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Gelf'and Inverse Problem for a Quadratic Operator Pencil

Kurylev Y.* and Lassas M.**

* Department of Mathematical Sciences, Loughborough University
Loughborough, LE11 3TU, UK

** Rolf Nevanlinna Institute, University of Helsinki
Helsinki, PL4, FIN-00014, Finland

1. Introduction. Main result. In the paper we deal with an inverse problem for a quadratic operator pencil

$$A(\lambda)u = a(x, D)u - ib_0\lambda u - \lambda^2 u, \tag{1}$$

$$Bu := \partial_\nu u - \sigma u|_{\partial M} = 0 \tag{2}$$

on a differentiable compact connected manifold $M, \dim M = m \geq 1$, with non-empty boundary $\partial M \neq \emptyset$. Here $a(x, D)$ is a uniformly elliptic symbol

$$a(x, D) = -g^{-1/2}(\partial_j + b_j)g^{1/2}g^{jl}(\partial_l + b_l) + q,$$

where $[g^{jl}]_{j,l=1}^m$ defines a C^∞-smooth Riemannian metric and $b = (b_1, ..., b_m)$ and q are, correspondingly, C^∞-smooth complex-valued 1-form and function on M. σ is a C^∞-smooth complex-valued function on ∂M and ∂_ν stands for the normal derivative.

Let R_λ be the resolvent of (1), (2) which is meromorphic for $\lambda \in \mathbb{C}$ (see Sect. 3 and [1]) and let $R_\lambda(x, y)$ be its Schwartz kernel. A natural analog of the Gel'fand inverse problem [2] is

Problem I. Let ∂M and $R_\lambda(x, y); \lambda \in \mathbb{C}, x, y \in \partial M$ be given. Do these data (Gel'fand boundary spectral data, GBSD) determine $(M, a(x, D), b_0, \sigma)$ uniquely?

Remark 1. Let $\mathcal{G}_\lambda$ be the Neumann-to-Dirichlet map $\mathcal{G}_\lambda f := u_\lambda^f|_{\partial M}$ where

$$A(\lambda)u^f(\lambda) = 0, \quad Bu^f(\lambda) = f. \tag{3}$$

Then GBSD means that $\mathcal{G}_\lambda$ are known for all λ.

Remark 2. By Fourier transform, $u(x, \lambda) \to u(x, t)$, Problem I is equivalent to the inverse boundary problem for the dissipative wave equation

$$u_{tt}^f + b_0 u_t^f + a(x, D)u^f = 0, \tag{4}$$

$$Bu^f = f|_{\partial M \times \mathbb{R}_+}; \quad u^f|_{t=0} = u_t^f|_{t=0} = 0, \tag{5}$$

where inverse data is given in the form of the response operator R^h;

$$R^h(f) := u^f|_{\partial M \times \mathbb{R}_+}. \tag{6}$$

This hyperbolic inverse problem and its analogs were considered in [3-5a]. Paper [3] dealt with the inverse scattering problem, $M = \mathbb{R}^m$, with $g^{jl} = \delta^{jl}$. It was generalised in [4] onto the Gel'fand inverse boundary problem in a bounded domain in $\mathbb{R}^m; g^{jl} = \delta^{jl}$. In [5] the uniqueness of the reconstruction of the conformally euclidian metric in $M \in \mathbb{R}^m$ and the lower order terms (with some restrictions upon these terms) was proven for the geodesically regular domains M. At last a local variant of the problem with data prescribed on a part of the boundary was studied in [5a]. As for the case $b_0 = 0$ and self-adjoint studied in full generality in [6,7].

In the paper we give the answer to Problem I assuming some geometric conditions upon (M, g). The main technique used is the boundary control (BC) method (see e.g. [8]) in the geometrical version [7].

Definition 1.(M, g) satifies Bardos-Lebeau-Rauch (BLR) condition if there is $t_* > 0$ and an open conic neighbourhood $\mathcal{O}$ of the set of not-nondiffractive points in $T^*(\partial M \times [0, t_*])$ such that any generalised bicharacteristic of the wave operator $\partial_t^2 - \Delta_g$ passes through a point of $T^*(\partial M \times [0, t_*]) \setminus \mathcal{O}$.

