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Abstract. The behaviour of the real roots of the Bernoulli polynomials $B_m(a)$ for large m is investigated. It is shown that if $N(m)$ is the number of these roots then

$$\lim_{m\to\infty} \frac{N(m)}{m} = \frac{2}{\pi e}$$

We also show that on the interval $I_m : -\left[\frac{m-2}{2\pi e}\right] < a < 1 + \left[\frac{m-2}{2\pi e}\right]$, the roots of $B_m(a)$ are close to the half-integer lattices: $a = 0, \pm 1/2, \pm 1, \pm 3/2, \pm 2, ...$ if m is odd, and $a = \pm 1/4, \pm 3/4, \pm 5/4, \pm 7/4, ...$ if m is even. The proof is based on the relation between $B_m(a)$ and the generalised Riemann zeta-function (Hurwitz zeta-function) $\zeta(s, a)$.

1 Introduction.

The classical *Bernoulli polynomials* $B_k(a)$ are defined through the generating function:

$$\frac{ze^{za}}{e^z - 1} = \sum_{k=0}^{\infty} \frac{B_k(a)}{k!} z^k :$$

$$B_0(a) = 1 \qquad B_1(a) = a - \frac{1}{2} \qquad B_2(a) = a^2 - a + \frac{1}{6} \qquad B_3(a) = a^3 - \frac{3a^2}{2} + \frac{a}{2}, ...$$

The *Bernoulli numbers* are defined through the relation: $B_k = B_k(0)$. Bernoulli polynomials and Bernoulli numbers possess many interesting properties and arise in many areas of mathematics (see [1, 2]).

In this paper we will discuss the behaviour of the real roots of the Bernoulli polynomials $B_m(a)$ for large values of the index m. We will prove that, for sufficiently large m

- on the interval $I_m : -\left[\frac{m-2}{2\pi e}\right] < a < 1 + \left[\frac{m-2}{2\pi e}\right]$, where the square brackets refer to the integer part, the roots of $B_m(a)$ are close to the half-integer lattices: $a = 0, \pm 1/2, \pm 1, \pm 3/2, \pm 2, ...$ if m is odd, and $a = \pm 1/4, \pm 3/4, \pm 5/4, \pm 7/4, ...$ if m is even;
- for any constant $\alpha > 1$ the largest real root $A(m)$ of $B_m(a)$ satisfies the inequality

$$\frac{m}{2\pi e} - \frac{1}{2} < A(m) < \frac{m}{2\pi e} + \frac{\alpha}{4\pi e} \log m \tag{1}$$

- the number $N(m)$ of the real roots of the Bernoulli polynomials $B_m(a)$ satisfies a similar inequality

$$1 + 4\left[\frac{m-2}{2\pi e}\right] < N(m) < \frac{2m}{\pi e} + \alpha \log m$$

In particular,

$$\lim_{m \to \infty} \frac{N(m)}{m} = \frac{2}{\pi e}. \tag{2}$$

We should mention that in the recent paper [4] by S.C.Woon the regular behaviour of the real roots of Bernoulli polynomials has been predicted but on a different larger interval. One of the Woon's conjectures (see [4], page 10) states:

$$\lim_{m \to \infty} \frac{N(m)}{m} = \frac{1}{5} + \frac{4}{105} = \frac{5}{21},$$

which is actually larger than $\frac{2}{\pi e}$: $\frac{5}{21} = 0.23809... > 0.234199... = \frac{2}{\pi e}$. The results of [4] were based on a computer analysis of the real roots of the Bernoulli polynomials $B_m(a)$ for $m \leq 80$.

