similarities between different models, as well as conditions for flux cloning are discussed. In addition, we study numerically the variety of dynamic behaviours of the flux that can be expected in a rotated T-Josephson junction system. A discussion and conclusion of the behaviour of soliton is given in the end of each parts.

Next **Chapter 5** is the second results chapter, which is devoted to investigation of the variety of dynamic behaviours of flux cloning that can be expected in T-shaped Josephson...
junctions with the energy dissipated, $\alpha \neq 0$, from the wave system. In section 5.1, the anomalies of flux cloning, in both T-shaped and rotated T-shaped junctions have been studied without applied magnetic field and bias current. The changing thicknesses of T-junctions lead to interesting effects in terms of their dynamics. In section 5.2, we confine our attention to find experimentally and described theoretically how fluxon cloning circuits can be used as a flux flow oscillator (FFO) operating without external magnetic field from linear overlap geometry. The numerical simulations focus on the flux cloning of conventional T-shaped junction when a soliton is generated in main junction (MJTL) by applying magnetic field. Therefore, each time the vortex passes the T-junction, then vortex cloning arises and the new cloned vortex moves along the transverse branch of the T-junction (AJTL), which can be used as a FFO operating without external magnetic field. Using numerical simulations the current-voltage characteristics of Josephson junctions of different magnetic field were measured. We collaborated with experimental scientists for experimental studying. These are Valery Koshelets and Pavel Dmitriev, Institute of Radio Engineering and Electronics (IREE), Moscow, Russia.

Chapter 6 concludes the main body of the thesis. Finally there are bibliography and then list of publications and contributions to conferences. The most works in this thesis have already been published (Flux Cloning Anomalies in damped Josephson nano-junction [fHK09e] and Spontaneous Movable Semifluxon Generation-New Phenomena in nanoelectronic superconducting system [HK10f]). The latest Rotated T-junction results have been submitted to New Journal of Physics for publication (Modelling of the high energy particle collisions in laboratory with the use of fluxon dynamics in T-Josephson Junctions [HK10e]). The FFO with fluxon cloning circuits results have been submitted for publication in Springer (Flux-Flow Oscillator (FFO) made with Fluxon Cloning Circuits [HGD10]). The latest FFO with fluxon cloning circuits results is still prepared for publication (Terahertz Generation from Nanostructures Superconducting Fluxon Cloning Circuit [HKK11]). Work contained in this thesis also contributes in a number of conferences (see list of publications and contributions to conferences).

Present work devoted to investigation of T-shaped Josephson junctions was distinguished by:

1. The First Prize Award For Outstanding Poster on Experimental And Theoretical Condensed Matter Physics and Nanoscience in 32nd International Workshop on Condensed Matter Theories [fHK08]. This progress was highlighted in [Kus09].
2. Distinguished Achievement in SIC03 held at University of Surry in June 2009 [fHK09b, fHK09c].

3. A Golden Prize Award Of the Best Poster Competition in applied science in SIC04 held at Manchester University in July 2010 [fHK10b].
Chapter 2

The Physics of Long Josephson Junctions

In this chapter the basic features of long Josephson junctions (LJJ) are briefly reviewed, what a long Josephson junction is and the electrodynamics of soliton in LJJ. In addition, the equations governing the complex and very interesting electrodynamics of long Josephson junctions are presented. The effects of applying an external magnetic field and bias current that are of importance for later chapters are discussed in some detail. In section 2.2, a brief description of $\pi$ Josephson junctions will be given what a $\pi$ long Josephson junction is and how this junction can be used to create spontaneous semifluxons. The most interesting of their properties and important effects will be discussed. The last section, 2.3, is a brief review of the basics of T-shaped Josephson junctions and the flux cloning phenomenon, which is the basis of the functionality in this thesis, will be introduced.

2.1 Long Josephson Junctions in 2D

A long Josephson junction, in which one direction (the $x$-direction) is large compared to the Josephson penetration depth, $\lambda_j = \sqrt{\phi_0 / 2\pi \mu_0 d J_c}$ ($d$ magnetic field thickness and $J_c$ critical current density), is a structure consisting of two superconducting electrodes, for example Nb, separated by a thin insulating layer (weak link) such as an oxide layer, usually of the order of 2-3 nanometers in thickness [CK88, Wal97]; see Fig. 2.1. In this weak link, which is called Josephson junction, electrons are able to tunnel through the barrier from one superconductor to another. This effect is called the Josephson effect, predicted by the British physicist Brian D Josephson in 1962. There are several types of long Josephson junctions: overlap, in-line and annular junctions geometry (Fig. 2.1 (b)). In this thesis, the junctions are patterned in an overlap geometry. For a detailed discussion of the junction geometrical
configuration, see Barone and Paterno [BP82]. The very long Josephson junction is also called the *Josephson transmission line* (JTL) in which the soliton can stay and move freely along [GK06, LKM’85]. As the width of the LJJ is much smaller than $\lambda_j$, $W \ll \lambda_j$, one can use a one-dimensional (1D) Josephson junctions. However, when the width of the junction is comparable with or larger than $\lambda_j$, $W \geq \lambda_j$, the system evolves to a large junction [CFG’96] (long Josephson junctions in 2D), referred as extended Josephson junction [Gué75] or 2D Josephson junctions [ELO’85, LFD’93, LOE’85, SZZ04].

The Josephson vortex, that exists in large or long Josephson junctions [LSL’05], is so-called a fluxon or soliton which carries a magnetic flux quantum ($\phi_0 = h/2e = 2.07 \times 10^{-15}$ Wb). The motion of Josephson vortices may be as fast as the speed of light and this has been exploited in various devices [BSY91, HWK’97, KP03, RN94, Ped91, Ust98b]. Many applications have been proposed of Josephson transmission lines (JTLs) such as digital computer logic [CGR’07, Hay84, MSS00] and memory functions [GMW80, Hay84]. Recently there has been a great deal of interest in the study of single flux quanta ($\phi_0$) on the JTLs [RLK’93] because of their potential applications in electronics [OUP93, Rai05, RN94, TYK’01], digital logic circuitry [Gué75, Kos91, NO78, NOO76, SK99]. Josephson junctions are used as potential switching elements for ultra-fast computers because they switch very fast, at extremely low power levels [Zap77]. The idea to use Josephson junctions for digital devices is based on the very short switching time from the superconducting to the normal state [Kos91]. In addition, the Josephson junctions use as local oscillators [BSY9, Jaw08, KSB’95, KSS93, MKM08, MP99, NEI’88, TUK06] for microwave detection or logical elements in high-speed super computers [LFD’93]. The investigation of 2D Josephson junctions is motivated due to one drawback of 1D junctions for local oscillators is the small power available or too large linewidth of the emitted radiation. Large junctions (2D junctions) are more promising because they carry a higher current, thus increasing the power available [LFD’93, Ped91].
2.1.1 Tunnelling Regimes in Josephson Tunnel Junctions

Different tunnelling processes in a Josephson junction can be identified by analyzing its current voltage characteristic (IV curve), which gives a complete picture of the dynamics in a long junction due to the Josephson effect, a flow of quasi-particles, and a current due to capacitative effects [Vis02, Wal97]. A standard measured IV curve of an underdamped Josephson junction is shown in Fig. 2.2. Four major tunnelling regimes can be identified:

1 Cooper pair tunnelling (or Zero voltage ZV): \((a \rightarrow b)\): In the absence of an applied voltage and starting from zero current, the zero voltage (ZV) current branch corresponds to the tunnelling of the Cooper pairs between the superconductors. The flow of current is called a Josephson current \((I_J)\) or a critical current constant that is the maximum supercurrent that can flow through the Josephson junction without bias current. Since this tunnelling happens in the absence of a potential difference between the superconductors, there will be zero...
resistance and the supercurrent flows without any voltage drop over the junction (see Fig. 2.2). This phenomenon is known as the dc-Josephson effect [Thy99, Wal97, Wei07].

The electrons in one superconductor can be described with one wave function. The phase difference (\(\phi\)) between the wave functions in the two superconductors induces a supercurrent (\(I_s\)) given by

\[
I_s(\phi) = I_c \sin(\phi) \quad (2.1)
\]

\[
\phi = \arcsin\left(\frac{\phi_0}{I_c}\right) = \arcsin(y) \quad (2.2)
\]

2 Gap voltage (GV): (\(b \rightarrow c\)): In a bulk superconductor, all electrons are condensed as Cooper pairs in a single quantum state. A Cooper pair is a bound state of two-electrons and has a charge -2e [Thy99]. If a bias current applied to the junction reaches the critical value \(I_c\), the lossless current transport breaks down and the system jumps into a voltage state that corresponds to the double gap energy \(2\Delta/e\). The gap energy expresses the binding energy of the Cooper pairs [Thy99, Wei07].

(c \(\rightarrow d\)): For the voltage \(V = 2\Delta/e\), the tunnelling current is dominated by quasiparticles, i.e. single electrons [Thy99, Wei07]. The gap voltage corresponds to the overlap of the quasiparticle densities of states of the two superconductors, giving rise to a large increase in the tunnelling current [Wal97].

3 Normal tunnelling (N) (\(d \rightarrow e\)): A further increase of the bias current merely displays Ohm’s law at \(V > 2\Delta/e\) with normal resistance \(R_n\) and direct tunnelling of single electrons. Therefore, a normal current will flow across the junction and the phase angles will increase linearly over time. This is called the ac-Josephson effect, because the voltage related to this normal current across the junction depends on time, i.e. it is an ac-voltage.

\[
V = \frac{h}{2e} \frac{\partial \phi}{\partial t} = \frac{(\phi_0)}{2\pi} \frac{\partial \phi}{\partial t} \quad (2.3)
\]

and the current will be an AC current with amplitude \(I_c\) and angular frequency

\[
\omega = \frac{\partial \phi}{\partial t} = \frac{2eV}{h} \quad (2.4)
\]

An applied dc current of 1 mV will produce a frequency \(f = \omega/2\pi = V/\phi_0 = 483.6\) GHz, which lies in the far infrared region [Wei07].

4 Sub-gap quasiparticle tunnelling (SG): (\(e \rightarrow f\)) When the bias current is decreased a hysteretic behaviour is observed. The tunnelling current remains to be carried by
quasiparticles and for zero temperature the current should decrease to zero before the system jumps into the zero voltage state again [Wal97]. (f → a) Upon decreasing the bias current such that the voltage drop across the junction is less than the gap voltage, a small sub–gap current is observed. This sub–gap current appears at finite temperatures and is due to the quasiparticles excited above the energy gap [Thy99].

**Fig. 2.2:** Typical I-V characteristic of Josephson tunnel junctions with different tunnelling regimes. Different tunnelling regimes are indicated by labels (ZV – zero voltage, SG– sub–gap, GV – gap voltage, N– normal tunnelling). (CP → CP) means Cooper pair tunnelling. (QP → QP) means quasiparticle tunnelling, $2\Delta/e > V > 0$. (CP → QP) means cooper pair dissociation and tunnelling into quasiparticle states, $> 2\Delta/e$. 
2.1.2 Electrodynamics of Long Josephson Junctions

The dynamics of fluxons in long Josephson junctions (LJJ) has received an increasing attention over the last decade because of possible applications in superconducting electronics [OUP93]. In general, the electrodynamics of the motion of a fluxon in LJJ is assumed to be described by a (2+1)-dimensional perturbed sine-Gordon equation (PSGE) together with the appropriate boundary conditions for the quantum mechanical phase difference, \( \phi(x, y, t) \), when both external current and magnetic field are applied. The magnetic field and the external current enter as a boundary condition on a perturbed sine-Gordon equation (PSGE) [BP82, CFG’96, LOE’85, LSC82b, Par93, Ust98a]. In normalized form this equation can be written

\[
\phi_{xx} + \phi_{yy} - \phi_{tt} - \sin(\phi) = \alpha \phi_t 
\]  

(2.5)

Here, \( x \) and \( y \) are two spatial variables normalized to the Josephson penetration depth \( \lambda_j \), \( t \) is time normalized to the inverse of the plasma frequency \( \omega_p \) (\( \lambda_j \omega_p = \bar{c} \), the maximum electromagnetic propagation velocity in the junction, called Swihart velocity or a speed of light), subscripts denote partial derivatives and \( \alpha \) is a dissipation coefficient (or the damping coefficient), which is assumed to be a real number with \( \alpha \geq 0 \). When \( \alpha = 0 \), eq. (2.5) reduces to the undamped sine-Gordon equation in two space variables, while when \( \alpha > 0 \), to the damped one.

The sine Gordon equation with its boundary conditions may appear in different ways, depending on the geometry of the junction and external conditions. The most often used geometry is the so-called overlap geometry described by eq. (2.5) with the boundary conditions, which depend on applied magnetic field and applied current [BP82, ELO’85, Par93, Ped86, PU95]. In an overlap structure, as shown in Fig. 2.3, the normalized total length and width of junctions are \( L \) and \( W \) along the \( x \) and \( y \) axes, respectively. The general boundary conditions for PSGE have the form

\[
\vec{n} \cdot \vec{\nabla} \phi |_{\partial \Omega} = \vec{n} \cdot [\hat{z} \times (\vec{H}_{\text{ext}} + \vec{H}_f^r)] |_{\partial \Omega} 
\]  

(2.6)

where \( \vec{n} \) is the outward normal to the boundary \( \partial \Omega \) of the junction region \( \Omega \), \( \vec{H}_{\text{ext}} \) is an external magnetic field and \( \vec{H}_f^r \) is the magnetic field caused by a current passing through the junction. When an external magnetic field \( \vec{H}_{\text{ext}} \) is applied to the plane of the junction parallel to the barrier and perpendicular to the length of junction and bias current is applied perpendicular through the junction, the boundary conditions may be written [BP82, CFG’96, ELO’85, LFD’93, LOE’85, Ped86, PU95, Ust98a] in the form
\begin{align}
H_y &= \frac{d\phi}{dx}_{x=0,L} = h \quad \text{(2.7a)} \\
H_x &= \frac{d\phi}{dy}_{y=0,W} = \mp \gamma \frac{W}{2} \quad \text{(2.7b)}
\end{align}

where $h = H_{\text{ext}}/J_c \lambda_j$ denotes the normalized measure of the external magnetic field and $\gamma$ is the normalized measure of the $y$-component of the external current injected at the junction boundary, see Fig. 2.3 (b) and (c).

\begin{figure}[h]
\centering
\includegraphics[width=\linewidth]{fig23.png}
\caption{(a) Josephson tunnel junction of overlap type. (b) Schematic diagram top of the overlap LJJ. The current from the upper electrode goes through the junction and comes in the lower electrode. (c) Overlap boundary conditions for the 2D sine-Gordon equation. The magnetic field, bias current and all lengths are dimensionless.}
\end{figure}

Without applied magnetic field and current, boundary conditions become
\begin{align}
H_y &= \phi_x(0, y, t) = \phi_x(L, y, t) = 0 \quad \text{(2.8a)} \\
H_x &= \phi_y(x, 0, t) = \phi_y(x, W, t) = 0 \quad \text{(2.8b)}
\end{align}

In the general case, no exact analytic solutions to the perturbed sine-Gordon equation (PSGE) are known. Therefore, PSGEs have to be solved numerically or by using analytical
approximation perturbation methods. However, non-trivial analytic solutions can be found for the unperturbed sine–Gordon Equation (SGE), is eq. (2.5) with zero right hand side,

\[ \phi_{xx} + \phi_{yy} - \phi_{tt} - \sin(\phi) = 0 \]  \hspace{1cm} (2.9)

A previous studies [ELO’85, LOE’85] show that for two-dimensional model of Josephson junctions of overlap geometry, even the width is much larger than \(\lambda_j\), the critical current and stationary fluxon velocities equal the critical current and stationary fluxon velocities obtained from one-dimensional model in the limit of a very narrow junction [CFG’96, ELO’85]. The two-dimensional model can reduce to the one dimensional model in the limit of very narrow junction when \((y/2)(W/2)^2 \ll 1\) [LOE’85] (\(W \leq 2\) [CFG’96]). Then, the results obtained from the 1D model can be used for large-area junctions [CFG’96, LOE’85], quasi-one-dimensional model. Therefore, analytical solutions of eq. (2.9) can be subdivided in three different classes of solutions as these on an infinitely long spatial interval [Thy99, Wal97, Wei07].