Theorem A. *Let $(\partial M; \mathcal{G}_\lambda, \lambda \in \mathbb{C})$ be GBSD for a quadratic operator pencil (1), (2). Assume that the corresponding Riemannian manifold (M, g) satisfies the BLR-condition. Then these data determine M and b_0 uniquely while $a(x, D)$ and σ to within a gauge transformation*

$$a(x, D) \longrightarrow \kappa a(x, D)\kappa^{-1}; \quad \kappa \in C^\infty(M; \mathbb{C}), \quad \kappa|_{\partial M} = 1, \quad \kappa \neq 0 \quad on \quad M.$$

2. Auxiliary constructions.. In view of the gauge invariance we can assume that $\sigma = 0$. By λ-linearisation;

$$u \to U = (u, \lambda u)^t,$$

the pencil (1), (2) takes the form

$$\mathcal{A}U = \lambda U; \quad \mathcal{A} = \mathcal{A}_0 + \mathcal{A}_1;$$

$$\mathcal{A}_0 = \begin{pmatrix} 0 & I \\ A_0 & 0 \end{pmatrix}; \quad \mathcal{A}_1 = \begin{pmatrix} 0 & 0 \\ a_1(x, D) & -ib_0 \end{pmatrix}.$$

Here $A_0 = -\Delta_g$ is the Laplace operator with Neumann boundary condition;

$$\mathcal{D}(A_0) = H^2_\nu(M) := \{u \in H^2(M) : \partial_\nu u|_{\partial M} = 0\}$$

and $a_1(x, D) = a(x, D) + \Delta_g$. Operators $\mathcal{A}_0, \mathcal{A}$ with

$$\mathcal{D}(\mathcal{A}_0) = \mathcal{D}(\mathcal{A}) = H^2_\lambda(M) \times L^2(M)$$

are closed in $\mathcal{H} = [L^2(M)]^2$. By the transformation $\lambda \to \lambda + id$; $A_0 \to A_0 + d^2$ we get

$$||A_0^{-1}|| < 1; \quad ||a_1(x, D)A_0^{-3/4}|| < 1/2. \tag{7}$$

The adjoint operator, $\mathcal{A}^*$ is then

$$\mathcal{A}^* = \begin{pmatrix} 0 & A^* \\ I & i\bar{b}_0 \end{pmatrix}, \quad \mathcal{D}(\mathcal{A}^*) = L^2(M) \times \mathcal{D}(A^*);$$

$$\mathcal{D}(A^*) = H^2_{\nu,b} := \{u \in H^2; \quad B^*u := \partial_\nu u - 2b_\nu u|_{\partial M} = 0\},$$

where $b_\nu = (\nu, b)$.

Using A^* instead of A we define operators $\mathcal{A}_{\rm ad}$ and $\mathcal{A}^*_{\rm ad}$;

$$\mathcal{A}_{\rm ad} = \begin{pmatrix} 0 & I \\ A^* & i\bar{b}_0 \end{pmatrix}, \quad \mathcal{D}(\mathcal{A}_{\rm ad}) = H^2_{\nu,b} \times L^2.$$

Our goal is to use eigenfunction expansion corresponding to $\mathcal{A}, \mathcal{A}^*$ and $\mathcal{A}_{\rm ad}, \mathcal{A}^*_{\rm ad}$. To this end we introduce operators $T_0, T = T_0 + T_1$ where

$$T_0 = \begin{pmatrix} 0 & A_0^{1/2} \\ A_0^{1/2} & 0 \end{pmatrix}, \quad T_1 = \begin{pmatrix} 0 & 0 \\ A_0^{-1/4} a_1 A_0^{-1/4} & -iA_0^{-1/4} b_0 A_0^{-1/4} \end{pmatrix}; \qquad (8)$$