Our approach is based on the relation between the Bernoulli polinomials and the *generalised Riemann zeta-function* also known as the *Hurwitz zeta-function* (see [1, 2, 3]):

$$\zeta(s, a) = \sum_{r=0}^{\infty} (a + r)^{-s} \qquad a \neq 0, -1, -2, \ldots$$

When $a = 1$ it reduces to the classical Riemann zeta-function. It is also known (see e.g. [2], volume 1, page 27) that in the special cases when s is a negative integer this function (as a function of the parameter a) reduces, up to a factor, to a Bernoulli polynomial: explicitly when $s = -m, m = 0, 1, 2, 3, ...$

$$\zeta(-m, a) = -\frac{B_{m+1}(a)}{m+1}.$$

Hurwitz [5] has found the following remarkable representation of the $\zeta(s, a)$ in the domain $0 < a \leq 1$ and $\mathrm{Re}(s) = \sigma < 0$:

$$\zeta(s, a) = \frac{2\Gamma(1-s)}{(2\pi)^{1-s}} \sum_{r=1}^{\infty} \frac{\sin(2\pi r a + \frac{1}{2}\pi s)}{r^{1-s}},$$

in which Γ is the classical Euler gamma-function (see e.g [2], volume 1, page 26).

Our main observation is that for a large negative σ this formula gives a good approximation for $\zeta(\sigma, a)$ on a much wider interval: $0 < a < -\frac{\sigma}{2\pi e}$. As a result we prove that

$$\frac{\zeta(\sigma,a)}{Q(\sigma)} = \sin(2\pi a + \frac{1}{2}\pi\sigma) + o(1) \qquad \text{as} \quad \sigma \to -\infty \qquad \text{provided} \quad 0 < a < \frac{-\sigma}{2\pi e}$$

where $Q(s) = \frac{2\Gamma(1-s)}{(2\pi)^{1-s}}$.

In particular for the integer $\sigma = -m+1$ we can prove that the Bernoulli polynomials $B_m(a)$ behave sinusoidally on the interval $I_m : -\left[\frac{m-1}{2\pi e}\right] < a < 1 + \left[\frac{m-1}{2\pi e}\right]$:

$$\frac{B_m(a)}{Q_m} = \cos(2\pi a - \frac{1}{2}\pi m) + o(1) \qquad \text{as} \quad m \to \infty,$$

where $Q_m = -\frac{2(m!)}{(2\pi)^m}$. This is in good agreement with the computer-generated graphs of the Bernoulli polynomials (see fig.1). We should mention that because of the rapid growth of the term $|Q_m|$ (for example $|Q_{30}| \approx 6 \times 10^8$), the sinusoidal behaviour in the Bernoulli polynomials is easy to overlook.

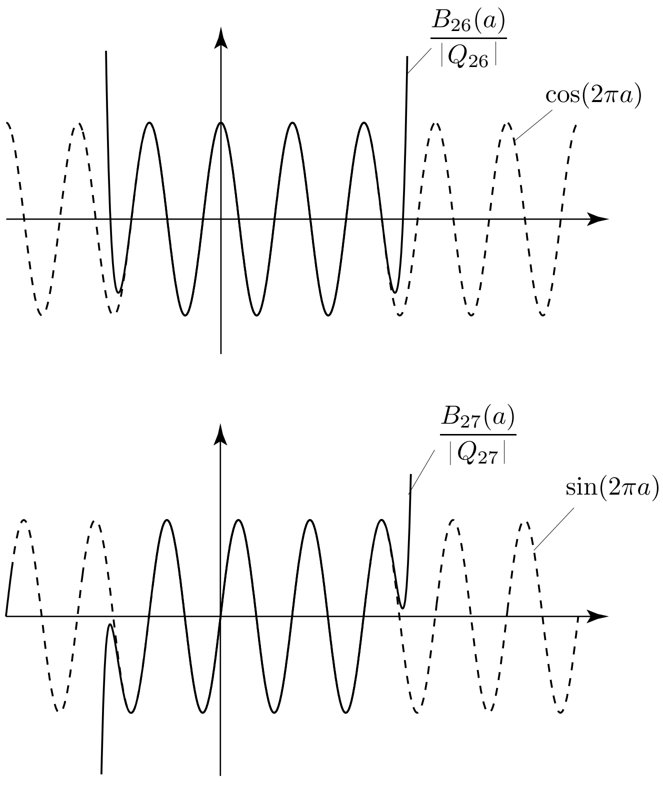

Figure 1

We show also that for any $c > \frac{1}{4\pi e}$ in the region $a > \frac{-\sigma}{2\pi e} + c\log(-\sigma)$ the Hurwitz zeta-function has no real roots for large negative σ. This implies the asymptotic formulas (1), (2) for the roots of the Bernoulli polynomials.