### A.1 Fluxon (Soliton)

The solution of unperturbed sine–Gordon equation, eq. (2.9), has a solitonic solution discussed in detail in [GK06] and [GK07]

\[ \phi(x, y, t) = 4 \tan^{-1}\left(\pm \exp\left(\frac{x-x_0-ut}{\sqrt{1-u^2}}\right)\right) \]  \hspace{1cm} (2.10)

With energy

\[ E = \int_{-\infty}^{\infty} dx \int_0^y dy \left[ \frac{\phi_x^2}{2} + \frac{\phi_y^2}{2} + \frac{\phi_t^2}{2} + 1 - \cos(\phi) \right] \]  \hspace{1cm} (2.11)

\[ E = \frac{8W}{\sqrt{1-u^2}} \]  \hspace{1cm} (2.12)

Depending on the sign \((\pm)\), this solution describes a soliton or an anti-soliton in phase difference \(\phi\) moving at normalized velocity \(u\) in the positive \(x\) direction. The soliton (anti-soliton) corresponds to a jump of \(\phi\) from 0 to \(2\pi\) (from \(2\pi\) to 0) [Abd04]. The initial condition for the soliton is taken at the time \(t = 0\) in the form of eq. (2.10). At this moment the soliton is positioned at the coordinate \(x = x_0\). The velocity \(u\) is measured in units of the Swihart velocity \(\tilde{c}\) and may take values in the range \(0 < u < 1\). At \(u = 0\), the soliton energy equals to \(8W\), which is identified with the normalized rest mass of the soliton. As known, \(\phi_x\)
represents a magnetic field component in the case of long overlap Josephson junctions, see Fig. 2.4. Thus the soliton solution which behaves as a quasi-particle with mass, energy and velocity is identified with the moving Josephson vortex (fluxon) containing one quantum of magnetic flux, which interacts with the environment of the junction, such as an externally applied magnetic field and injected currents, see next section 2.1.3 [Wei07].

![Fig. 2.4: (a) Phase distribution of a soliton. (b) Magnetic field distribution of a soliton. (c) Supercurrent across the junction corresponding to a soliton.](image)

In the case of 2D Josephson junctions, the propagation of a 2D vortex line is perpendicular to the longer direction of junction for narrow junction [ELO’85, SZZ04]. For broad junction, however, there exists a class of excitations propagating of arbitrary shape along a Josephson vortex line. These excitations are associated with the distortion of a Josephson vortex line of an arbitrary profile [GKS’09]. The distortions of a Josephson vortex line are referred to as shape waves. This phenomenon is explained by the Gulevich group [GKS’08, GKS’09, GSY’08]. They pointed out that under some conditions, a moving vortex with the shape excitation can have less energy than the same vortex without it. In addition, shape waves can have almost any shape and retain it for a long time while the wave is propagating [GKS’08].
A.2 Plasma Waves

Besides soliton solutions discussed above, the sine-Gordon equation (eq. 2.9) has the simplest solution of the type $\phi = \phi_0 e^{i(kx - \omega t)}$ by assuming that $|\phi| \ll 1$. The electromagnetic small amplitude wave, which travel along the junction, are called Josephson plasma waves (JPWs), or sometimes plasmons [Abd04, Vis02]. Plasma waves are generated by moving fluxons through some inhomogeneous regions in the junction such as tunnel barrier inhomogeneities or imperfect junction boundaries [Ust98a]. In addition, plasma waves interact with fluxon [MKM08] and leads to important experimental observations in the current–voltage curve, which are called Fiske modes (see section 2.1.3).

A.3 Breather

Another type of nonlinear excitations are breather oscillations or just breathers. These solutions are described as a “bound state oscillations of soliton – anti-soliton pair”. They appear in two forms: (a) soliton anti-soliton bound states located near the center of the junction and (b) solitons bound to virtual anti-solitons at the open end of the junction [Abd04]. Breathers are unstable and therefore decay after some transient time [Wal00]. Due to the decay (annihilation of soliton and anti-soliton) of this solution, it is not taken into account for time averaged measurements [Thy99].

In reality, the energy dissipated from the wave system cannot be ignored, i.e. $\alpha > 0$. For $\alpha = 0$, the energy for the undamped SGE given by eq. (2.12) is conserved, while for $\alpha > 0$, the energy is not conserved. This loss, as a small perturbation, leads to decrease the velocity of the flux. For small perturbations, the soliton solution stays stable with some modifications. According to the perturbational approach, treating the energy dissipating terms as small perturbations, the soliton solution for the system is given by

$$\phi(x, y, t, u, x_0) = 4 \tan^{-1} \left( \pm \exp \left( \frac{x - X(t)}{\sqrt{1 - u(t)^2}} \right) \right)$$  \hspace{1cm} (2.13)

with the perturbation $X(t) = x_0 + \int_0^t u(\tau) d\tau$. Such solitons behave like relativistic particles with an energy equal to

$$E(\phi) = \frac{bw}{\sqrt{1 - \eta^2}}$$  \hspace{1cm} (2.14)
The investigated SIS junctions with typically $\alpha \leq 0.05$ are well described by the perturbed sine-Gordon equation with small perturbations [Thy99].

2.1.3 Long Josephson Junctions in Applied External Magnetic Field and Bias Current

In linear junctions moving fluxons cannot avoid interaction with the environment of the junction. The simplest type of the interaction with environment is the interaction with the boundaries of the system. This interaction strongly influences the dynamics of fluxons inside the junction. In long junctions, three major regimes of fluxon motion can be observed [Wal97, Wal00].

**B.1 Zero-Field Steps (ZFSs).** If no external magnetic field is applied to a Josephson junction, $h = 0$, and a low bias current, $0 < \gamma < 1$, is applied through the junction, fluxons can be derived through the junction. They will be reflected at the boundaries with a change of its polarity (fluxon antifluxon and vice versa) at its ends [Abd04]. The fluxon dynamics are again reflected by the shape of the voltage step. When increasing the bias current again, more fluxons are trapped in the junction. This process gives rise to zero-field steps (ZFSs) in the current-voltage characteristic of the junction, which are the most common signatures of fluxon motion [LSC82a, LSC82b, Thy99, Wal97, Wal00], see Fig. 2.5.

**B.2 Fiske Steps (FSs).** If a magnetic field exceeding a certain critical value is applied to a long linear junction perpendicular to the direction of propagation of the vortices, vortices are nucleated at one end and are annihilated at the other end of the junction. In the process of annihilation of the fluxon, plasma waves are generated, which resonate with the junction cavity [Wei07]. This process gives rise to so-called Fiske resonances.

**B.3 Flux Flow Steps (FFSs).** In high magnetic fields, which break the symmetry of the junction, the resonant Fiske states overlap with each other and the dynamics are purely determined by the flux–flow (FF), i.e. to the continuous injection of fluxons nucleated at a high rate from one end of the junction and flow viscously in a dense chain to the other end, where they annihilate at the other boundary of the junction [PU95, Wal97]. This process is termed flux-flow [Jaw08, MP99, TUK06]. FF is interesting for applications in Flux-Flow Oscillators (FFO), conventional FFO, in long Josephson junctions [BSY9, KSS93, Jaw08, MKM08, MP99, TUK06], which constitute a source of millimeter and submillimeter wave radiation [KSB‘95, KSS93, KSS‘96, NEI‘83, NEI‘84, NEI‘85, NEI‘88, US98]. The power...
and the frequency of the emitted radiation can be adjusted by the external magnetic field [KM01, Thy99]. FF have been already successfully implemented as local oscillators for integrated receivers [dLBB’10, FSD’01, KM01, KS00, KSB’95, KSF’04, Thy99]. In a Josephson FFO, when an external magnetic field is applied in plane to the junction, the field penetrates into the junction in the form of Josephson fluxons (solitons). They are propagated across the junction under the influence of the bias current. Ideally, if the bias current is appropriated, a Lorentz force will drive fluxons to move along the junction. Otherwise, fluxons are static [Par93, SYH’99]. Then they are annihilated at the other edge. At this edge, when fluxons reach the junction edge each fluxon radiates electromagnetic waves (terahertz radiation) [AFU05, BK06, GWU00, KY97, MS95, SYR’06]. The frequency $f$ of the radiation emitted by a moving fluxon chain is $f = V/\phi_0$, where $V$ is a dc voltage induced by the fluxon motion [BK97, BSY91, KSS93, CFG’96, GMU96, MP99, PU95, Ust98a].

ZFSs, FSs and the FF are observed in the experimental I-V characteristic. The Fig. 2.5 shows I-V curves for zero magnetic field and the Fig. 2.6 shows I-V curves obtained by continuously varying the external magnetic field. A more detailed discussion of these dynamical regimes is found in the vast literature on long junctions [CGP88, Par93, PU95, Thy99, Ust98b, Wal00].

![Fig. 2.5: IVCs of an overlap junction (1D) without external magnetic field][Thy99].
The effect of the width on the dynamical behavior of the large junction under a bias current and damping is reflected in the IV characteristics because of self-coupling of Josephson radiation [KS00]. The effect of Josephson self coupling (JSC) is absorption of the FFO-emitted radiation by the quasi-particles in the cavity of the long junction [KS00]. Typical I-V characteristics of the FFO were recorded at different magnetic fields, which subsequently splits into a series of resonant Fiske steps, are shown in Fig. 2.7.

In Fig. 2.7, there is a boundary voltage, which is about one third of the superconductor gap voltage $V_b = 1/3 V_g$. It clearly divides the IVCs into two regions with a boundary at $V_b$, where the FS’s disappear. As also seen in Fig. 2.7, for $V < V_b$, Fiske steps appear in the IVC, which consists of nearly vertical, equally spaced voltage spikes [PSK’07]. For $V > V_b$, the damping parameter is significantly increased due to the self-pumping effect [KSS’97]. The FFSs become smooth and continuous tuning of the FFO frequency up to the gap voltage [KS00, PSK’07].
2.2 Josephson $\pi$-Junctions

In a SIS Josephson junction (JJ), the supercurrent $I_s$ and Josephson phase are given by eq. (2.1) and eq. (2.2), respectively. Usually $I_c$ is positive and the Josephson phase $\phi = 0$ corresponds to the minimum energy of JJ, $U = E_j(1 - \cos \phi)$. This is the so-called 0 Josephson junction (0 JJ) or a conventional Josephson junction. However, a negative supercurrent, $I_c < 0$, is predicted through a JJ with ferromagnetic impurities in the tunnel barrier, i.e. $-I_c \sin (\phi) = 0$. In this case, the solution $\phi = 0$ is unstable and corresponds to the maximum energy, $U = E_j(1 + \cos \phi)$, while $\phi = \pi$ is stable and corresponds to the ground state. This is the so-called $\pi$ Josephson junction ($\pi$ JJ), in contrast to conventional 0 junctions with $\phi = 0$. In case of a $\pi$ Josephson junction, the Josephson relation, eq. (2.1), is modified to $I_s(\phi) = -I_c \sin (\phi) = I_c \sin (\phi + \pi)$. In experiment it is unlikely to distinguish 0 JJs from $\pi$ JJs from the current–voltage characteristic (IVC) of a single junction due to the measured critical current in junction is always positive and is equal to $|I_c|$ [Hil08, Wei07, WKW’07].
2.2.1 Semifluxon

During last years it was shown both theoretically [KBM95, XMJ’95] and experimentally [KTM99, SYI02], that there was also significant interest in vortex excitations carrying an arbitrary fraction of the standard flux quantum as \( \phi = \phi_0 \kappa/2 \), where \( \kappa \) is a the value of the discontinuity [GKK04]. For \( \pi \) discontinuity, Josephson vortices carry only half of the magnetic flux quantum \( \phi_0 \). This half-vortex has been studied theoretically by several authors [DG’08, GB’07, GKK03, GKK04, GSG’04, GSK’04, GSK’05, GVS’08, SvG04, SvGV’03, XMJ’95, WGC’08]. Because of containing one-half of the flux quantum, they are called “semifluxon”. In other words, a \( 2\pi \) vortex is a fluxon carrying \( \phi_0 \), while a \( \pi \) vortex is a semifluxon carrying \( \phi_0/2 \), see Fig. 2.8 [GKK04]. A single semifluxon can have positive or negative polarity carrying the flux \( +\phi_0/2 \) (semifluxon) or \( -\phi_0/2 \) (anti-semifluxon), respectively [GSK’04, KBM95, SvGV’03, SVG04, Wei07].

Semifluxons exist in specially designed long Josephson junctions (LJJ), which are so-called “0-\( \pi \)” LJJ, see Fig. 2.8. This 0-\( \pi \) junction consists of two joint parts. The 0-junction which is a conventional Josephson junction (one may literally say, “with positive critical current”) and the other second part is a \( \pi \)-junction, which is a standard junction “with a negative critical current” [Hil08, GKK04, SvGV’03, Wei07]. Semifluxons, which carry spontaneous half flux quntum, can be created at the boundaries between the 0 and \( \pi \) regions provided the Josephson junction is long enough [GSK’05, GVS’08, Wei07]. The 0-\( \pi \) boundaries are sometimes called discontinuity points, see Fig. 2.8 [GSK’05]. The most popular \( \pi \)-junction configuration currently is the superconductor–(insulator)-ferromagnet-superconductor (S(I)FS) junction [Hil08, Wei07].
Semifluxons are very interesting objects which are not studied in detail, especially experimentally, because of difficulties with a fabrication of 0–π junctions [GSK’04, SVG04]. Although the classical properties of semifluxons are understood, their quantum behaviour and their possible applications in the quantum domain still have to be studied [GKK02, GKK03, GSG’04, GSK’05, GVS’08]. Whereas, semifluxons and fluxons can be described by sine-Gordon-equation, their properties are very different, fluxons are topologically stable solitons with very long life time while the semifluxons created dynamically have a very short life time. Semifluxons often represent the ground state of the system with a topological defect associated with the 0–π boundary at which a spontaneously vortex is formed and pinned, which emerges in the absence of a driving bias current or an external field $H$, i.e. in the ground state. In contrast, normal fluxons represent an excited state of the system and they are moving in the LJJ with a speed close to the speed of light [GSG’04, Wei07]. This property of semifluxons makes them attractive, e.g., for information storage and processing in classical
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and quantum regimes [GSG’04, GSK’04]. For more details see references [GKK03, GKK04, SVG04, SYI02, XMJ’95, Wei07, WKW’07].