$$\mathcal{D}(T) = \mathcal{D}(T_0) = [\mathcal{D}(A_0^{1/2})]^2 = [H^1(M)]^2.$$

By (7) T ia bounded-invertible. We have

$$T_0 U = L^{-1}\mathcal{A}_0 L U; \quad TU = L^{-1}\mathcal{A} L U \quad \text{for} \quad U \in \mathcal{D}(A_0^{3/4}) \times \mathcal{D}(A_0^{1/2}); \qquad (9)$$

$$L = \begin{pmatrix} A_0^{-1/4} & 0 \\ 0 & A_0^{1/4} \end{pmatrix}.$$

3. Abel-Lidskii expansion. From (18) $T_0^{-1} \in \Sigma_p, \quad p > m$ where Σ_p is the Schatten-von Neumann class (see e.g. [9]). As T_1 is bounded $T = T_0 + T_1$ is a weak perturbation of T_0. Due to the general theory of weak perturbations of self-adjoint operators (see e.g. [1, Sect.6.2-6.4]) the spectrum $\sigma(T)$ of T is normal .

Let $\beta > m$ be an even integer, $\tau > 0$ and Γ - a finite contour in $\mathbb{C}$, $\Gamma \cap \sigma(T) = \emptyset$. Denote by $P^\beta_{\Gamma,\tau}(T)$ the modified Riesz projector for T;

$$P^\beta_{\Gamma,\tau}(T) = -\frac{1}{2\pi i}\int_\Gamma e^{-\tau z^\beta}(T - z)^{-1} dz,$$

and by $P^\beta_{\Gamma,\tau}(T_0)$ -the analogous projector for T_0.

Let Γ be a contour in $\mathbb{C}$ consisting of two segments $Imz = \pm a, Rez \in [-b, b]$, and four semiaxes $Imz = \pm cRez$ (see Fig. 1).

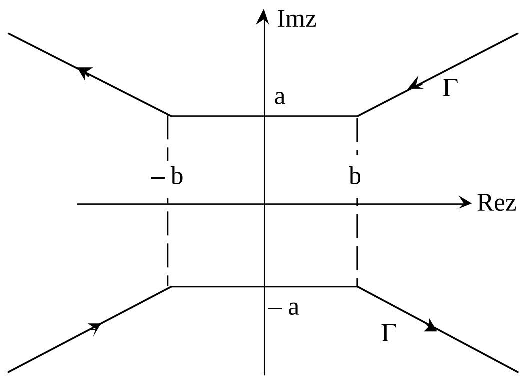

Fig. 1

Parameters a, b, c are chosen so that

i) $\sigma(T)$ lies inside Γ;

ii) $Rez^\beta \geq c_0 |z^\beta|, c_0 > 0$ for $|Imz| \leq c|Rez|$.

Theorem 1 (Abel-Lidskii convergence). *There exist real numbers* $\alpha_N > 0$, $N = 1, 2, ...,$ *which depend only upon* $\sigma(T)$ *such that*

$$Y = \lim_{\tau \to +0} \lim_{N \to \infty} P^{\beta}_{N,\tau}(T)Y. \tag{10}$$

The convergence in (10) takes place in $[H^s]^2$, $s \in [-1/2, 1/2]$ when $Y \in [H^s]^2$ and in the graph norm of T^n when $Y \in \mathcal{D}(T^n), n = 1, 2, ...$ Here $P^{\beta}_{N,\tau}(T)$ correspond to the contours Γ_N obtained from Γ by cutting it by vertical lines $Rez = \pm\alpha_N$ (see Fig.2).