2 Asymptotic behaviour of the Hurwitz zeta-function $\zeta(\sigma, a)$ for large negative σ.

The Hurwitz zeta-function (or generalised Riemann zeta-function) is defined as a series

$$\zeta(s,a) = \sum_{r=0}^{\infty} (a+r)^{-s} \qquad a \neq 0, -1, -2, \dots$$

in the complex domain $\mathrm{Re}(s) > 1$ and can be analytically continued to a meromorphic function in the whole complex plane with the only pole at $s = 1$ (see [1],[2],[3]). When $a = 1$ it reduces to the classical Riemann zeta-function

$$\zeta(s) = \sum_{k=1}^{\infty} k^{-s}.$$

The Hurwitz zeta-function can be extended to the whole of the complex s-plane through the formula

$$\zeta(s,a) = -\frac{\Gamma(1-s)}{2\pi \mathrm{i}} \int_{\infty}^{(0+)} \frac{(-z)^{s-1} e^{-az}}{1-e^{-z}} \mathrm{d}z \qquad a > 0$$

in which the integral is taken over a curve starting at 'infinity' on the real axis, encircles the origin in a positive direction and returns to the starting point (see [2]). By using an alternative integral formulation for $\zeta(s,a)$ it can be shown that $\zeta(s,a)$ is analytic everywhere except for the simple pole at $s = 1$.

The Hurwitz zeta-function obviously satisfies the functional relation:

$$\zeta(s,a) = \zeta(s, n+a) + \sum_{r=0}^{n-1} (r+a)^{-s} \qquad n = 1, 2, \dots \tag{3}$$

Since each term in this relation is analytic we can assume this relation is true for the whole of the complex s plane, except for $s = 1$.

In this paper we restrict ourselves to the case when s is real: $s = \sigma \in \mathbb{R}$. When σ is negative Hurwitz has found the following Fourier representation for $\zeta(\sigma, a)$ on the interval $0 < a \leq 1$:

$$\zeta(\sigma, a) = \frac{2\Gamma(1-\sigma)}{(2\pi)^{1-\sigma}} \sum_{r=1}^{\infty} \frac{\sin(2\pi r a + \frac{1}{2}\pi\sigma)}{r^{1-\sigma}} \tag{4}$$

From this formula we see that

$$\frac{\zeta(\sigma,a)}{Q(\sigma)} = \sin(2\pi a + \frac{1}{2}\pi\sigma) + o(1) \qquad \text{when} \quad \sigma \to -\infty,$$

where $Q(\sigma) = \frac{2\Gamma(1-\sigma)}{(2\pi)^{1-\sigma}}$. Our first theorem proves that this is actually true on a much larger interval.

As part of the theorem proofs we will use the following inequality for the function $S(p,n) = 1^p + 2^p + \ldots + n^p$:

$$S(p,n) < n^p \left(\frac{1-e^{-p}}{1-e^{-p/n}} \right)$$

Indeed,

$$S(p,n) = n^p \left(1 + (1 - \frac{1}{n})^p + (1 - \frac{2}{n})^p + \ldots + (1 - \frac{n-1}{n})^p \right)$$

But since $1 - x < e^{-x}$ for $x < 1$ we have

$$1 - \frac{1}{n} < e^{-1/n} \qquad 1 - \frac{2}{n} < e^{-2/n} \qquad \ldots \qquad 1 - \frac{n-1}{n} < e^{-(n-1)/n}$$

and therefore

$$S(p,n) < n^p \left(1 + e^{-p/n} + e^{-2p/n} + \ldots + e^{-(n-1)p/n} \right) = n^p \left(\frac{1-e^{-p}}{1-e^{-p/n}} \right).$$