2.2.2 Analytic Dynamics of Semifluxons in 0- π Long Josephson Junctions

The electrodynamics of a long Josephson junction with π -discontinuity point, as shown in Fig. 2.8 (a), is described by SGE as

$$\phi_{xx} - \phi_{tt} - \sin(\phi) = \theta_{xx},$$  \hspace{1cm} (2.15)

where \(\theta(x)\) is a function, which is constant everywhere and jumps by π at the discontinuity point \((x = 0)\) [SvGV’03]. Then the solution \(\phi(x)\) should be discontinuous at \(x = 0\). It is not very convenient to deal with discontinuous functions such as \(\phi(x)\), and with singular functions such as \(\theta_{xx}(x)\) [GKK03, GKK04]. To simplify the analysis, it is convenient to present the phase \(\phi\) as a sum of two components: the magnetic one \(\mu(x)\) and the order-parameter related one \(\theta(x)\), i.e.,

$$\phi(x, t) = \mu(x, t) + \theta(x)$$  \hspace{1cm} (2.16)

Then the sine-Gordon equation reads [GKK03, GKK04]

$$\mu_{xx} - \mu_{tt} - \sin(\mu + \theta) = 0,$$  \hspace{1cm} (2.17)

where

$$\theta(x) = \begin{cases} 0, & x < 0 \\ \pi, & x > 0 \end{cases}$$  \hspace{1cm} (2.18)

The static version of eq. (2.15), \(\mu_{xx} - \sin(\mu + \theta) = 0\), is solved [GKK03]. The analytical expression for the semifluxon is

$$\mu(x) = \begin{cases} 4 \arctan(ge^x), & x < 0 \\ \pi - 4 \arctan(ge^{-x}), & x > 0 \end{cases}$$  \hspace{1cm} (2.19)

where \(g = \tan(\pi/8) = \sqrt{2} - 1\). The final expression for the semifluxon shape in terms of the total phase \(\phi(x)\) can be written in a more compact form as [XMJ’95]

$$\phi(x) = -4 \text{sgn}(x) \arctan(ge^{-|x|})$$  \hspace{1cm} (2.20)
These eqs. (2.19) and (2.20) are presented in Fig. 2.8 (b) and compared with the fluxon. It is important to note that the eq. (2.19) is a unique semifluxon solution when $\gamma = 0$, but the semifluxon is not unique when $\gamma \neq 0$. There are semifluxons with a hump [SVG04] - a bending (large hump) appearing in phase of semifluxon [SvGV’03, SVG04]. The plot of the semifluxons with a hump is presented in Fig. 2.9.

![Fig. 2.9: A plot of the phase of the two semifluxons (SFs) with a hump (solid-lines) for $\gamma = 0.05$ [SVG04].](image)

### 2.3. Flux Cloning Circuits: T-shaped Josephson Junctions

In nature, vortices may arise suddenly anywhere. Their prediction has crucial importance for our life. Vortex nucleation has been most studied in superconductors. A common belief written in all textbooks [for example, BP82, Tin96, Sch97] is that a single vortex cannot be nucleated inside a superconductor. It may only penetrate from the border [BP82]. However, for multi-connected weak superconductors (Josephson junction), there may arise a fluxon cloning. This new phenomenon, which is called flux cloning, was predicted theoretically in the Refs. [GK06, GK07]. In addition, it has been observed in numerical experiments [GK06, GK07].

In Josephson electronic circuits, a Josephson transmission line (JTL) can be connected together as a T-shaped junction, which can be a very promising element for logic Josephson electronics. This kind of multiple JTLs is also called the turning points of the Josephson line [NOO76], which generally has three arms (strip) [ND82]. By using T-shaped JTLs, it is possible to design, not only a logic circuit, but also a flux pump [GGK’08], which can be used as a T-ray generator. In general, T-junctions have two different perpendicular widths; a main Josephson transmission line (MJTL) of width $W_0$ along the $x$-axis, and an additional Josephson transmission line (AJTL) of width $W$ along the $y$-axis (see Fig. 2.10). They can be connected to form two kinds of T junctions (see Fig. 2.10 (a) and (b)).
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Fig. 2.10: The T-junction consists of two attached Josephson transmission lines. (a) Conventional T-junction which is the general shape of the Josephson T-junctions, (b) Rotated T-junction which is the T-junctions rotated by -90°.

According to the T-junction of Gulevich and Kusmartsev, an additional JTL is connected to the main waveguide at the centre [GK06, GK07], which is sometimes called here a conventional T-junction (see Fig. 2.10 (a)). In a T-junction, a single vortex propagates from the left side of the T-junction and finishes with a two-vortex state. The baby vortex is nucleated at the moment when a mother vortex passes the branching T-shaped junction. In order to give birth to a new vortex, the mother vortex, in main LJJ, must have enough kinetic energy. The part of this energy the mother vortex loses at the T-junction is used to generate the baby vortex.

The original vortex line (denoted by 1 in the Fig. 2.11 (a)) lying in the plane of the junction moves towards the T-junction. The superconducting contour around the vortex covers a single flux quantum or single vorticity. When approaching the T-Junction the vortex is stretching and bending and its length increases. At the T-junction the vortex length is maximal, see curved line 2 in Fig. 2.11 (a). This state is associated with the vortex nucleation barrier. At this moment the bent vortex line touches the right corner of the T-junction and the new baby vortex is nucleated. This nucleation happens by the corner cutting a piece from the bent (pregnant) mother vortex, see the Fig. 2.11 (a). As result, the original mother vortex of the enlarged length is split into two pieces of a smaller length. After the vortex division the length of the mother vortex returns to the original size $W_0$, while the size of the nucleated vortex is adapted to the width of the second branch of the junction $W$ (see, the state 3, presented in Fig. 2.11 (a)). The dynamics of the Josephson vortex can be traced on the colour plots (Fig. 2.11 (b), where the snapshots for the vortex positions are presented at different time intervals $t$. The vortex can be seen here as a green-yellow stripe, while the superconducting states without vortices are shown in blue and red.
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Fig. 2.11: (a) A schematic picture of the vortex positions in the T-shape Josephson junction (presented in the bold lines) during the vortex nucleation process. The thin arrows indicate the direction of the vortex motion. The vortex state before nucleation is denoted by 1; the moment of nucleation is denoted by 2; the vortex state after the nucleation is denoted by 3. Widths of main and additional branches of the T-junction are $W_0$ and $W$, respectively (b) Division of the vortex moving with high velocity. Widths of the main and additional branches of the T-junction are $W_0 = 10 \lambda_j$ and $W = 5 \lambda_j$ respectively, where $\lambda_j$ is a Josephson length. The blue and red colours are two equivalent states of a superconductor without vortices.

Gulevich and Kusmartsev [GK06, GK07] studied the dynamic behaviour of a soliton in conventional T-junctions in the absence of magnetic field, $H = 0$. They pointed out that in the connecting area, there is the possibility of one of two different types of behaviour on the part of the soliton. The first behaviour is a reflection of the soliton from the T junction (Fig. 2.12 (a)). The second behaviour is a flux cloning (Fig. 2.12 (b)) [GK06, GK07]. In flux cloning, when the fluxon (the mother vortex) in the main JTL approaches the fork and satisfies certain conditions, it can split to generate a new flux (a baby vortex) in the additional JTL and the mother vortex will still propagate continuously in the main JTL. There are two conditions required to satisfy flux cloning, depending on the velocity of the soliton and the bias current. The first condition, the critical velocity, is

$$u \geq u_c = \frac{\sqrt{W(W+2W_0)}}{W+W_0}$$

(2.21)

The second condition, the critical driving current, is
\[ \gamma \geq \gamma_c = \frac{4W}{\pi(2W_\theta + W)} \]  (2.22)

Otherwise, the soliton will be reflected if \( u \) and \( \gamma \) are less than \( u_c \) and \( \gamma_c \), respectively [GK06, GK07].

Therefore, the dynamics of fluxon cloning circuits may be used to build up a flux flow oscillator (FFO) which can operate without magnetic field. We have developed a new design of the FFO with cloning circuits in order to avoid the Fiske resonance for an FFO with standard rectangular overlap geometry, which have used only overlap structure as FFOs operate by applying external magnetic field [Jaw08, MP99, PU95, TUK06].

**Fig. 2.12:** Numerical simulations of the time evolution of superconducting phase difference; (a) The time snapshots of the reflection of an incident fluxon propagating without cloning and (b) Cloning of the fluxon propagating through the T-junction. After the time \( t=4 \) due to the cloning there are created two fluxons. The colour scale on a right side represents the superconducting phase difference \( \phi \) [GK06].

### 2.4. Conclusion

The flux cloning phenomenon was studied in narrow conventional T-junction, \( W \leq W_0 \). In this thesis, we investigate the variety of dynamic behaviours of flux cloning that can be expected in conventional T-shaped Josephson junctions with the energy dissipated, \( \alpha \neq 0 \), from the wave system. In addition, the flux cloning phenomenon is studied in extended conventional T-junction, \( W > W_0 \). Moreover, we propose a new kind of T-junction, which is a \( 90^\circ \) rotated conventional T-junction and attempt to estimate theoretically and numerically the critical velocities. In next chapter, we clarify the method of modelling and numerical simulation problems.
Chapter 3

Numerical Simulations Method

In this chapter the method of modelling and simulation problems is presented by using the finite element numerical technique (using COMSOL Multiphysics 3.5 software). Some examples are solved.

3.1 Numerical Simulations Using Finite Element Methods

The numerical method is still the basis of work with general initial conditions and geometries [PU95]. The finite element method (FEM), its practical application often known as finite element analysis (FEA), is one of the most widely utilized computer aided mathematical technique for finding approximate numerical solutions of partial differential equations (PDE) that model physical problems that would otherwise be difficult to obtain such as structural, electric field and fluid flow. The approximation comes about when transforming the differential equation to a linear algebraic problem, in which the finite elements have all the complex equations solved. The physical effects are determined by a given set of boundary conditions, which can be forces, temperatures, hydrostatic pressures, centrifugal pressures, torques, and displacements. The basic idea behind finite element modeling is dividing a domain up into subdomains, i.e. the geometry being modelled will always be divided into smaller divisions known as elements. The elements are connected together to form the finite element mesh, see Fig. 3.1. Each element contains nodes which are points where the elements are mathematically connected to one another. The elements also have defined any physical properties that are necessary for the model to define how the
structure will react to certain boundary conditions. In addition, boundary conditions are added to the mesh to simulate the real world application of the geometry. In a structural simulation, the elements at the nodes are connected to form an approximate system of equations for the whole structure, i.e. forming element matrix, which is called the stiffness matrix. Stiffness matrices, which define the geometric and material properties, are a fundamental part of FEA. Then FEA programs solve the problem to obtain the required variables such as stress, strain or velocity flow depending on the application. Hence, FEA allows initial design analysis to be done with minimal time. Present day supercomputers are now able to produce accurate results due to increase in mesh density [PH05, Zim06].

The numerical simulations have been performed with the use of the finite element program package FEMLAB, which has been implemented using COMSOL Multiphysics 3.5 software [Zim06]. In setting up a model in COMSOL Multiphysics, there are conceptually five stages [Zim06]:

1. **Drawing**: Specifying the domains and constructing the geometries, see Fig. 3.1.
2. **Mathematical equations**: Specifying the equations satisfied internally within the geometries (subdomain physics) and those on the boundaries.
3. **Meshing**: Specifying the number of elements in the domains, which is important, i.e. more elements (higher density mesh) means higher accuracy. COMSOL Multiphysics has a powerful mesh generator. One historic distinguishing feature of finite element method is the permission of irregular and arbitrary meshes, see Fig. 3.1. Also it permits user control over domains to turn out the meshing to be appropriate enough for the more accurate results.
4. **Solving**: The meshing geometries and boundaries are set to solve the problem and calculate desired quantities. The program calculates the interactions using physics equations and recording resultants
5. **Post-processing**: Representing the numerical simulation graphical and computational processing. The output screen will now display the solution in the form of a graph in 2D and 3D, which is sometimes used colours. The colour domains represent the change value of variables, see, for example figures from Fig. 3.2 to Fig. 3.10. This will be clarified afterward.

In this software, the sine-Gordon-equation, eq. (2.5), in 1D, 2D and 3D, which is a non-linear differential equation, is included as one of its standard equation-based models. Since solitons are a specific solution of sine-Gordon-equation, the dynamic behaviour of solitons motion in Josephson junctions can be detected. For example, using a time-dependent solver, it
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demonstrates a succession of faster solitons passing through a slower one, all reforming after the collision, the change of the value of the superconducting phase difference and soliton moving.

The post-processed domain colour plot shows the solution extrapolated to later time steps for soliton or magnetic field. The colour scale represents the distribution of the magnetic field, $|\nabla \phi|$ or $\phi_x$, or solitons, $\Delta \phi = 2\pi$, as the dark blue colour represents the minimum value and red colour represents the maximum (see the colour scale in figures from Fig. 3.2 to Fig. 3.10). The time-dependent sine-Gordon-equation has been studied with boundary conditions depending and independent on external magnetic field and bias current. The dynamical behaviour of Josephson vortices can be indicated by time dependent change of the value of the superconducting phase $\phi$, which is represented in the program by a change in colour domains. The solution can be presented on the figures from Fig. 3.2 to Fig. 3.10. It is convenient for each value of the phase difference to be represented by a colour. The dark blue colour represents the minimum value of the phase $\phi$ and red colour represents the maximum value of the phase $\phi$. For instance, the colour scale in Fig. 3.2 and Fig. 3.8 represents the total phase difference, which varies from 0 to 6.3 in Fig. 3.2 while in Fig. 3.8 varies from $-47.45$ to $47.46$, where the colour changes from blue to red in two figures. In general, semifluxons are presented as the phase difference change, $\Delta \phi = \pi$, while fluxons are presented as the phase difference change, $\Delta \phi = 2\pi$ (the intermediate colour between these values represents the core of the vortex, see the colour scale in figures from Fig. 3.2 to Fig. 3.10).