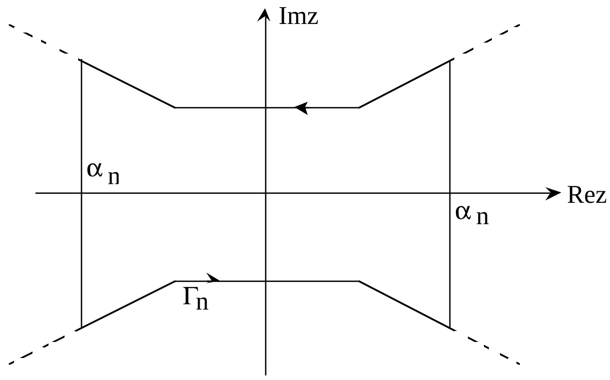

Fig.2

Proof. Since $T_0 \in \Sigma_p, \quad p > m$ and T_1 is bounded the results of [1, Sect. 6.2-6.4] (see also [10]) show the existence of α'_N which depend upon $\sigma(T_0), \sigma(T)$ such that

$$P^{\beta}_{N,\tau}(T) \xrightarrow[N \to \infty]{s} P^{\beta}_{\tau}(T).$$

The proof of the strong convergence is based upon exponential estimates for $(T - z)^{-1}, (T_0 - z)^{-1}$. However since $P^{\beta}_{N,\tau}(T)$ remains intact under small deviations of α'_N it is possible to choose α_N independent of $\sigma(T_0)$. Moreover the results of [1] show that

$$P^{\beta}_{\tau}(T) - P^{\beta}_{\tau}(T_0) = -\frac{1}{2\pi i} \int_{\Gamma} e^{-\tau z^{\beta}} (T - z)^{-1} T_1 (T_0 - z)^{-1} dz; \tag{11}$$

$$||(T - z)^{-1} T_1 (T_0 - z)^{-1}||_s \leq c_s |z|^{-3/2}, \quad s \in [-1/2, 1/2], \quad z \quad \text{lies outside} \Gamma, \tag{12}$$

where $|| \cdot ||_s$ stands for the operator norm in $[H^s]^2$. As $\mathrm{s} - \lim P^{\beta}_{\tau}(T_0) = I$ and the rhs of (11) tends to 0 when $\tau \to +0$ the statement follows for $Y \in [H^s]^2$.

The last part of Theorem follows from the case $s = 0$ since for $Y \in \mathcal{D}(T^n)$

$$T^n P^{\beta}_{N,\tau}(T)Y = P^{\beta}_{N,\tau}(T) T^n Y.$$

Since $\mathcal{A}$ has only point spectrum and $\sigma_p(\mathcal{A}) = \sigma(T)$ equation (9) yields that $\mathcal{A}$ has normal spectrum.

Lemma 1. *Let* $U=(u^1,u^2)^t\in H^1(M)\times L^2(M)$ *or* $[C_0^\infty(M)]^2$. *Then*

$$U=\lim_{\tau\to 0}\lim_{N\to\infty}P^\beta_{N,\tau}(\mathcal{A})U,$$

where the convergenve takes place in $H^1\times L^2$ *when* U *lies in this space or in* $C^N(\Omega)$ *for any* $N>0,\Omega\ll M$ *when* $U\in[C_0^\infty(M)]^2$.

Proof. As $Y=L^{-1}U\in[H^{1/2}]^2$ when $U\in H^1\times L^2$ Theorem 1, $s=1/2$ proves the statement for this case. As $L^{-1}[C_0^\infty(M)]^2\subset\mathcal{D}(T^n)$ for any $n>0$ and $\mathcal{D}(T^n)\subset[H^n]^2$ this case also follows from Theorem 1 and the fact that L is a pseudodifferential operator of the order 1/2.

Corollary 1. *Let* $U\in L^2(M)\times H^1(M)$ *or* $[C_0^\infty(M)]^2$. *Then*

$$U=\lim_{\tau\to 0}\lim_{N\to\infty}P^\beta_{N,\tau}(\mathcal{A}^*)U, \tag{13}$$

where the convergenve takes place in $L^2\times H^1$ *and* $C^N(\Omega)$ *for any* $N>0,\Omega\ll M$, *respectively.*

Proof. As $||(T^*-\bar z)^{-1}-(T_0-\bar z)^{-1}||_s=||(T-z)^{-1}-(T_0-z)^{-1}||_{-s}$ estimate (12) remains valid for T^*,T_0 and $s=1/2$ for z outside Γ. The same arguments as in Theorem 1 show that

$$Y=\lim_{\tau\to +0}\lim_{N\to\infty}P^\beta_{N,\tau}(T^*)Y \quad \text{in} \quad [H^{1/2}]^2.$$

As $Y=LU\in[H^{1/2}]^2$ when $U\in L^2\times H^1$ (13) follows. As for the case $U\in[C_0^\infty(M)]^2$ the arguments are the same as in Lemma 1.