We shall also need an estimate for the sum of the series: $\sum_{r=2}^{\infty} \frac{1}{r^{1+p}}$. Now

$$\frac{1}{2^{1+p}} + \frac{1}{3^{1+p}} + \frac{1}{4^{1+p}} + \ldots = \frac{1}{2^p} \left(\frac{1}{2} + \frac{1}{3}(\frac{2}{3})^p + \frac{1}{4}(\frac{2}{4})^p + \ldots \right)$$

But $(\frac{2}{n})^p < \frac{1}{n}$ if $p > 4$ and $n \geq 3$ so we easily deduce

$$\sum_{r=2}^{\infty} \frac{1}{r^{1+p}} < (\zeta(2) - \frac{3}{4})2^{-p} \qquad p > 4$$

Finally, we will also make particular use of Stirling's inequality for the gamma function:

$$(2\pi p)^{\frac{1}{2}} p^p e^{-p} < \Gamma(1+p) < (2\pi p)^{\frac{1}{2}} p^p e^{-p} e^{\frac{1}{12p}} \qquad p \gg 1$$

Theorem 1. *Let $\sigma = -p, p \geq 0$ and $0 < a < \alpha p$ where $\alpha < \frac{1}{2\pi e}$ then the Hurwitz zeta-function satisfies the inequality*

$$\left| \frac{\zeta(-p,a)}{Q(-p)} - \sin(2\pi a - \frac{1}{2}\pi p) \right| < C_1 p^{-1/2} (2\pi e \alpha)^p + C_2 2^{-p}, \tag{5}$$

where C_1, C_2 are constants, which do not depend on p. In particular, on the interval $0 < a < \frac{1}{2\pi e} p$ we have the asymptotic behaviour

$$\frac{\zeta(-p,a)}{Q(-p)} = \sin(2\pi a - \frac{1}{2}\pi p) + o(1) \qquad \text{when} \quad p \to \infty.$$

Proof

Using the functional relation (3) we have for any integer n

$$\left| \frac{\zeta(\sigma, n+a)}{Q(\sigma)} - \frac{\zeta(\sigma,a)}{Q(\sigma)} \right| \leq \frac{1}{|Q(\sigma)|} \sum_{r=0}^{n-1} (r+a)^{-\sigma}$$

Now assuming that $0 < a \leq 1$ we obviously have

$$\sum_{r=0}^{n-1} (r+a)^{-\sigma} \leq S(p,n) \qquad 0 < p = -\sigma$$

and, as we obtained above,

$$S(p,n) < n^p \left(\frac{1-e^{-p}}{1-e^{-p/n}} \right).$$

Also, from the Stirling formula for the Γ-function we have the following asymptotically exact inequality $\Gamma(1+p) > (2\pi p)^{1/2} p^p e^{-p}$ and therefore

$$\frac{1}{Q(-p)} = \frac{(2\pi)^{1+p}}{2\Gamma(1+p)} < \{\frac{2\pi e}{p}\}^p \frac{\pi}{\sqrt{2\pi p}}$$

Thus for a large p

$$\left| \frac{\zeta(-p,n+a)}{Q(-p)} - \frac{\zeta(-p,a)}{Q(-p)} \right| < \left\{ \frac{2\pi e n}{p} \right\}^p \frac{\pi}{\sqrt{2\pi p}} \left(\frac{1-e^{-p}}{1-e^{-p/n}} \right)$$

$$< \left\{ \frac{2\pi e n}{p} \right\}^p \frac{\pi}{\sqrt{2\pi p}} \frac{1}{(1-e^{-\frac{1}{\alpha}})} \qquad \text{if} \quad \frac{n}{p} < \alpha \tag{6}$$

However, from Hurwitz' formula (4) it follows that

$$\frac{\zeta(-p,a)}{Q(-p)} = \sum_{r=1}^{\infty} \frac{\sin(2\pi r a - \frac{1}{2}\pi p)}{r^{1+p}} = \sin(2\pi a - \frac{1}{2}\pi p) + \frac{\sin(4\pi a - \frac{1}{2}\pi p)}{2^{1+p}} + \frac{\sin(6\pi a - \frac{1}{2}\pi p)}{3^{1+p}} + \dots.$$