In addition, the general boundary conditions, eq. (2.6), for PSGE or SGE, which represent the applied current and magnetic field on the junction, is presented in program in general form as

$$- n \cdot \Gamma = G \quad (3.1)$$

For example, in the case of overlap Josephson junction, the boundary condition for applied magnetic field and bias current, eq. (2.7) can be presented as

$$\vec{n} \cdot \vec{\nabla} \phi \bigg|_{x=0,L} = h \quad (3.2)$$

$$\vec{n} \cdot \vec{\nabla} \phi \bigg|_{y=0,W} = \mp \gamma \frac{W}{2}, \quad (3.3)$$

or without applied magnetic field and bias current as

$$\vec{n} \cdot \vec{\nabla} \phi \bigg|_{x=0,L} = 0. \quad (3.4a)$$
\[ \mathbf{n} \cdot \nabla \phi \big|_{y=0,W} = 0. \quad (3.4b) \]

In addition, the boundary conditions in conventional T-Josephson junction, Fig. 2.11 (a), with no applied external magnetic field and bias current can be presented as:

\[ \mathbf{n} \cdot \nabla \phi \big|_{A'-H} = 0 \quad (3.5) \]

In all numerical simulations for the case of zero applied magnetic field, we have used initial conditions with the following form: for the phase

\[ \phi(x, y, t)\big|_{t=0} = \phi_{\text{soliton}}(x) \quad (3.6) \]

and for its time derivative

\[ \partial \phi(x, y, t)/\partial t\big|_{t=0} = u \partial \phi_{\text{soliton}}(x)/\partial x \quad (3.7) \]

where

\[ \phi(x, y, t) = \phi_{\text{soliton}}(x, y, 0) = 4 \tan^{-1} \left( \frac{x - x_0}{\sqrt{1 - u^2}} \right) \quad (3.8) \]

In addition, we limit the analysis to the case when the main Josephson junction width \( W_0 \) is constant and the additional Josephson junction width, \( W \), is variable. It is worth noting that extended T-Josephson junctions, in general, mean the widths of MJTL \( (W_0) \) and AJTL \( (W) \) are larger than the Josephson length \( (\lambda_j) \). Here, extended T-Josephson junctions are denoted to both \( W_0, W \geq \lambda_j \) and \( W > W_0 \).

### 3.2 Numerical Simulations of Soliton in Josephson junction

The most efficient method of learning is by example. Therefore, I would like to present to you a simple Josephson junction model and simulation: we have studied the time-dependent sine-Gordon-equation with boundary conditions either dependent or independent of external magnetic field or a current.

**Example A: Single Soliton**

Here is an example of dynamical behaviour of a fluxon in a 2D Josephson junction. By using COMSOL Multiphysics, a Josephson junction (isolator layer), Fig. 2.3 (c), can be
studied numerically. First, a geometry is drawn. Then, the SGE is applied, eq. (2.9), with boundary condition of the absence of an externally magnetic field, eq. (3.4) and initial condition, eqs. (3.6) and (3.7) with $u = 0.8$ and $x_0 = -1$. The simulation, for example, is taken by using a mesh consisting of 1184 elements, see Fig. 3.1. Then, the model is solved and some relations can be plotted. For example, at $t=1$, soliton (fluxon) in 1D and 2D are represented in Fig. 3.2. In Fig. 3.2, dark blue colour represents the phase $\phi$ equal to 0, red colour represents the phase $\phi$ equal to $2\pi$ and the intermediate colour between blue and red represents the Josephson vortices (see the colour scale in Fig. 3.2). The change of phase during the time is showed in Fig. 3.3. It is clear soliton change to anti-soliton after reflecting at the boundary.

Fig. 3.1: Drawing and Meshing the Josephson junction.
Fig. 3.2: Snapshots of numerical simulations of phase difference $\phi$ (fluxon) in 2D at $t=1$. The colour scale represents the superconducting phase difference $\phi$. The dark blue colour represents the minimum value of phase $\phi$ (equal to 0), and red colour represents the maximum value of phase $\phi$ (equal to $2\pi$), while the area when the colour change represents the vortex (b) The behaviour of phase difference $\phi(x)$ in the Josephson junction along $x$-axis at $t=1$. 
Fig. 3.3: (a) The dynamic behaviour of a moving soliton in JJ during $t=0$-$10$ in $x$ and $y$ vs $t$. The colour scale represents the superconducting phase difference $\phi$. The soliton changes into an anti-soliton after reflection by the boundary. (b) The behaviour of phase difference $\phi(x)$ in Josephson junction along $x$-axis at $t=0$ (soliton) and (c) at $t=8$ (anti-soliton).
In addition, the magnetic field (fluxon) at \( t=1 \) is shown in 2D in Fig. 3.4. The motion of the magnetic field \( \phi_x \) along \( x \)-axis during period 0 to 10 is plotted Fig. 3.5. This figure shows that the fluxon arriving at the junction boundary undergoes reflection into an anti-fluxon which is driven then back into the junction. Because of studying in 2D, however, this change (fluxon to anti-fluxon) may be not observed when we plotted the magnitude of magnetic field in \( x \) and \( y \) directions, which is represent in COMSOL as \( |\nabla \phi| = \sqrt{(\phi_x)^2 + (\phi_y)^2} \), see Fig. 3.6.

![Fig. 3.4: Numerical simulations of the dynamic behaviour of the magnitude of magnetic field (|\nabla \phi|) at \( t=1 \). The colour scale represents total magnitude of magnetic field.](image)
Fig. 3.5: The dynamic behaviour of a moving soliton in JJ during $t=0-10$ in $x$ and $y$ vs $t$. The colour scale represents the $x$ component of magnetic field, $\phi_x$. The fluxon changes into an anti-fluxon after reflection by boundary. The behaviour of magnetic field along $x$-axis, $\phi_x(x)$ (b) at $t=0$ (fluxon) and (c) at $t=8$ (anti-fluxon).
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Fig. 3.6: Numerical simulations of the dynamic behaviour of the magnitude of magnetic field ($|\nabla \phi|$) during $t=0$-$10$ for $x$ and $y$ VS $t$. The colour scale represents the magnitude of magnetic field ($|\nabla \phi|$). The behaviour of magnetic field, $|\nabla \phi|$, along $x$-axis (b) at $t=0$ (fluxon) and (c) at $t=8$ (fluxon).
Example B: Multi-Soliton

When a magnetic field, for example $h=2$, is applied to the junction boundaries, eqs. (2.7a) and (2.8b), the existence of $n$ ($n > 1$) solitons in the junction is possible. Snapshots of the magnetic vortex positions are presented at different time intervals $t$, see Fig. 3.7. In the figure, numerical results show that solitons penetrate and move into the junction from the boundaries towards other edges. Vortices on the left and right sides approach and strike each other. Then, they will be trapped inside the junction ($t=60$, 80 and 100). At $t=100$, there are 10 solitons inside the junction. Also, ten solitons are noticed when the phase difference is plotted (Fig. 3.8).

![Snapshots of numerical simulations of Josephson vortices (|$\nabla \phi|$) for applied magnetic field, $h=2$. Many vortices can pass through the junction and they are bonded in the centre of junction. (b) The magnetic field ($|\nabla \phi|$) along x-axis at $t=100$. There are 10 vortices (multifluxon) in the junction.](image-url)
Fig. 3.8: Numerical simulations of phase difference ($\phi$) when applied magnetic field, $h=2$, for multisoliton at $t=100$. The range of colour of phase difference in junction is nearly from -30 to 30. Each fluxons are represented by phase differences as $\Delta \phi = 2\pi$. Therefore, there are 10 vortices in this range.

To investigate what happens to the dynamic behaviour of fluxons when a bias current is applied, for example $\gamma=0.1$, the boundary conditions are added as Fig. 2.3 (c), eq. (2.7). The behaviour of solitons can be seen from the snapshots for the magnetic vortex positions which are presented at different time intervals $t$ shown in Fig. 3.9. At $t=40$, it is clear that the vortices on the right side of junction are accelerated and move faster than vortices entering from the left side of the junction. Then, the fast vortices on the right side strike and push to the left the slow vortices on the left side. Consequently, the fluxons flow from right to left. Moreover, if the direction of bias current in Fig. 2.3 is reversed i.e. down to up, the dynamic behaviour of vortices is still the same but changing only in direction. Vortices on the left side of junction are accelerated and move faster than vortices entering from the right side of the junction. Therefore, the fluxons flow from left to right, see Fig. 3.10.
Fig. 3.9: Snapshots of numerical simulations of Josephson vortices in an applied magnetic field, $h=2$, and bias current, $γ=0.1$. The acceleration of vortices from the right side increases while the vortices from the left side slowdowns. More vortices enter from the right side of the junction. Then, they push to the left the vortices on the left side of junction. Therefore, after a long time, all fluxons in the junction move from right to left.

Fig. 3.10: Snapshots of numerical simulations of Josephson vortices when the direction of bias current in Fig. 2.3 is reversed in direction (down to up), $h=2$ and $γ=0.1$. The acceleration of vortices from the left side increases while the vortices from the right side slowdowns. More vortices enter from the left side of the junction. Then, they push to the right vortices on the right side of junction. Therefore, after a long time, all fluxons in the junction move from left to right.
Chapter 4

Flux Cloning Anomalies in Extended T-Shaped Josephson Nano-Junctions Undamped System

In previous studies, as discussed in Chapter 2, the flux cloning phenomenon was studied in narrow conventional T-junction, i.e. where the width of additional Josephson transmission line (AJTL), $W$, is less or and equal to the width of main Josephson transmission line (MJTL), $W_0$. In this chapter, it will be shown that the conventional flux cloning may disappear and even more interesting phenomena can be observed if the width AJTL becomes extensive. Section 4.1 studies the behaviour of solitons in an extended conventional T-junction, $W > W_0$. In section 4.2, we propose a new kind of T-junction, which is a $90^\circ$ rotated conventional T-junction and attempt to estimate theoretically and numerically the critical velocities. In the following, we study the effect of widths on the behaviour of solitons in rotated T-junctions by a set of 2D geometries with varying JTL widths. We distinguish between small widths, $W \leq W_0$, and the large widths, $W > W_0$.

4.1 Extended Conventional T-shaped Josephson Junctions

4.1.1 Semifluxons

In conventional T-shaped JJ, Fig. 2.10 (a), the dynamic behaviour of fluxons was studied [GK06, GK07] when the width of AJTL, $W$, is narrower than that of MJTL, $W_0$. There we study the dynamic behaviour of fluxons in the case $W > W_0$. The behaviour of soliton changes because the critical velocity becomes very high, close to the Swihart velocity. In addition, there is a discrepancy between analytical estimation based on the soliton energy
arguments [GK06, GK07] and numerical evaluation of critical velocities when $W > W_0$ (see Fig. 4.1). The theoretical prediction of the critical velocity can be calculated by eq. (2.21), while the numerical value is evaluated when the soliton changes behaviour from reflection to splitting [GK06, GK07] (see Fig. 2.12). Here, we show that new movable semifluxons can be excited in extended conventional T-Josephson junctions, $W_0, W > \lambda_j$ and $W > W_0$, when a fluxon crosses T-junctions.

In a special case, when the energy of incident fluxon is very large, the T-shaped Josephson junction allows vortex excitations carrying a half of the standard flux quantum. When the width $W$ is too wide, the mother vortex shows strange behaviour in both MJTL and AJTL. In contrast to original prediction ($u_c$) indicating that fluxon moving with velocity higher than critical will be cloned [GK06, GK07], we show that in some cases the mother vortex is reflected from the T-junction while at the same time at the moment of the reflection some new excitations will be created and continuously propagate in MJTL. In connection area of T-junction, the soliton starts to divide and to clone but it does not have enough energy, i.e. it cannot generate a complete quantized fluxon (baby and mother vortices). Then, the fluxon will be trapped in the T-junction and semifluxons will be created. Then it becomes possible for them to propagate in AJTL and MJTL. In MJTL, the semifluxon after creation moves and
its stability looks similar to that the original fluxon. In AJTL, however, a massive shape wave (tsunamis) will appear together with a semifluxon which becomes unstable and fast decays into plasma waves. Because of the broad width of AJTL, the propagation of an unstable semifluxon over a large distance requires a high energy and high velocity to move along the AJTL, which can be estimated from the eqs. (2.12) and (2.21), respectively. Therefore, an unstable semifluxon will struggle to continue its motion due to the condition that its velocity is not large enough. Then, the unstable semifluxon, which moves along the AJTL, will be reflected and move back to the MJTL in $x$ and $-x$ directions. In connection area of the T-junction, the trapped fluxon will be reflected as well and then starts propagating in $-x$ direction. On the other hand, in $x$ direction, the unstable semifluxon starts moving and passing through the MJTL at the same time transforming as an unstable anti-semifluxon. An unstable anti-semifluxon moves faster than stable semifluxon. Consequently, the semifluxon and anti-semifluxon are strongly attracted to each other, like quarks in high energy physics. This results typically in the fast enough annihilation between them. Hereafter, the generation of periodic stable semifluxon and anti-semifluxon chains may spontaneously appear and propagate in MJTL. As a result, a series annihilation and consequent re-creation of such pairs will take place indicating on a complex dynamics existing between these new solitons, semifluxons and anti-semifluxons.

4.1.2 Numerical Studies of Semifluxon

First of all, a $0-\pi$-long Josephson junction, which discussed in chapter 2, is modelled with COMSOL package with eqs. (2.19) and (2.20), which describe the behaviour of semifluxon in junction, see Fig. 2.8. The results, calculated here, are similar to previous results [GKK03, GKK04, WKW+07] (see Fig. 2.8 and Fig. 4.2).
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Flux Cloning Anomalies in Undamped System

Furthermore, we consider the case when the fluxon cannot be cloned, for example, when $W=2.2$. The critical velocity ($u_c$) is equal to 0.94. Note, although the fluxon is moving with the speed $u \cong 0.99$, flux cloning does not occur. The numerical simulations show that in this case the soliton is reflected. In Fig. 4.3, the mother vortex starts moving with high velocity ($u=0.985$) and then it is reflected from the T-junction. Fig. 4.3 shows that semifluxons can be created when soliton crosses the T-shaped junction, i.e. at the time $t=5$. After creation, this stable semifluxon moves in MJTL, while at the same time another semifluxon created together with a massive shape wave is moves in AJTL (see the snapshot of the semifluxon dynamics presented in the Fig. 4.3 at the time $t=6$). In addition, an unstable semifluxon in AJTL is reflected and moved in MJTL as an unstable anti-semifluxon (presented Fig. 4.3 at time $t=8$). Finally at time $t=12$, the semifluxon and anti-semifluxon meet and annihilate. Then, spontaneous series of movable pairs consisting of a semifluxon and an anti-semifluxon are continuously generated and annihilated in MJTL.
Fig. 4.3: Numerical simulations of the creation of movable semifluxons. In this case no flux cloning occurs. An incident fluxon (created at $t=0$) is propagating with the velocity $u = 0.985$ when $W = 2.2$. For a comparison in this case the estimated critical velocity is $u_c = 0.94$. A movable semifluxon is created in MJTL after soliton crosses the T-junction (at $t=5$). In a short time after this event, the anti-semifluxon begins to propagate in MJTL. The semifluxon and anti-semifluxon strongly interact. The colour scale represents the superconducting phase difference $\phi$ in unit of $\pi$. A soliton is represented by $0$ to $2\pi$ phase differences whereas (anti-)semifluxons are represented by phase differences as $\Delta\phi = \pi$. 
To simplify the result, the phase difference $\phi$ is plotted along $x$-axis in Fig. 4.4 and Fig. 4.5. In Fig. 4.4, the phase difference of the incident fluxon (at time $t = 0$) varies from $0$ to $2\pi$, whereas the phase difference of the movable semifluxon, which is created by T-junction at time $t = 6$, varies from $0$ to $\pi$. Fig. 4.5 shows that at time $t = 15$, the series of pairs consisting of the semifluxon and anti-semifluxon appear and in the next moment their annihilation event takes place.