Using the representation

$$\mathcal{A}^*_{\rm ad}=J\mathcal{A}J^{-1};\quad \mathcal{A}^*=J^*\mathcal{A}_{\rm ad}[J^*]^{-1}; \tag{14}$$

$$J\left[(u^1,u^2)^t\right]=(u^2+ib_0u^1,u^1)^t,$$

we come to

Corollary 2. *The statement of Lemma 1 is valid for* $\mathcal{A}^*_{\rm ad}$. *The statement of Corollary 2 is valid for* $\mathcal{A}_{\rm ad}$.

4. Root functions and boundary spectral data.. Let $\mu_j:=dim\mathcal{H}_j=dim\mathcal{H}^*_j$ where $\mathcal{H}_j:=P_{\lambda_j}(\mathcal{A})\mathcal{H}$; $\mathcal{H}^*_j:=P_{\bar\lambda_j}(\mathcal{A}^*)\mathcal{H}$ and $r_j:=dimKer(\mathcal{A}-\lambda_j)=dimKer(\mathcal{A}^*-\bar\lambda_j)$. Denote by $\Phi_{j,k,0}=(\phi^1_{j,k,0},\phi^2_{j,k,0})^t$, $\Psi_{j,k,0},k=1,...,r_j$ the eigenvectors of $\mathcal{A},\mathcal{A}^*$ at $\lambda_j,\bar\lambda_j$, correspondingly, and by $n_{j,k},n_{j,1}\ge n_{j,2}\ge...\ge n_{j,r_j}$, their partial null multiplicities; $\mu_j=n_{j,1}+...+n_{j,r_j}$. Let $\Phi_{j,k,l},\Psi_{j,k,l},l=1,...,n_{j,k}$ be the root functions associated with $\Phi_{j,k,0},\Psi_{j,k,0}$;

$$(\mathcal{A}-\lambda_j)\Phi_{j,k,l}=\Phi_{j,k,l-1};\quad (\mathcal{A}^*-\bar\lambda_j)\Psi_{j,k,l}=\Psi_{j,k,l-1}. \tag{15}$$

It is possible to choose $\Phi_{j,k,l},\Psi_{j,k,l};j=1,2,...,k=1,...,r_j,l=1,...,n_{j,k}$ so that

$$(\Phi_{j,k,l},\Psi_{j',k',l'})_{\mathcal{H}}=\delta_{j,j'}\delta_{k,k'}\delta_{l,n_{j,k}-l'-1} \tag{16}$$

(see e.g. [11; Sect. 2] or [12; Sect. 1.2]). The choice of $\Phi_{j,k,l}, \Psi_{j,k,l}$ when j is fixed is non-unique. The group of admissible transformations form a subgroup in $GL(\mu_j, \mathbb{C})$ defined by conditions (15), (16) (see e.g. [11; sect. 2]).

Let $U, V \in \mathcal{H}$. Denote by

$$\mathcal{F}(U) = \mathcal{U} := \{U_{j,k,l}; U_{j,k,l} = (U, \Psi_{j,k,n_{j,k}-l-1})\};$$

$$\mathcal{F}^*(V) = \mathcal{V}^* := \{V^*_{j,k,l}; V^*_{j,k,l} = (V, \Phi_{j,k,n_{j,k}-l-1})\}$$

their Fourier transforms with respect to $\mathcal{A}, \mathcal{A}^*$, correspondingly. Using Lemma 1 and Corollary 2 we obtain

Corollary 3. *Let $U \in H^1 \times L^2, V \in L^2 \times H^1$. Then their Fourier transforms $\mathcal{U}, \mathcal{V}^*$ determine (U, V) uniquely.*