Therefore

$$\left| \frac{\zeta(-p,a)}{Q(-p)} - \sin(2\pi a - \frac{1}{2}\pi p) \right| < \sum_{r=2}^{\infty} r^{-p-1} < (\zeta(2) - \frac{3}{4}) 2^{-p} \tag{7}$$

if $p > 4$. The estimates (6) and (7) imply the theorem.
By a slight modification of the previous arguments we can prove the following:
Theorem 2 *For any constant* $c > \frac{1}{4\pi e}$ *there exists* $p_0(c)$ *such that* $\zeta(-p,a)$ *is negative for any* $p > p_0$ *and* $a > \frac{p}{2\pi e} + c\log p$.
Proof From the same functional relation (3)

$$\frac{\zeta(-p,a+n)}{Q(-p)} < \frac{\zeta(-p,a)}{Q(-p)} - \frac{S(p,n-1)}{Q(-p)} < \frac{\zeta(-p,a)}{Q(-p)} - \frac{(n-1)^p}{Q(-p)}$$

Now we can use Stirling's inequality $\Gamma(1+p) < \sqrt{2\pi p}\,(\frac{p}{e})^p e^{\frac{1}{12p}}$, so

$$Q(-p) = \frac{2\Gamma(1+p)}{(2\pi)^{1+p}} < \frac{\sqrt{2\pi p}}{\pi} \left(\frac{p}{2\pi e} \right)^p e^{\frac{1}{12p}}$$

However, we know from the Hurwitz formula, that when $p \to \infty$

$$\frac{\zeta(-p,a)}{Q(-p)} = \sin(2\pi a - \frac{1}{2}\pi p) + o(1)$$

Therefore if $\frac{(n-1)^p}{Q(-p)} > 1$ and p large enough then $\zeta(-p,a+n) < 0$. But as we have shown

$$\frac{(n-1)^p}{Q(-p)} > \left(\frac{2\pi e(n-1)}{p} \right)^p \sqrt{\frac{\pi}{2p}} e^{-\frac{1}{12p}}$$

which is greater than 1 if

$$n-1 > \frac{p}{2\pi e}\left(\frac{2p}{\pi}\right)^{\frac{1}{2p}} e^{\frac{1}{12p^2}} = \frac{p}{2\pi e} e^{\{\frac{(\log 2p - \log \pi)}{2p} + \frac{1}{12p^2}\}}$$

Now, using the inequality $e^x < 1 + \alpha x$ if $\alpha > 1$ and x is sufficiently small we find that $\frac{(n-1)^p}{Q(-p)} > 1$ if

$$n-1 > \frac{p}{2\pi e} + \frac{\alpha}{4\pi e}\log p$$

where $\alpha > 1$ and p is sufficiently large. This implies the theorem.

3 The Real Roots of the Bernoulli Polynomials.

As we have already mentioned, when $p = -\sigma = m,\ m \in \mathbb{Z}^+$ the Hurwitz-zeta function reduces to certain polynomials related in a simple way to the Bernoulli polynomials:

$$\zeta(-m, a) = -\frac{B_{m+1}(a)}{m+1} \tag{8}$$

Because of the well-known symmetry properties of the Bernoulli polynomials

$$B_m(1-a) = (-1)^m B_m(a) \tag{9}$$

we can extend all of the results we have for positive roots of $B_m(a)$ to the general case. In particular we can apply the results of the previous section to the problem of real roots of Bernoulli polynomials.

Let $N(m)$ be the number of real roots of $B_m(a)$, and $A(m)$ be the largest of these roots.