Fig. 4.4: The change of the phase $\phi$ along $x$-axis. An incident fluxon is propagating with a velocity higher than the critical one. At the time $t = 0$ (solid line), the change of the phase difference of incident fluxon as usual, is equal to $2\pi$. At $t = 6$ (dotted line), after fluxon has just crossed the T-junction, the change of the phase difference is equal to $\pi$ as it is needed for a semifluxon, which is located at approximately $x = 4$. It is movable semifluxon.

Fig. 4.5: The change of the phase $\phi$ along $x$-axis at $t = 15$. An incident fluxon is propagating with a velocity higher than the critical one. The series of the semifluxon and anti-semifluxon pairs appears. Each semifluxon has the change of the phase difference equal to $\pi$. The semifluxon and anti-semifluxon are colliding at approximately $x = 12$ and then again are annihilated by each other.
Next, we consider the case when flux can be cloned (Fig. 4.6 (b)). In the case $W=1.7$, the theoretical prediction gives $u_c=0.93$ while the numerical evaluated gives $u_c=0.979$. A movable semifluxon can be temporarily formed by a T-junction only if its velocity lies in a specific range. The values of velocities from this range should be slightly less than the numerical critical velocity ($u \approx u_c$). In Fig. 4.7, an incident fluxon propagates with velocity $u=0.97$. The numerical simulations show the creation of movable semifluxons in MJTL and AJTL at time $t=5$ (see Fig. 4.7 and Fig. 4.8). Annihilation of a semifluxon and a reflected fluxon appear in Fig. 4.7 at time $t=13$. Therefore, there are three types of the dynamical behaviour for soliton in extended T-Josephson junction are possible: reflection (Fig. 4.6 (a)), flux cloning (Fig. 4.6 (b)) and spontaneous movable semifluxon generation which arises during a reflection (Fig. 4.7).

**Fig. 4.6:** Numerical simulations of the superconducting phase difference when $W_0=1, W=1.7$ and $u_c=0.979$ (a) Reflection of an incident fluxon propagating with velocity $u=0.9$. The colour scale represents the superconducting phase difference $\phi$ from 0 to $2\pi$. (b) Cloning of a fluxon propagating with velocity $u=0.984$. The colour scale represents the superconducting phase difference $\phi$ from -1 to 7.5. In (a) and (b), solitons are represented by phase difference ranging from 0 to $2\pi$. 
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Flux Cloning Anomalities in Undamped System

Fig. 4.7: Numerical simulations of spontaneous movable semifluxon generation arising during a reflection of an incident high energy fluxon propagating with the velocity \( v = 0.97, \theta = 1.7 \). The critical velocity is here equal to \( v_c = 0.979 \). A movable semifluxon is here temporarily formed after the initial fluxon has crossed the T-junction. Then, it is annihilated after a very short time. The colour scale represents the superconducting phase difference \( \phi \). A soliton is represented by 0 to \( 2\pi \) phase differences whereas (anti-)semifluxons are represented by phase differences as \( \Delta \phi = \pi \).

Fig. 4.8 The phase difference \( \phi \) along x-axis at \( t = 0 \) (solid line), \( t = 5 \) (dotted line), and \( t = 6 \) (dashed line). A movable semifluxon has been temporarily formed after the fluxon has crossed the T-junction. For a movable semifluxon the change of the phase difference is equal to \( \pi \).
4.1.3 Conclusion:

We have demonstrated numerically that in conventional T-Josephson junctions it is possible to observe movable semifluxons. In general, all previous studies have only shown localised magnetic vortices carrying half flux quantum $\pm \phi_0/2$, i.e. immovable semifluxons and pinned by topological defects in 0-π JJ. In the case of the very large extended T-shaped Josephson junctions, when $W > W_0$, our numerical calculations show clear evidence of three different interesting and important features for semifluxon generation: (a) (anti-)semifluxons can spontaneously appear at the T-shaped junctions when a high energy fluxon is crossing this junction, (b) (anti-)semifluxon is not localized but it may move freely along the long branches of the Josephson junction, (c) a spontaneous generation of periodic movable (anti-)semifluxon chains may occur. Also there are spontaneous series of annihilation and creation between anti- and semifluxons. These semifluxons and their dynamics are very useful an understanding the behaviour and decay of high energy elementary particles. If the fluxon is equivalent to such particle, then the semifluxon will be equivalent to a quark. Thus, the T-shaped Josephson junction may be considered as a very economic laboratory for studying such a quark creation and other high energy phenomena [fHK10].

4.2 Rotated T-shaped Josephson Junctions

On the other hand, there is another design with regard to the T-shaped geometry. The main JTL is connected to the centre of the additional JTL to form a T-junction. A 90$^\circ$ anticlockwise rotation the previous T-junction (Fig. 2.10 (a)) to give a rotated T-junction (Fig. 4.9). In this case, the properties of the vortices after cloning are slightly different from the previous method of cloning due to the changing situation between the T-shaped arms. In this case the longitudinal momentum of the soliton after cloning vanishes. Both the mother fluxon (after cloning) and the baby vortex will propagate as vortex and anti-vortex in the additional JTL (up and down, respectively) in the direction perpendicular to the original one. They have the same physical properties because of the symmetry of the JTL. In this thesis, we focus on this kind of junction in the absence of both bias current and a magnetic field.
4.2.1. Analytical Calculations

In rotated T-shaped junctions, two different types of behaviour are also possible - either reflection from the rotated T-junction (Fig. 4.10 (a)) or flux cloning (Fig. 4.10 (b)). Nevertheless, the critical velocity is changed because of the change of geometry and the change of the width and the propagation direction of the mother vortex after cloning. It can be estimated from the following energy considerations. In general, the energy, eq. (2.12), during flux cloning in rotated T-junction is conserved

\[
\frac{8W_{\phi}}{\sqrt{1-u_i^2}} = \frac{8W + 6W}{\sqrt{1-u_f^2}}
\]  

(4.1)

where \(u_i\) and \(u_f\) are initial and final velocity of fluxon, respectively. The final velocity for cloned vortices is

\[
u_f = \sqrt{1 - \frac{4W^2}{w_0^2} (1 - u_i^2)}
\]  

(4.2)

To calculate the critical velocity at the T-fork area, the minimal energy required for the fluxon splitting in the MJTL is equal to twice the rest mass (8W) of the fluxon in the AJTL.
\[
\frac{8W_o}{\sqrt{1-u_i^2}} = 8W + 8W. \tag{4.3}
\]

This gives the critical velocity as

\[
u_c = \frac{\sqrt{4W^2 - W_i^2}}{2W}. \tag{4.4}
\]

This means the initial velocity of soliton is larger than the critical velocity, \(u_i > u_c\). In addition, flux cloning takes place and vortices are propagated if the initial energy is greater than the final energy

\[
\frac{W_o}{\sqrt{1-u_i^2}} > \frac{2W}{\sqrt{1-u_i^2}}. \tag{4.5}
\]

At the moment of cloning, \(u_f = 0\)

\[
W < \frac{W_o}{2 \sqrt{1-u_i^2}} \tag{4.6}
\]

From eq. (4.6), we see that if the width AJTL, \(W\), is small enough, then there are more than enough energy for flux cloning, i.e. flux cloning must occur at any vortex speed. In this case, vortices, which are generated, are speedily propagated in AJTL. When the width AJTL, \(W\), is increased, then the speed of cloned vortices is decreases.

However, eq. (4.4) imposes the restriction between the widths in the MJTL and the AJTL. For a finite critical speed, it is necessary that the width of the AJTL should be greater than half the width of the MJTL, \(W > W_o/2\). In principle, theoretically, when \(W < W_o/2\), the critical velocity is zero. The vortex moving at any speed will be cloned. In reality, however, the initial soliton speed should be greater than critical value, \(u_i > u_c\). We attempt to evaluate it by using an approximate method. Because of the energy barrier (strong curve, nearly circular) in the area of the T-junction, we predict the critical velocity from the curvature. In the case \(W \ll W_o/2\), however, the critical velocity cannot be predicted although the final velocities does not equal to 0, \(u_f \neq 0\). Also cloning still occurs because the initial soliton has enough energy to clone.
4.2.2 Numerical Simulations

The first purpose of this section, section A.1, is to study numerically the variety of dynamic behaviours of the flux that can be expected in a rotated T-Josephson junction system. In addition, we investigate whether or not flux cloning will occur when \( W < W_0/2 \) and attempt to estimate numerically the critical velocities in this specific case. The same way, the numerical value for the critical velocity has been evaluated when the change of behaviour of a fluxon from reflection [Fig. 4.10 (a)] to splitting [Fig. 4.10 (b)] has occurred. The second purpose, section A.2, is to investigate the dynamics behaviour of the high energy soliton when \( W > W_0 \). For these we assume that the stripes of which Josephson junction is made of are narrow, \( W_0 \) and \( W \leq 1 \), then we have studied the behaviour of a Josephson vortex in the T-shaped Josephson junction and the effect of widths \( W \) and \( W_0 \) on the flux cloning. In order to understand our finding, we limit our analysis to the change in the width of the additional JTL, \( W \), only, (changing it from 0 to 1 using units of \( \lambda_f \)) and focus on the soliton behaviour in cases \( W \leq W_0/2 \), \( W > W_0/2 \) and \( W > W_0 \). Without loss of generality, the width of the main JTL, \( W_0 \), has been fixed at 0.8.

A.1 Flux Cloning in Rotated T-shaped Josephson Junctions

In case \( W > W_0/2 \), flux cloning has been clarified. For instance, Fig. 4.10 shows numerical simulations of the time-dependent two dimensional sine-Gordon equations in a rotated T-junction, when \( W_0 = 0.8, W = 0.5 \) and \( u_c = 0.6 \). In addition, the two different behaviours of the fluxon (reflection and flux cloning) are illustrated when it approaches the T-junction. We have used initial conditions, eqs. (3.6) and (3.7) with the initial soliton position \( x_0 = -3 \), and initial velocities \( u = 0.55 \) in case of reflection (Fig. 4.10 (a)) and \( u = 0.65 \) in case of flux cloning (Fig. 4.10 (b)).
The following result of the numerical simulations shows a very clear flux cloning phenomenon when \( W \leq W_0/2 \), (see Fig. 4.11 (a)). In this range, although the critical velocity cannot be predicted by eq. (4.4), rather, it is evaluated numerically. For instance, when the width of AJTL, \( W \), equals 0.4, the critical velocity, \( u_c \), is evaluated numerically and equals 0.37. The baby vortex starts splitting and may propagate in a direction opposite to the mother vortex propagation when initial conditions are the same as those above. According to the conventional T-junction, the value of the critical velocity, which is calculated using eq. (2.21), is different although the widths of the MJTL and the AJTL are similar to the rotated T-junctions. In Fig. 4.11 (b), the flux cloning occurs when the critical velocity is equal to 0.75, which is much greater than critical velocity, \( u_c \), in the rotated T-junction. Consequently, it is clear that in the rotated T-junction, flux cloning can occur from a slow soliton and does not require high levels of energy comparable with what is needed in T-junction.
Flux Cloning Anomalies in Undamped System

Chapter 4

Fig. 4.11: (a) Flux cloning arising in rotated T-junction when a fluxon is propagating with the velocity \( u = 0.4 \) when \( W_0 = 0.8 \), \( W = 0.4 \) and the critical velocity \( u_c = 0.37 \). (b) Flux cloning arising in T-junction when a fluxon is propagating with the velocity \( u = 0.8 \) when \( W_0 = 0.8 \), \( W = 0.4 \) and the critical velocity \( u_c = 0.75 \). The critical velocities are different in these two cases although \( W_0 \) and \( W \) are the same. The colour scale represents the superconducting phase difference \( \phi \).

The critical velocity is a function of the width of the additional JTL, \( W \), shown in Fig. 4.12 when \( W_0 = 0.8 \). When \( W \leq W_0 / 2 \), numerical values of the critical velocity, \( u_c \), are also evaluated and shown in Fig. 4.12 by dots. When \( W > W_0 / 2 \), the comparison between the numerical simulations and critical velocities obtained from analytical expressions is presented in the same figure. The value of critical velocity, \( u_c \), can be calculated analytically using eq. (4.4) from the energy conservation relation. One may notice that although the results obtained numerically and using analytical expressions are strictly coincidental when \( W > W_0 / 2 \), they are divergent after \( W \geq W_0 = 0.8 \). Therefore, if the value of the critical velocity (\( u_c \)) satisfies the condition \( 0 < u < 1 \) and approaches the limit (\( u_c \approx 1 \)), flux cloning occasionally may not occur. When the width of the AJTL, \( W \), is greater than that of the MJTL, \( W_0 \), \( u_c > 0.87 \), numerical simulations show that a mother vortex will struggle to clone vortices in AJTL, but there are fractional vortices, i.e. semifluxon and others can be temporary created, when the energy of the initial soliton is high enough. The high energy cloning may occur together with a spontaneous creation of shape waves or with some other excitations such as plasma waves. For example, when the width, \( W = 1 \), the critical velocity is calculated by eq. (4.4) and is equal \( u_c = 0.92 \). However, our numerics show that when the mother vortex approaches the T-junction with such a speed it is just reflected (see Fig. 4.13), and flux cloning does not occur. However, when the width \( W = 0.9 \), the critical velocity calculated with the use of eq. (4.4) is
equal to \( u_c = 0.9 \). However, instead of the expected flux cloning we see the formation of fractional vortices (see Fig. 4.14). These fractional vortices such as semifluxon and anti-semifluxon and other fractions (fractional semifluxons) live for a short time and after created they are transformed into each other on a very long time scale. We investigate their dynamical behaviour below when we consider a very long AJTL.

**Fig. 4.12:** Schematic representation of the dependence of the critical velocity \((u_c)\) and the width of AJTL \((W)\) when \(W_c=0.8\). The red dots correspond to critical velocity obtained in numerical simulations of the superconducting phase dynamics with the use of the 2D sine-Gordon equation. The solid line represents the theoretical predictions based on analytical calculations by using eq. (4.4).