Due to the relations (14) the analogous results take place for $\mathcal{A}_{\text{ad}}, \mathcal{A}^*_{\text{ad}}$ with basis

$$\tilde{\Psi}_{j,k,l} = J\Phi_{j,k,l}; \quad \tilde{\Phi}_{j,k,l} = (J^*)^{-1}\Psi_{j,k,l}. \tag{17}$$

The basis $\Phi_{j,k,l}, \Psi_{j,k,l}$ makes sense to the following

Definition. Boundary spectral data (BSD) of the pencil (1), (2) is the collection $(\partial M; \lambda_j, \phi^1_{j,k,l}|_{\partial M}, \psi^2_{j,k,l}|_{\partial M},\ j = 1, 2, ..., k = 1, ..., r_j, l = 1, ..., n_{j,k})$.

Theorem 2. *GBSD determine BSD to within the group of transformations of the biorthogonal basis which preserve properties (15), (16).*

Proof. Given $R_\lambda(x, y), x, y \in \partial M$ it is possible to find $u^f_\lambda|_{\partial M}$ where u^f_λ is the solution to (3). Consider $U^f_\lambda = (u^f_\lambda, \lambda u^f_\lambda)^t$. Then

$$(a - \lambda)U^f_\lambda = 0,$$

where a is an operator on $H^2 \times L^2$;

$$a = \begin{pmatrix} 0 & I \\ a(x, D) & -ib_0 \end{pmatrix}.$$

Let $e \in H^2, \partial_\nu e|_{\partial M} = f$ and $E = (e, 0)^t$. Then

$$U^f_\lambda = E - (\mathcal{A} - \lambda)^{-1}(a - \lambda)E.$$

U^f_λ is a meromorphic function of λ with possible singularities only at $\lambda_j \in \sigma(\mathcal{A})$ and $U^f_\lambda - P_{\lambda_j}(\mathcal{A})U^f_\lambda$ is analytic at λ_j. But

$$[P_{\lambda_j}(\mathcal{A})U^f_\lambda]^1|_{\partial M} = \sum_{k=1}^{r_j} \sum_{l=0}^{n_{j,k}-1} U^f_{j,k,l}(\lambda)\phi^1_{j,k,l}|_{\partial M}.$$

By Green's formula

$$(\lambda - \lambda_j)(U^f_\lambda, \Psi_{j,k,n_{j,k}-l-1}) = \int_{\partial M} f(\psi^2_{j,k,n_{j,k}-l-1})|_{\partial M} dS - \tag{18}$$

$$-(U^f_\lambda, \Psi_{j,k,n_{j,k}-l-2}).$$

By means of equation (18) (with different f) it is possible to find all $\lambda_j \in \sigma(\mathcal{A}) = \sigma(A(\lambda))$ as well as the boundary values $\phi^1_{j,k,l}|_{\partial M}, \psi^2_{j,k,l}|_{\partial M}$ to within a linear transformation preserving (15), (16) (for details see e.g. [11; Sect. 3]).

Let $u^f(x,t)$ be the solution to (4), (5) and $v^g(x,s)$ be the solution to the initial-boundary value problem

$$v^g_{ss} - \bar{b}_0 v^g_s + a^*(x,D)v^g = 0, \tag{19}$$

$$B^* v|_{\partial M\times\mathbb{R}_+} = g, \quad v^g|_{s=0} = v^g_s|_{s=0} = 0, \tag{20}$$

which is associated with $\mathcal{A}_{\rm ad}$. Let

$$U^f(t) = (u^f(t), iu^f_t(t))^t, \quad V^g(s) = (v^g(s), iv^g_s(s))^t.$$

Then

$$U^f_t + i\mathcal{A}U^f = 0, \quad V^g_s + i\mathcal{A}_{\rm ad}V^g = 0.$$

Lemma 3. *For any $f, g \in L^2(\partial M\times\mathbb{R}_+)$ BSD $\{\lambda_j, \phi^1_{j,k,l}|_{\partial M}, \psi^2_{j,k,l}|_{\partial M}\}$ determine $\mathcal{F}U^f(t)$ and $\mathcal{F}_{\rm ad}V^g(s) = \mathcal{V}_{\rm ad} = \{(V^g(s), \tilde{\Psi}_{j,k,n_{j,k}-l-1})\}$.*