Theorem 3 *Let α be any constant such that $\alpha > 1$. For m sufficiently large*

$$1 + 4\left[\frac{m-2}{2\pi e}\right] < N(m) < \frac{2m}{\pi e} + \alpha \log m \tag{10}$$

$$\frac{m}{2\pi e} - \frac{1}{2} < A(m) < \frac{m}{2\pi e} + \frac{\alpha}{4\pi e}\log m \tag{11}$$

The roots of $B_m(a)$ on the interval $-\left[\frac{m-2}{2\pi e}\right] < a < 1 + \left[\frac{m-2}{2\pi e}\right]$ are simple and close to the haf-integer lattices:

$a = 0, \pm 1/2, \pm 1, \pm 3/2, \pm 2, ...$ *if m is odd, and*

$a = \pm 1/4, \pm 3/4, \pm 5/4, \pm 7/4, ...$ *if m is even;*

Proof From theorem 1 and (8) it follows that

$$\frac{B_m(a)}{Q_m} = \cos(2\pi a - \frac{1}{2}\pi m) + o(1)$$

on the interval $I_m:\ -\left[\frac{m-1}{2\pi e}\right] < a < 1 + \left[\frac{m-1}{2\pi e}\right]$ when $m \to \infty$. Actually this is true also for the k^{th} derivative of $\frac{B_m(a)}{Q_m}$ but on a smaller interval: I_{m-k-1}. Indeed,

$$B_m^{(k)}(a) = m(m-1)\dots(m-k+1)B_{m-k}(a)$$

and

$$Q_m = -\frac{2m!}{(2\pi)^m} = \frac{m(m-1)\dots(m-k+1)}{(2\pi)^k}Q_{m-k}$$

Now, according to theorem 1 and to (8) when $m \to \infty$

$$\frac{B_m^{(k)}(a)}{Q_m} = \frac{m(m-1)\dots(m-k+1)B_{m-k}(a)}{m(m-1)\dots(m-k+1)Q_{m-k}}(2\pi)^k$$
$$= (2\pi)^k \cos(2\pi a - \frac{1}{2}\pi(m-k)) + o(1) = \cos^{(k)}(2\pi a - \frac{1}{2}\pi m) + o(1)$$
$$\text{if} \quad -\left[\frac{m-k-1}{2\pi e}\right] < a < 1 + \left[\frac{m-k-1}{2\pi e}\right].$$

In particular, on the interval I_{m-2} the function $\frac{B_m(a)}{Q_m}$ (and its derivative) tend to $\cos(2\pi a - \frac{1}{2}\pi m)$ (and its derivative) when $m \to \infty$ which ensures that for large m all the roots of $B_m(a)$ on this interval are simple and located near the points $a = \frac{m+1}{4} + \frac{k}{2} \quad k \in \mathbb{Z}^+$. This implies the last statement of the theorem and the lower estimates of (10) and (11).

The upper estimates for $A(m)$ follows directly from theorem 2. To prove the upper estimates for $N(m)$ we need the following simple lemma.

Lemma If a function $f(x)$ (with a continuous n^{th} derivative) on some interval (a, b) has the property that the sign of the n^{th} derivative is constant throughout the interval then f has no more than n roots on this interval.

Now we apply this lemma to the function $f(a) \equiv B_{m+1}(a)$ on the interval J_m : $(\frac{m}{2\pi e}, \frac{m}{2\pi e} + \frac{\alpha}{4\pi e}\log m)$ to estimate the number of roots there. The idea of this calculation is clear from figure 2 (in which $\kappa \equiv \frac{1}{2\pi e}$).