**Fig. 4.13:** Numerical simulations of the reflection, in rotated T-junction, of a fluxon propagating with velocity \( u = 0.99 \) when \( W_0 = 0.8 \), \( W = 1 \) and the critical velocity \( u_c = 0.92 \). The colour scale represents the superconducting phase difference \( \phi \). Solitons are represented by the region of the fast phase change which ranging from 0 to \( 2\pi \).
Fig. 4.14: Numerical simulations of flux cloning of a fluxon, which was propagating with velocity $u = 0.99$ when $W_0 = 0.8$, $W = 0.9$ and the critical velocity $u_c = 0.9$. The shape waves are arising both in mother and baby vortex lines and will propagate along AJTL together with some other excitation such as plasma waves, that will also arise, propagate and reflected in MJTL. At time $t = 7$, we may see already the formation of the solitons on both sides, up and down, along the AJTL, but the structure of each of these solitons is not trivial. One may see that each of them consists of two $\pi$-solitons separated nor far that means that they are strongly bound.

A.2 Dynamic Behaviours and Explosions of High Energy Josephson Vortices in Extended Rotated T-junctions

At width $W = 0.9$, when the soliton moves with high energy and velocity $u = 0.99$, we observe the formation of fractional vortices which arise in the process of the vortex fission. It is very interesting to investigate in detail the fission process of the initial soliton and its evolution in the rotated T-junction. For this purpose let us first investigate how the electrodynamic of fluxons are changing along $x$-axis, MJTL, when the bent soliton enters to the T-junction. The first stage of this process happens for example during the time interval between the time $t = 3.3$ until the time $t = 7.3$. In the time $t = 3.3$, the soliton begin to divide and move on the AJTL up and down (Fig. 4.15).
3.3. A fluxon was propagating with velocity $v = 0.99$ when $W_0 = 0.8$ and $W = 0.9$. The length of red arrows represents the value of magnetic flux carried by this nonlinear wave. The longest arrows are associated with the largest flux density.

Probably, here the soliton does not have enough energy to complete the formation of real stable vortex-antivortex pair. As a result, therefore, instead there is a cloned pair of fractional vortices which begin to move along AJTL. This is clearly illustrated on the next snapshot taken at about next moment of time, at $t = 3.4$ shown in Fig. 4.16. At this moment one may see a clear formation of an instantaneous semifluxon and anti-semifluxon pair which are slightly disturbed by a shape wave. In addition, we see here also a formation of another semisoliton which is formed in the MJTL and pinned near the T-junction. Also, in Fig. 4.16, we notice a formation of a new nonlinear wave originated from the central point (see Fig. 4.9). This nonlinear wave may give a birth to a new fractional anti-soliton. This central point is in fact the touching point for initial soliton, which during the fission process reaches opposite wall of T-junction, AJTL boundary (1) (see Fig. 4.15).

![Fig. 4.15](image-url)
Then, this new nonlinear wave, starting from the central point, or in other words the new fractional anti-soliton is expanded as exploding in the whole T-junction area in the opposite direction of the initial soliton, (see Fig. 4.17). During the time interval from $t = 3.5$ to 4.3, in the turbulent fission area, the fractional vortices are strongly interacting and one may see, for example, that the new fractional anti-soliton will push the semifluxon and all together will be moved in the MJTL. Meanwhile, the other pair of fractional vortices moves in the AJTL as a friendly pair consisting of the fractional soliton and anti-soliton. The snapshot of the evolution of this pair is presented in Fig. 4.18.
Fig. 4.17: Snapshot of soliton fission and the formation of the fractional vortices (the two semifluxons and two anti-semifluxons) at time $t = 3.5$. On the square insert we present the phase profile of the fractional pair taken along the horizontal line and central point. There one can clearly see that the phase change on the wave front is equal to $\pi$ that is associated with the semifluxons or anti-semifluxons.
Fig. 4.18: Snapshot of the phase evolution during the high-energy soliton fission process arising at the T-junction in the time interval from $t=3.6$ to $t=4.3$. On the square insert we present the phase profile of the fractional pair taken along the horizontal line and central point. During this time interval after the creation of the fractional vortices there arises an internal phase explosion started from the central point which is indicated by arrows. The central point is here surrounded by a semicircle associated with a profile of the extended fractional fluxon. The radius of this semicircle increases with time and does so the length of the fluxon. This expansion of the fractional vortex continues until it reaches the opposite side of the T-junction, AJTL boundaries (2) and (3), that happen at the time $t=4.3$. In the central region this fractional anti-vortex collides with fractional vortex which was originally pinned by the junction and localized in the MJTL. Then the pair becomes nearly localized in the MJTL as the original first semifluxon. It is of note that this localization of the semifluxon pair in the MJTL exists for a short time and does not depend on or only vaguely depends on the dynamical nonstationary processes existing at the same time in other parts of the T-junction. At the next time interval two nonlinear back waves will originate from the corners of the T-junction (see Fig. 4.19).
A long time after the fission, the fractional vortices transform into a semisoliton and anti-semisoliton as shown in Fig. 4.19 at time $t=7.3$ (this is explained in more details in second stage of the fission process, the snapshots of the phase was taken at the time $t=7.3$ when the evolution is mostly focused along the AJTL). There, at this stage double pairs of semivortex and anti-semivortex are created. They are strongly bound. In addition at very initial stage of those pairs' formation, there are nucleations of small fractional vortices (fractional semifluxons), that are located on boundaries (2) and (3). These small fractional vortices are also expanded in AJTL (see Fig. 4.20).

Fig. 4.19: The typical distribution of the phase arising at long-time scales. Here it is shown at time $t=7.3$. We may see already the formation of the solitons on both sides, up and down, along the AJTL, but the structure of each of these solitons is not trivial. One may see that each of them consists of two strongly bound $\pi$-solitons.

Fig. 4.20: Snapshot taken at time $t=4.5$ indicating fractional semifluxon creation. In Fig. 4.18, the first one may see the formation the bound semifluxon-anti-semifluxon pair which is localized near and left side from the T-junction. This pair is practically not moved. On the other hand from the right side of this pair one may see a strong front of the $\pi$-wave which is slightly curved along the vertical direction and reminds two joint semi-circles propagating to the right. The steepness of this wave and/or the value of the magnetic flux carried by this wave is indicated by the length of the arrows. With the next time interval at time $t=4.5$ this from will reach the opposite side of the T-junction on boundaries (2) and (3) with the creation of two fractional semifluxons, one of which will be propagating up, while the second will be propagating to down along the AJTL.
In MJTL, an appearing temporary semifluxon and fluctuating fractional anti-semifluxons were propagated here for a short time (see Fig. 4.17). Then, the semisoliton gradually decayed because of fractional anti-semisoliton originated from the opposite wall of the T-junction, AJTL boundary (1). The evolution of phase on $x$-axis is shown these changes of the phase dynamics during the time interval $t = 2.7$ to $t = 6$ (see Fig. 4.21).

![Image](image.png)

**Fig. 4.21:** The evolution of phase along $x$-axis from the time $t = 2.7$ to $t = 6$. Soliton transfers to temporary semifluxon and fluctuating fractional anti-semifluxon. Different colours correspond to different moments of time, with time step 0.1.

Next, we consider the evolution of phase along $y$-axis, AJTL boundary (1), at the initial time interval when the original bent soliton is entering the T-junction and reaches the opposite wall (boundary (1)). This happens for example during the time interval $t = 3.3$ to $t = 7.3$. The evolution profile of the superconducting phase on the AJTL boundary (1), (see, Fig. 4.9), is presented in Fig. 4.22. Here a central point corresponds to zero, $y = 0$. It is clear from the evolution of this profile that a dynamical pair of semifluxon anti-semifluxon and double pairs of fractional, close to semifluxon anti-semifluxon pair, are generated. In more details, during the time interval $t = 3.3$ to 5 (see Fig. 4.23), a pair of fractional vortices is initially created. At time $t = 3.6$ the dip in the phase is the largest and one may believe that at this moment the vortex-antivortex pair appears almost likely to be created. Indeed, the dip in the dependence of the phase on the coordinate $y$ at the value $y = 0$ is nearly equal to $2\pi$. However, at the next moment of time the depth of the dip decreases until it reaches the nearly flat region $\pi$. And one may say that this moment corresponds to a formation of an instantaneous semifluxon and anti-semifluxon pair which is separated by a distance of four Josephson lengths, see Fig. 4.24.
Fig. 4.22: The evolution of phase ($\phi$) along $y$-axis in the time interval $t = 3.3$ to $t = 7.3$. In beginning, a pair of semifluxon anti-semifluxon were generated and then a double pairs of fractional, close to semifluxon anti-semifluxon pairs were created. Different colours correspond to different moments of time when time step is taken as 0.1.
Fig. 4.23: The evolution of phase ($\phi$) along $y$-axis with the time from $t = 3.3$ to $t = 5$. Different colours correspond to different moments of time with time step is 0.1.

Fig. 4.24: Snapshot of the phase taken at the time $t = 4.8$, indicating the formation of the semifluxon and anti-semifluxon separated by a distance of 4 Josephson lengths.
The formation of such temporary semifluxons and anti-semifluxons may be justified when we investigate the flux density arising in the course of such a phase evolution. Since the magnetic field component in \( y \)-direction is equal to \( H_y = \partial \phi / \partial y \), we immediately find the distribution of magnetic flux is presented in Fig. 4.25. The maxima correspond to vortices and minima to anti-vortices. The curve with the highest maximum and deepest minimum corresponds to the time \( t = 3.6 \). It is clear that when time increases the height of the maximum decreases. From this figure we see, first, that indeed the fractional vortex-antivortex pair has been created. There the bump corresponds to the vortex while a deep minimum (anti-bump) corresponds to an anti-vortex. During this time interval the distance between vortices in the pair increases and the flux carried by each of these solitons decreases.

![Fig. 4.25: The evolution of the magnetic flux density during the time interval from \( t = 3.6 \) to 4.7 for the created semifluxon pair. Different colours on this figure correspond to different moments of time.](image)

In the next time interval, from \( t = 5 \) to \( t = 7.3 \), the overall evolution of phase shows that the new fractional soliton anti-soliton pairs are generated (Fig. 4.26). Then, the phase gradually transforms from the fractional vortex anti-vortex pair to the newly created semifluxon- anti-semifluxon pairs with a hump [SVG04] as shown in Fig. 4.26 and Fig. 4.27. In contrast to usual semifluxons with a hump, which can be constructed when \( \gamma \neq 0 \), these semifluxons with a hump may be constructed because of moving semifluxons.
Fig. 4.26: The evolution of phase ($\phi$) along $y$-axis within the time, from $t = 5$ to $t = 7.3$. At $t = 5$ there is only a pair of semifluxons. Then one sees here that later during the course of the phase evolution double pairs of fractional soliton and anti-soliton are generated. Different colours correspond to different moments of time, with time step 0.1.

Fig. 4.27: Snapshot of the phase taken at time $t = 7.3$ and indicating the formation of double pairs of semifluxon and anti-semifluxon with a hump.
Now, specifically, in the time interval from $t = 4.8$ to $t = 5.3$ the evolution of the phase indicates that the pair of semifluxons which have just been created begins to decay after a very short time interval (Fig. 4.28). This happens because there is still a flow of energy in the form of plasma waves along the MJTL that leads to their focusing and a formation of the new nucleated fractional (not fully developed) vortex antivortex pair located in the region between these original semifluxons (Fig. 4.29), where the flux density associated with this phase evolution is presented. There we see the additional two maxima and two minima arising between the semifluxons. The maxima correspond to the nucleated fractional vortices while the minima correspond to the nucleated fractional anti-fluxons. We see that with the time evolution from $t = 4.8$ associated with the dark colour to $t = 5.3$ associated with the lightest colour the flux gradually transforms from the semifluxons to the newly created fractional vortices.

**Fig. 4.28**: The evolution of phase ($\phi$) along $y$-axis with the time from $t = 4.8$ to $t = 5.3$. The pair of semifluxons are created at time $t = 4.8$ and then they begin to decay after a very short time with the formation of other fractional pair located just around the central point.
Fig. 4.29: The evolution of the magnetic flux density during the time interval from $t = 4.8$ to $t = 5.3$ for the created double semifluxon pair. Different colours correspond to different moments of time with time step 0.1.

In the next stage of the time evolution, in the time interval from $t = 5.6$ to $t = 6.3$, we see the formation of a strongly fluctuating region, which is associated with the bottom of the following dip in the dependence of the phase on the coordinate $y$. Its evolution is shown in Fig. 4.30 and Fig. 4.31. Due to their fast dynamics the formation and evolution of these fluctuations are also accompanied with the creation of inhomogeneous electric potential (Fig. 4.32). The maximal fluctuation of the electric potential arises exactly in the area of the magnetic flux fluctuations. At the later stage of the evolution all these electromagnetic fluctuations are slowly dispersed and vanish, transforming into radiating plasma waves.
Fig. 4.30: The evolution of phase during the time interval from $t = 5.6$ to 6.3. Different colours correspond to different moments of time with time step 0.1.

Fig. 4.31: The evolution of the magnetic flux density during the time interval from $t = 5.6$ to $t = 6.3$. Different colours correspond to different moments of time with time step 0.1.
Fig. 4.32: The evolution of the current on y-axis during the time interval, from \( t = 5.6 \) to \( t = 6.3 \). Different colours correspond to different moments of time with time step 0.1.

4.2.3 Conclusion

In summary, we have demonstrated that high energy processes such as those arising in LHC can be modelled and investigated in condensed matter physics laboratory. We model the LHC in a form of the T-shaped Josephson junction, rotated T-junction, where two Josephson transmission lines are crossed. The high energy particles have been modelled with the use of Josephson fluxons accelerated to a very high relativistic speed. The fluxons have been accelerated along the main transmission line (MJTL) and entering the area of the T-junction with a speed closed to speed of light. We show that in this case such a collision of the fluxon with the T-junction causes it to be split into two \( \pi \) vortices (semifluxons), like an elementary particle split into other new particles. At this moment, these \( \pi \) vortices do not move far because of strong force between them. However these \( \pi \) vortices are dynamical and evolve or are transformed fast into other ones ever separating from each other because fractional semifluxons generate and break the bound between pair semifluxons. The situation is somewhat reminiscent of the splitting of an elementary particle into quarks, which are confined by strong forces and never separating far from each other. Nevertheless, when the fluxon speed is not as large, we found a conventional flux cloning where one initial transmitting vortex is cloned and divided into two other vortices propagating through two branches of the Josephson junction forks. There the two new vortices have the same properties because of the symmetry of the T-junction. We checked numerically the theoretical
predictions of critical velocity for a set of 2D geometries with varying JTL widths. In general, we observe here that to have the high energy vortex splitting into fractional vortices (as one may observe in future LHC) the two symmetrical branches of the T-junction must have a width larger than the initial one (i.e. \( W > W_0 \)) where the original vortex is accelerated to relativistic speeds.