Proof. Part integration together with relation (15) for Ψ yields that

$$i\partial_t(U^f(t), \Psi_{j,k,n_{j,k}-l-1}) = \lambda_j(U^f(t), \Psi_{j,k,n_{j,k}-l-1}) + (U^f(t), \Psi_{j,k,n_{j,k}-l-2}) +$$

$$+\int_{\partial M} f(t)\psi^2_{j,k,n_{j,k}-l-1}|_{\partial M}dS.$$

As $U^f|_{t=0} = 0$ this equation proves Lemma for $U^f(t)$. Taking into account (17) the same considerations prove Lemma for $V^g(s)$.

Corollary 3. *Let $f, g \in L^2(\partial M\times\mathbb{R}_+)$. Given BSD and $t, s \geq 0$ it is possible to evaluate*

$$(U^f(t), J^*V^g(s)) =$$

$$= i\int_M [u^f_t(x,t)\bar{v}^g(x,s) - u^f(t)\bar{v}^g_s(x,s) + b_0(x)u^f(x,t)\bar{v}^g(x,s)]dx.$$

Proof. The statement is an immediate corollary of the fact that $U^f(t) \in H^1\times L^2$, $J^*V^g(s) \in L^2\times H^1$, Lemma 1, Corollary 1, definition (14), and Lemma 3.

5. Reconstruction of (M,g). Denote by $\mathcal{L}^s, s\in\mathbb{R}$ the subspace in $H^{s+1}\times H^s$ of the functions which satisfy natural compatibility conditions for the hyperbolic problem (4), (5) (see e.g [13]) and by $\mathcal{L}^s_{\rm ad}$ the analogous subspace for (19), (20).

Theorem 2 [14]**.** *Let (M,g) satisfies the BLR-condition. Then*

$$\{U^f(T); f\in H^s_0(\partial M, [0,T])\} = \mathcal{L}^s, \quad T > t_*, s \geq -1/2.$$

Corollary 4. *Let (M,g) satisfies the BLR-condition. Then BSD determine $\mathcal{F}(\mathcal{L}^s)$, $\mathcal{F}_{\rm ad}(\mathcal{L}^s_{\rm ad})$, $\quad s \geq -1/2$.*

Proof. The statement follows from Lemma 3 and Theorem 2.

Let $\Gamma \subset M$ be open, $t \geq 0$. Denote

$$M(\Gamma,t) = \{x \in M : d(x,\Gamma) \leq t\}.$$

Lemma 4. *Let $\mathcal{U} \in \mathcal{F}(\mathcal{L}^s), s \geq 0$, $\mathcal{U} = \mathcal{F}U$. Then for any $\Gamma \subset \partial M, t_0 \geq 0$ BSD determine whether $m_g(suppU \cap M(\Gamma,t)) = 0$ or not. Analogous statement takes place for $\mathcal{V}_{\rm ad}$.*

Here m_g is the measure on (M,g).

Proof. Consider $\mathcal{U}(t) = \{U_{j,k,l}(t)\}$ where

$$\frac{d}{dt}U_{j,k,l}(t) + i\lambda_j U_{j,k,l}(t) + iU_{j,k,l+1}(t) = 0, \quad t \in \mathbb{R}, \tag{21}$$

$$U_{j,k,l}(0) = U_{0;j,k,l}, \tag{22}$$

where $\{U_{0;j,k,l}\} = \mathcal{U}_0 \in \mathcal{F}(\mathcal{L}^s)$. Then $\mathcal{U}(t) \in \mathcal{F}(\mathcal{L}^s)$ for all t and $\mathcal{U}(t) = \mathcal{F}U(t)$ where