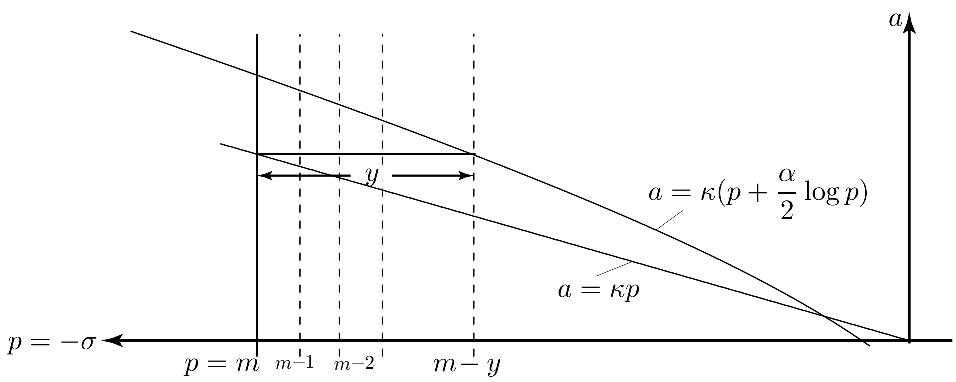

Figure 2

Using the well-known fact that

$$B'_{m+1}(a) = (m+1)B_m(a)$$

(see e.g. [2]) we differentiate B_{m+1} many times until we have a positive function and then apply the lemma. As one can see from Fig. 2 if $n > [y]$ then $B_{m+1}^{(n)}(a)$ will be positive in the interval J_m and, as such, cannot have more than n simple roots in this interval. Now y is the solution to the equation

$$\kappa\Big((m-y) + \frac{\alpha}{2}\log(m-y)\Big) = \kappa m$$

or

$$-y + \frac{\alpha}{2} \log(m - y) = 0$$

We claim that the solution to this equation satisfies the inequality

$$y < \frac{\alpha}{2} \log m$$

Indeed the function $F(y) = -y + \frac{\alpha}{2} \log(m - y)$ is monotonically decreasing and

$$F(\frac{\alpha}{2} \log m) = -\frac{\alpha}{2} \log m + \frac{\alpha}{2} \log(m - \frac{\alpha}{2} \log m) < 0.$$

Thus according to the lemma $B_{m+1}(a)$ has no more than $\frac{\alpha}{2} \log m$ roots on the interval $(\frac{m}{2\pi e}, \frac{m}{2\pi e} + \frac{\alpha}{4\pi e} \log m)$. Using the symmetry relations (9) of $B_{m+1}(a)$ and the regular behaviour of the roots on the middle interval we obtain the inequality (10). This proves theorem 3.

4 Concluding Remarks.

In this paper we have proved the regular lattice behaviour of almost all of the real roots of the Bernoulli polynomials $B_m(t)$ and found asymptotic formulae for the number of these roots, for large values of the index m. It is surprising that these results seem to have not been noted before, particularly as the proofs are elementary. The only relevant paper we have found is [4] by S.C.Woon, who observed the regular behaviour of the real roots of $B_m(t)$ using a numerical investigation. We should mention that our interest also started from a graphical analysis of the Bernoulli polynomials using computer-generated graphs.

Computer experiments carried out by S.Woon also demonstrate a remarkably regular structure of the complex roots of $B_m(t)$. It would be very interesting to find a mathematical explanation for this and to find explicitly the curves along which the complex roots of $B_m(t)$ lie.

As we have seen, this problem is closely related to the problem of the behaviour of the zeroes of the Hurwitz-zeta function $\zeta(s, a)$, and therefore belongs to the circle of problems connected to the famous Riemann hypothesis (see [3]). However, we believe that our problem is not as difficult and we hope to make progress in this direction in the future.

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References

[1] E.T.Whittaker and G.N.Watson, *A Course of Modern Analysis.*, Cambridge University Press, 1963.

[2] A.Erdelyi (Editor) *Higher Transcendental Functions.*, Vol.1-3. McGraw-Hill Book Company, 1953.

[3] A.Ivic *The Riemann Zeta-function. The Theory of the Riemann Zeta-function with Applications.* John Wiley & Sons, 1985.

[4] S.C.Woon *Analytic Continuation of Bernoulli Numbers, a New Formula for the Riemann Zeta-function, and the Phenomenon of Scattering of Zeroes.* physics/9705021, July 1997.

[5] A.Hurwitz *Einige Eigenschaften der Dirichlet'schen Functionen* $F(s) = \Sigma(\frac{D}{n})\frac{1}{n^s}$, *die bei der Bestimmung der Classenzahlen binärer quadratischer Formen auftreten.* Zeitschrift für Math. und Phys., XXVII (1882), 86-101.