In addition, we point out that the differences between flux cloning in two kinds of T-junction may be crucial. In conventional T-junction, the original vortex is divided into two vortices and propagated through two branches of T-junction. Therefore, one of the two vortices has the same properties of the original one. In rotated T-junction, however, the vortex is divided into two other vortices propagating through one branch of T-junction, which is not the same branch of the transmitting vortex. Therefore, two new vortices have the same properties because of the symmetry of the T-junction. In addition, in narrow AJTL, \( W > W_0 \), the initial vortex in conventional T-junction are required to clone higher energy than that in rotated T-junctions, i.e., flux cloning can arise from slow vortex.
Chapter 5

Flux Cloning Anomalies in T-Shaped Josephson Nano-Junctions Damped System

In the previous chapter, numerical simulations were presented for undamped SGE in both kinds of T-shaped Josephson junctions. The case of damped SGE may be different because energy is not conserved. The loss of energy as a small perturbation leads to an exponential decay of the energy when the soliton moves within the junction. Therefore, the velocity of the flux decreases [BP82]. The balance between the net energy input and dissipation requires applying bias current to determine the condition of a stable soliton motion [WWY92]. In this situation, in general, biased by both applied bias current and applied magnetic field, the junction has important applications as flux flow oscillator (tunable high frequency oscillator) in the millimeter and the submillimeter range. Here, conventional T-junction can also be used as a FFO. In this chapter, we study numerically a variety of dynamic behaviours of flux cloning that can be expected in conventional T-shaped Josephson junctions system with energy dissipation, \( \alpha = 0.1 \) - this is the value used in experimental study [fHGD*10], from the wave system when an external magnetic field and a bias current are applied.

5.1 Extended Conventional T-shaped and Rotated T-junctions

As we found numerically in chapter 4, in undamped SGE system, the fluxon moving with velocity higher than critical velocity can generate another fluxon by means of T-junction. However, when the value of the critical velocity (\( u_c \)) approaches the limit (\( u_c \approx 1 \)), \( W > W_0 \), the soliton may clone or reflect with generation of fractional flux. The purpose of this section
is a numerical study of the variety of dynamic behaviours of flux cloning that can be expected in T-junctions system with the energy dissipated from the system when there is no applied bias current. In addition, the relation between the thicknesses, $W_0$ and $W$, to satisfy flux cloning will be investigated. In the numerical investigations, we assume both, $W_0$ and $W \leq 1$ in units of Josephson penetration depth ($\lambda_J$). In order to understand the behaviour of the Josephson vortex in extended Josephson junctions we have limited our analysis to the change in the width of the additional JTL only, and have focused on the behaviour in three cases $W > W_0$, $W = W_0$, and $W < W_0$.

5.1.1 Numerical studies of flux cloning

We study briefly the time-dependent perturbed sine-Gordon-equation, eq. (2.5) with $\alpha = 0.1$, with boundary conditions in the case where there no applied magnetic field and bias current, eq. (3.5), and initial condition, eqs. (3.6) and (3.7). We take $W_0 = 0.6$ in T-shaped Josephson junctions of both kinds in Fig. 2.10. The critical velocity is calculated, using eq.(2.21) or eq. (4.4), to satisfy the flux cloning in conventional T-Josephson junction or rotated T-Josephson junction, respectively.

In conventional T-shaped junction, when $W < W_0$, the mother vortex can give birth to a new vortex and flux cloning occurs as shown in Fig. 5.1. However, for $W > W_0$, the numerical simulations illustrate that a mother vortex struggles to create a baby vortex. Under these circumstances, the cloning phenomenon does not occur. Although the vortex is moving fast, $u = 0.99$, and it has a huge energy, it still does not give birth to a new vortex. Consequently, the mother vortex is reflected when it approaches the T junction (see Fig. 5.2 (a)). Similarly, the same strange result is obtained when $W = W_0$ (see Fig. 5.2 (b)). These results may be due to another kind of excitation entitled breather soliton.
Fig. 5.1: Cloning of a fluxon propagating with velocity $u = 0.85$ when $W = 0.4 < W_0 = 0.6$ and $u_c = 0.8$.

Fig. 5.2: Reflection of mother vortex propagating with velocity $u = 0.99$ when (a) $W = 1 > W_0$ and $u_c = 0.93$. (b) $W = W_0 = 0.6$ and $u_c = 0.87$.

The results for rotated T-junction are represented in Fig. 5.3. The numerical investigations show that flux cloning can occur only in case $W < W_0$. Therefore, in general, flux cloning will occur without a distribution vortex only if a copy of the original vortex moves to the narrow width of AJTLs either T-junction or rotated T-junction. In other words, the ratio between the widths, $W_0$ and $W$, should be less than one, $W/W_0 < 1$. 


Fig. 5.3: (a) Reflection of mother vortex propagating with velocity $u = 0.99$ when $W = 1 > W_0 = 0.6$ and $u_c = 0.95$. (b) Reflection of mother vortex propagating with velocity $u = 0.99$ when $W = W_0 = 0.6$ and $u_c = 0.87$. (c) Cloning of fluxon propagating with velocity $u = 0.99$ when $W = 0.4 < W_0 = 0.6$ and $u_c = 0.66$.

5.1.2 Conclusion

We investigated the behaviour of soliton in damped system without applying a bias current. We have found that when an additional Josephson transmission line is larger than the main Josephson transmission line, numerical simulations do not show the cloning phenomenon at all and soliton is reflected at the T junction. This strange result may happen because the soliton loses more energy in the sharp edge. Although the vortex is moving very fast and has huge energy, it still does not give birth to a new vortex. We have investigated conditions at which flux cloning occurs when both widths, $W_0$ and $W$, are changing. This leads to conclusion that the width of AJTL should be less than that of MJTL [fHK09].
5.2 Flux Flow Oscillators (FFO) with Cloning Circuits

In general, fluxon cloning circuits provide the producing fluxon (in AJTL) without an applied magnetic field. Therefore, they can be used as a FFO operating without external magnetic field. It is worth noting that there are many interesting configurations for along Josephson junctions such as overlap, inline and annular geometries. However, the inline and annular structures are not suitable for application as flux flow oscillators (FFO) [Jaw08]. Therefore, previous studies have used only overlap structure as FFOs operate by applying external magnetic fields. A standard overlap junction is called a conventional FFO junction. In this structure, there arise Fiske resonances which are associated with the emission of electromagnetic radiation. Recently, it was shown that the cloned vortices may be ordered to form a train of fluxons, which is eventually operating as a flux flow oscillator created without an external magnetic field for annular geometry by means of a conventional T-junction [GKS’08]. Here, we will confine our attention for first time to study theoretically and show experimentally how a fluxon cloning circuits can be used as a FFO operating without an external magnetic field from linear overlap geometry shown in Fig. 2.3. In fluxon cloning circuits design, when the magnetic field, which is applied only to main FFO, and the bias current are introduced in the direction perpendicular to the long dimension of the junction, the FFO is tapered from both main FFO and additional FFO side. Therefore, the additional FFO side can be used as a FFO operating without external magnetic field. As a result of this modification, the Fiske resonant structure is almost totally suppressed.

5.2.1 Theoretical Study of FFOs with Fluxon Cloning Circuits

In this design the FFO consider a fluxon cloning circuit by means of T-shaped Josephson junction as shown in Fig. 5.4. The boundaries A’, D’ and G’ in Fig. 2.11 are converted to the sharp constrictions A, D and G as shown in Fig. 5.4, which are added to minimize reflection of vortices from the ends of the junction as shown, for example, in Fig. 5.5, Fig. 5.6 and Fig. 5.7. The magnetic vortices only penetrate from boundaries A and G of the main JTL. In these circuits, a main JTL branch (width $W_0$) is a main FFO junction (conventional FFO) and an additional JTL branch (width $W$) is a side’ FFO junction (additional FFO). In addition, all part of a Josephson junction oscillates with the same frequency (i.e. the mean voltage in all parts of a Josephson junction is the same).
When the applied external magnetic field exceeds the critical value, $h_{\text{min}}$, which is required for the fluxon to penetrate inside the junction in the absence of a bias current, the magnetic flux begins to enter through both boundaries A and G of the main FFO and pushed into the junction in the form of separated vortices. These vortices are pinned on the inhomogeneity of the junction (see Fig. 5.5 (a)). As the field increases, see Fig. 5.5 (b), new vortices penetrate through the boundaries and push the previous ones towards the T-junction [DS94]. At the T-junction, vortices are stuck because they do not have enough energy to overcome the barrier energy of the T-junction. At a high value of $h$, although fluxons may flow fast into the junction, they still cannot fission; see Fig. 5.5 (c).

Furthermore, fluxons injected into the main FFO junction boundaries by an external magnetic field are accelerated into the interior of the junction by an externally applied bias current. If the current exceeds its critical value, the flux into the junction will necessarily begin to flow [Par93, Ust98a] and fission at the T-junction. With $\gamma > 0$, the critical value of the magnetic field is reduced [Par93, CDL'97]. When the combination of $\gamma$ and $h$ is appropriately above threshold values, fluxons may have enough kinetic energy to fission at the T-junction and each time when the vortex is passing the T-junction vortex cloning may arise and the new cloned vortex moves along the transverse branch of the T-junction (additional JTL, see Fig. 5.4). First, fluxons penetrate the main FFO junction from both boundaries A and G, as shown in Fig. 5.6. These fluxons, which are propagated in the junction, are controlled by two forces: the driving force, which acts on vortices in the $-x$ direction and the magnetic force, which acts in the $x$ direction on entering vortices from boundaries A and G. Second, at a low bias current, the effect of a Lorentz force is weak. Vortices penetrate the junction from both boundaries A and G and accelerate towards the opposite side. Nevertheless, the fluxons’ motion from boundary G is faster than that from boundary A. Therefore, the number of fluxons on the right side is higher than on the left side. In addition, the vortices on the left side...
may be pinned in the junction (Fig. 5.6 (a) and (b)), depending on the strength of the magnetic field. Meanwhile, the fluxons on the right side begin to fission very slowly by the T-junction and move in main and additional JTL toward boundaries A and D, respectively. Then, the cloned fluxons, which move in main FFO, begin propagating to the left and push the vortices which are pinned in the left side of the junction. After a long time, therefore, all fluxons in main FFO move to the left.

Eventually, as the value of $\gamma$ is increased still further, the fluxons may start to flow through the main FFO junction only from boundary G (from right to left) because of a Lorentz force (see Fig. 5.6 (c)). At the T-junction, the first fluxon begins splitting into two vortices in main FFO and additional FFO. Then, they continue propagating to the junction boundaries A and D. The same process happens with the second, third and following fluxons. Therefore, the train of fluxons flows into additional JTL, which may be used as an FFO operating without an applied external magnetic field.

### 5.2.2 Numerical Results

We have studied the time-dependent sine-Gordon-equation, eq. (2.5), with boundary conditions in T-Josephson junction. The boundary conditions, eq. (2.6), which represent applied current and magnetic field on junction, can be presented in the programme as

$$\vec{n} \cdot \vec{\nabla} \phi_{B,F,H} = -\gamma \frac{W_0}{2}$$

(5.1)

$$\vec{n} \cdot \vec{\nabla} \phi_{c,E} = -\gamma \frac{W}{2}$$

(5.2)

$$\vec{n} \cdot \vec{\nabla} \phi_{A,G} = \pm \frac{h}{\sqrt{a^2+b^2}}$$

(5.3)

$$\vec{n} \cdot \vec{\nabla} \phi_D = 0$$

(5.4)

Long Josephson Nb-AlO$_x$-Nb junctions with overlap geometry are used as FFOs with splitting circuits. These circuits have dimensions as $L = 400 \mu m$ and $W_0 = 16 \mu m$ for the main FFO and $L = 200 \mu m$ and $W = 8 \mu m$ for the “side” FFO. All lengths are represented in the programme in units of the Josephson penetration depth, $\lambda_J \approx 8 \mu m$ [KS00]. The damping parameter used is 0.1 and magnetic field and electric current bias are chosen qualitatively.
First of all, the dynamics of vortices have been studied in flux cloning circuits at zero driving current and at different values of the magnetic field, which were applied to boundaries A and G of the main FFO. Fig. 5.5 shows examples of the penetration of magnetic flux into the junction at different values of magnetic field \((h=1.4, 2, \text{ and } 3)\). At \(h=1.4\), only a single fluxon can enter but it stops after moving a short distance. When the magnetic field is increased, see Fig. 5.5 (b) and (c), one may see that, although the magnetic field is high, the vortices cannot clone at the T-junction. In Fig. 5.5 (b), \(h=2\), many fluxons can pass through the junction at \(t=40\), but these vortices are bond when they approach the T-junction. In case \(h=3\), although the concentration of the fluxons increases, they are trapped by the T-junction, see Fig. 5.5 (c). In addition, fluctuations in the external magnetic field affect the average distance between fluxons. It is clear in Fig. 5.5 (b) and (c) that the spacing between the moving fluxons is inversely proportional to \(h\).

![Fig. 5.5: Numerical simulations for the penetration of magnetic flux into junction without applied bias current. (a) At \(h=1.4\), only one vortex can enter from each of the boundaries A and G. They penetrate a small distance into the junction and then are pinned. (b) At \(h=2\), many vortices can pass through the junction. At the T-junctions, these vortices are bond and are not cloned. (c) At \(h=3\), the penetrated vortices move fast. The concentration of fluxons in the junction also increases. Vortex fission does still not occur. The colour scale represents the distribution of the magnetic field from -0.5 to 4.](image-url)
Further, we have investigated the efficiency of side FFO in flux cloning circuits when external bias current and magnetic field are applied to main FFO. First, the external magnetic field is fixed, for example $h=2$. The dynamics of magnetic flux at different bias currents can be traced on the colour plots in Fig. 5.6. At a low bias current, $\gamma=0.04$, fluxons penetrating from the right side of the T-junction are more numerous than those from the left side (see Fig. 5.6(a) at $t=40$). In addition, the vortices on the right side move at high speed, whereas vortices on the left side move slowly. The motion of these vortices, on the left side, gradually decreases when they approach the T-junction. Sometimes they may nearly hold inside the junction, as shown in Fig. 5.6 (a) at $t=70$ (the vortices are not pinned if $h=3$ as shown in Fig. 5.7). On the right side of the T-junction, the fluxons approaching the T-junction start fission slowly by the T-junction. Therefore, fluxons tend to concentrate and slow down before the T-junction. Then, the fluxons that have split accelerate and propagate in both main JTL and additional JTL toward the ends of the JTL. On the left side of the T-junction, vortices (in the main FFO) accelerate after passing the T-junction and push the pinned fluxons to move in $-x$ direction. As a result, after a long time, all fluxons in the main FFO move from right to left. When the bias current increases, the speed of vortices on the right side increases; however, the speed of vortices on the left side reduces. For instance, if $\gamma=0.08$, the dynamics of the magnetic vortices are exactly the same as if $\gamma=0.04$ but succession divisions require less time, as shown in Fig. 5.6 (a) and (b). In addition, the distance of the propagated fluxons from the left side is decreased (see Fig. 5.6 (a) at $t=70$ and Fig. 5.6 (b) at $t=50$).
Fig. 5.6: Numerical simulations for penetration of magnetic flux into junction with variable bias current at $h=2$. (a) At $\gamma = 0.04$, more vortices are entering from the right side of the main FFO. On left side of the main FFO, they struggle to approach the T-junction. Then, they are pushed to left by cloned vortices, which enter from T-junction. Vortex fission occurs slowly. (b) At $\gamma = 0.08$, the acceleration of vortices from the right side increases while the vortices from the left side slowdowns. The flowing of vortices and their fission take less time. (c) At $\gamma = 0.3$, vortices in main FFO flow from the right to the left and vortex fission occurs faster than in previous case. The colour scale represents the distribution of the magnetic field.
**Fig. 5.7:** Numerical simulations of magnetic flux penetration into the junction when $h=3$ and $\gamma = 0.04$. In main FFO, fast vortices enter from each boundaries A and G and move toward T-junction. Vortices on the left side continue to move toward T-junction without stopping (no pinning is observed). Because of speedy vortices originated from the right side, they pass through T-junction and continue moving to the left with pushing vortices on left side and moving to left direction. Succession divisions are a bit faster than in the case $h=2$ in Fig. 5.6 (a). The colour scale represents the distribution of the magnetic field.

At a high bias current of $\gamma = 0.3$, the magnetic fluxons viscously flow from right to left in main FFO. In addition, when vortices approach the T-junction, they can split effortlessly (see Fig. 5.6 (c)). From Fig. 5.6 one may see that the time required for sequencing divisions is inversely proportional to the value of bias current. In case $\gamma = 0.3$, a series of cloned fluxons occurs quickly. Therefore, the chain of fluxons can be produced in an additional JTL. In this case, the additional JTL can be used as a FFO operating without applying an external magnetic field to this stripe.

In addition, after proceeding to the FFO from vortex fission by using a T-junction, we briefly study numerically the dynamics of vortices in the conventional FFO in long Josephson junctions. Fig. 5.8 shows schematically the junction geometry used in the conventional FFO. At zero driving current, when a low magnetic field is applied to boundaries A and G of the conventional FFO, only a single fluxon can enter but it stops after moving a short distance, see Fig. 5.8. As the magnetic field is increased (Fig. 5.9 (a), $h=2$), many fluxons can pass through the junction. However, when vortices entering from boundary A approach those entering from boundary G, all vortices will be bond inside junction. If the external magnetic field is fixed, for example $h=2$. The dynamics of the magnetic flux at different bias currents can be traced on the colour plots in Fig. 5.9. At a low bias current, $\gamma =0.04$, fluxons penetrating from the right side of the junction are more numerous than those from the left side (see Fig. 5.9 (b) at $t = 40$).
In the case of applied bias current, the vortices on the right side move at a higher speed than those on the left side. The motion of these vortices, on the left side, gradually decreases. Sometimes they may nearly hold inside the junction, as shown in Fig. 5.9 (b) at t=70. Then, the fast fluxons, which enter from the right side of the junction, approach and push the slow fluxons on the left side to move in $-x$ direction. As a result, after a long time, all fluxons in the junction move from right to left. When the bias current increases, the speed of vortices on the right side increases; however, the speed of vortices on the left side reduces and the vortices pin, see, for example, Fig. 5.9 (c) and (d). In addition, the distance of the propagated fluxons from the left side is decreased, see Fig. 5.9 (c), and may hold in the sharp edge (see Fig. 5.9 (d)). At a high bias current, $\gamma = 0.3$, the fluxons fluently flow from right to left in conventional FFO, see Fig. 5.9 (d).

![Figure 5.8](image-url)  
**Fig. 5.8:** Numerical simulations of Josephson vortices in conventional FFO when applied magnetic field $h=1.4$. Only one vortex can enter from each of the boundaries A and G. They penetrate a small distance into the junction and then are pinned.
Chapter 5  
Flux Cloning Anomalies in Damped System

(a) $\gamma = 0$

(b) $\gamma = 0.04$

(c) $\gamma = 0.08$

(d) $\gamma = 0.3$

Fig. 5.9: Numerical simulations of the penetration of Josephson vortices into conventional FFO when applied magnetic field $B = 2$ and variable bias current. (a) At $\gamma = 0$, many vortices can pass through the junction but these vortices are bond in the centre of junction. (b) At $\gamma = 0.04$, the acceleration of vortices from the right side increases while the vortices from the left side slowdowns. Then, fast vortices pushed to left the slow vortices on the left side. (c) At $\gamma = 0.08$, the penetrated vortices from the left side of the FFO are moved slowly and then pinned, while the penetrated vortices from the right side of the FFO are moved fast. Then, they push to the left the pinned vortices on the left side. (d) At $\gamma = 0.3$, vortices in FFO flow from the right to the left.

IVCs for both conventional FFO and FFO with flux cloning circuits are shown in Fig. 5.10 (a) and (b), respectively. In general, from the established equilibrium dynamics the mean voltage can be extracted,

$$\bar{V} = \frac{h \omega \omega_p}{2e},$$

where $\omega$ is the frequency of oscillations in units of the plasma frequency $\omega_p$, which equals to $1.28 \times 10^{12}$Hz [PSK’07]. Measurements of I-V curves are made at different fixed values of the magnetic field (over the range $h = 2$ to $h = 5$) for a Nb-AlO$_x$-Nb FFO in both conventional FFO and FFO with flux cloning circuit see Fig. 5.10 (a) and (b). In this figure,
numerical simulations of conventional FFO give a good qualitative results with the numerical simulations of FFO with flux cloning circuits. In addition, it is clear that if a bias current is fixed, the voltage and thus the frequency of the FFO increases linearly with incremented magnetic fields. However, the Fiske steps and boundary voltage \( V_b = 0.9 \text{ mV} \) [KS00]) cannot be identified in IV curves in Fig. 5.10 due to Fiske resonance steps are caused by the interaction between the bias current and the microwave cavity modes in the dielectric barrier of the junction [GHF’93], which is not modelled at the moment in this investigation.

Fig. 5.10: I-V characteristics for a Nb-AlO\(_x\)-Nb FFO with incremented magnetic field. (a) Conventional FFO, (b) FFO with flux cloning circuits.
5.2.3 Comparison with Experimental Results

The described flux cloning circuits above have been designed and studied experimentally by using Nb-AlOₓ-Nb junctions [HGD'10, FSD'01, KS00] by Koshelets group. Each of two samples described in this thesis comprises additional JTL (width 4 or 8 μm, length 200 μm), is connected to the main JTL (width 16 μm, length 400 μm). On each flux cloning circuit two separate SIS junctions are attached to both main and additional FFOs providing the possibility to measure the power delivered by FFOs as well as the spectra of the FFO radiation. In this design the flat bottom electrode of the FFO is used as a control line in which the current $I_{cl}$ produces the magnetic field, which mainly is applied perpendicular to the long side of the main FFO junction.

IVCs of the conventional FFO of traditional design (fabricated on the same substrate with flux cloning circuits) are presented in Fig. 5.11 (a); each curve was measured at different magnetic field produced by the integrated control line and the colour of the lines corresponds to the power. The feature at about 930 μV (450 GHz), where the curves get denser, is a JSC (Josephson Self-Coupling) boundary voltage [KSS’97]. It considerably modifies the FFO properties at the voltages $V \approx V_b = 1/3 * V_g$; the dumping induced by JSC completely suppresses the Fiske steps at $V > V_g/3$. For linewidth measurements of the standard FFO [GMU96, KSS’96], a harmonic mixer (HM) is connected to the main FFO that gives a possibility to measure emission from this line. All obtained results are very similar to measured for FFO developed [dLB’10, KSB’95].

Preliminary results for the circuits developed to study flux cloning are presented below; IVCs of the circuit with narrow (4 μm) and wide (8 μm) additional FFOs are shown in Fig. 5.11 (b) and (c). Again, the different curves correspond to different values of magnetic field; the colour of lines correspond to power detected by the HM connected to main (Fig. 5.11 (b)) and additional (Fig. 5.11 (c)) JTL. From these figures, it is clear that because of presence of the splitting point there is non-zero return current on the FFO IV curves even at large magnetic fields. Presumably it means that fluxons are trapped at the inhomogeneity at the T-junction and considerable current is required to move them in the presence of the splitting point. In addition, an important feature of most of the FFO IVCs is the almost complete absence of the Fiske steps as we predict. Such steps arise only at high magnetic fields for flux cloning circuit with wide additional FFO, when the density of fluxons is large and they are associated with resonance created on additional JTL. Spacing between these steps (of about 50 μV) is close enough to expected distance between Fiske steps for 200 μm FFO.
It is possible to detect emitted radiation from both the main and additional FFOs. Narrow-band Josephson radiation has been measured from the main FFO only at some specific conditions; presumably the FFO linewidth considerably increases due to the presence of the inhomogeneity that modifies fluxon motion. However, for the additional FFO a very narrow-band radiation has been measured on Fiske steps (see Fig. 5.12). The frequency of the FFO is determined by its voltage according to the Josephson relation. The measured peak behaves as regular Josephson radiation peaks. Often higher harmonics of emitted radiation can be detected at biasing on these steps. No narrow-band radiation was detected at other FFO biasing except these steps; only at high voltages very wide peaks were measured [fHGD’10].
Fig. 5.11: Typical set of IVCs for a Nb-AlO$_x$-Nb FFO of traditional design recorded with incremented magnetic fields. Each IVC is measured for a fixed control line current, $I_{CL}$ which is then incremented by $\Delta I_{CL}$ before the next IVC is recorded ($\Delta I_{CL} = 1$ mA for (a) and (c), while for (b) $\Delta I_{CL} = 2$ mA): (a) experimental IVCs for conventional FFO; IVCs of the flux cloning circuits with different width of additional FFO ((b) 4 $\mu$m, (c) 8 $\mu$m).
5.2.4 Conclusion

In the present work we have developed a conventional FFO which we call Flux Flow Oscillator with Cloning Circuit (FFO with CC). After some optimization, it may serve as an effective source of terahertz radiation. The coherent radiation is formed by a train of fluxons cloned in the T-shaped Josephson junction. Therewith, for the first time we describe theoretically that indeed vortex cloning does exist and, as a result, many single Josephson vortices can be born inside a superconductor and for a dense vortex train even without an application of magnetic field despite common belief. We found that when these cloned vortices are ordered in a line to form a train with period commensurate with the size of the additional Josephson transmission line (AJTL), there may arises a strong coherent terahertz radiation intensively emitted along the AJTL. Thus, such a system eventually forms a flux flow oscillator created without external magnetic field. We compare our results with experimental results, which are performed by collaborating with experimental scientists [fHGD’10]. They have also identified a series of resonance frequencies where such trains from cloned vortices generate a strong coherent terahertz radiation. They have measured the shape of the spectral lines and found that they are very narrow and the FFO with CC can serve as FFO in system where application of magnetic field is limited.
Chapter 6

Summary and Conclusions

The body of work presented in this thesis demonstrates the usefulness and versatility of the dynamic behaviours of vortices in 2D T-Josephson junctions in both undamped and damped regimes, see Fig. 7.1. The interest to study vortex dynamics in T-junctions is boosted by the fact that vortices in such junctions can nucleate without applying a magnetic field. We presented two results in T-junctions. The first is devoted to investigation of the problem of undamped SGE system. The second is dedicated to the problem of damped SGE system of conventional T-junction when external magnetic field and bias current are applied.

In the main part of this work, extended T-Josephson junctions were considered. In a conventional T-junction, a flux cloning phenomenon was investigated as long as its width is larger than the Josephson penetration length, $W_0, W > \lambda_J$. However, as an additional junction width increases, $W > W_0$, division of vortex in T-area becomes more important and many novelties appear. We study the effect of geometrical widths of T-junction and dynamic behaviours of Josephson vortices in wide widths regimes, $W > W_0$, which is the objective of work in conventional T-junction.

Meanwhile, we proposed a new kind of T-junction. Rotated T-junction is the rotated conventional T-junction by 90 degrees (counter clockwise). An analytical expression for critical velocity, which is required to clone moving vortices, has been obtained. By treating the vortices as quasi-particles, we derive an equation of motion by using an energy balance. Then, the theoretical predictions of critical velocity were checked numerically for a set of 2D geometries with varying JTL widths. In addition, the high energy particles have been modelled with the use of Josephson fluxons accelerated to relativistic speeds. Then, we observe here that to have the high energy for vortex splitting into fractional vortices (as one may observe in the future LHC), the transverse branch of the T-junction must have a width
larger than the initial one (i.e. \( W > W_0 \)) where the original vortex is accelerated to the relativistic speeds. There are created double pairs of semifluxon and anti-semifluxon, which are strongly bound. In addition at very initial stage of those pairs' formation, there are nucleations of small fractional semifluxons, which are also expanded in the junction.

In the last part of the current work, FFO with flux cloning circuits give an opportunity for use as FFO operating without applied magnetic field. It was explained theoretically and investigated experimentally for the first time by using Nb/Al-AlOx/Nb technology Josephson tunnel junctions. First, we numerically investigated the dynamics of both conventional FFO and FFO with flux cloning with overlap geometry in the presence of an external magnetic field. From numerical simulations of both conventional FFO and FFO with flux cloning, better understanding of the vortices mechanism in FFO with flux cloning circuits was gained. Second, the flux-flow dynamics in both conventional FFO and FFO with flux cloning were performed experimentally by collaborating with experimental scientists [fHGD'10]. In order to compare between conventional FFO and FFO with flux cloning, the IVCs in magnetic field was determined experimentally. From the shape of the spectral lines, it was shown that FFO with flux cloning can serve as FFO in a system which the application of magnetic field is limited.

**Future Work**

The treatment of T-shaped Josephson junctions in this thesis is necessarily incomplete. Extended T-shaped Josephson junctions display a lot of interesting features, such as appearance of a movable (anti-)semifluxon in undamped system, that are worth further investigation. Many questions still remain open for the fluxon and semifluxon dynamics in these systems. In addition, the demonstration of coherent effects of effective widths in this system seems to be a realistic goal for future experiments. We optimized the fabrication process of these junctions to achieve a movable (anti-)semifluxon. In addition, the FFO with flux cloning needs to be numerically investigated more realistically. A model based on surface losses and the Josephson self-pumping effect will be proposed for an explanation of the theoretical current-voltage characteristics. The study will be performed in the frame of a modified sine-Gordon model, which includes surface losses and self-pumping effect.
Fig. 7.1: Mind Mapping of Conclusion.
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