$$U_t(t) + i\mathcal{A}U(t) = 0, \quad U(0) = U_0.$$

As $s \geq 0$ Lemma 1 and Sobolev embedding theorem show that

$$u^1(t)|_{\partial M} = \lim_{\tau \to 0} \lim_{N \to \infty} [P^\beta_\tau(\mathcal{A})U(t)]^1, \tag{23}$$

where the convergence takes place in $L^2(\partial M)$. In view of the Homgren-John theorem [15] the fact that $m_g(suppU \cap M(\Gamma,t)) = 0$ is equivalent to the fact that

$$suppu^1|_{\partial M \times \mathbb{R}} \cap (\Gamma \times [-t_0,t_0]) = \emptyset. \tag{24}$$

However $\phi^1_{j,k,l}|_{\partial M}$ are known so that the statement follows from (21), (22) and (23), (24).

Corollary 5. *Let $\Gamma \subset \partial M, t_0 \geq 0$ and $s \geq 0$. Then BSD determine subspaces $\mathcal{F}(\mathcal{L}^s(\Gamma,t_0)), \mathcal{F}([\mathcal{L}^s(\Gamma,t_0)]^c)$, and $\mathcal{F}_{\rm ad}(\mathcal{L}^s_{\rm ad}(\Gamma,t_0)), \mathcal{F}_{\rm ad}([\mathcal{L}^s_{\rm ad}(\Gamma,t_0)]^c)$, where*

$$\mathcal{L}^s(\Gamma,t_0) = \{U \in \mathcal{L}^s : suppU \subset cl(M(\Gamma,t_0))\};$$

$$[\mathcal{L}^s(\Gamma,t_0)]^c = \{U \in \mathcal{L}^s : suppU \subset cl(M \setminus M(\Gamma,t_0)\}$$

and analogous definitions are valid for $\mathcal{L}^s_{\rm ad}(\Gamma,t_0), [\mathcal{L}^s_{\rm ad}(\Gamma,t_0)]^c$.

Proof. By Lemma 4 BSD determine $[\mathcal{L}^s(\Gamma,t_0)]^c, [\mathcal{L}^s_{\rm ad}(\Gamma,t_0)]^c$. As $U \in \mathcal{L}^s(\Gamma,t_0)$ is equivalent to the fact that $(U,J^*V) = 0$ for all $V \in [\mathcal{L}^s_{\rm ad}(\Gamma,t_0)]^c$ the remaining part of Corollary 5 follows from Corollary 3.

Corollary 6. *Let* $\Gamma_i \subset \partial M, t_i^+ > t_i^- \geq 0;\ i = 1, ..., I$. *Denote by* M_I *the set*

$$M_I = \cap_{i=1}^I (M(\Gamma, t_i^+) \setminus M(\Gamma, t_i^-)). \tag{25}$$

Then BSD determine whether $m_g(M_I) = 0$ *or not.*

Corollary 6 is the basic analytic tool in the reconstruction of (M, g). For this end introduce $\mathcal{R} : M \to L^\infty(\partial M)$;

$$\mathcal{R}(x) = r_x(y) = d(x, y), \quad y \in \partial M.$$

It is shown in [7] that $\mathcal{R}(M) \subset L^\infty(\partial M)$ has a natural structure of a Riemannian manifold such that $\mathcal{R} : M \to \mathcal{R}(M)$ is an isometry.

Theorem 3. *BSD of the operator pencil (1), (2) which satisfies the BLR-condition determine* (M, g) *uniquely.*

Proof. In view of the above remark about isometry between (M, g) and $\mathcal{R}(M)$ it is sufficient to show that BSD determine $\mathcal{R}(M)$. Choose $\delta > 0$ and a collection of $\Gamma_i, i = 1, ..., I(\delta)$ such that $\mathrm{diam}(\Gamma_i) \leq \delta$, $\cup \Gamma_i = \partial M$. Let

$$p = (p_1, ..., p_{I(\delta)}), \quad p_i \in \mathbb{N}, \quad t_i^+ = (p_i + 1)\delta; \quad t_i^- = (p_i - 1)\delta. \tag{26}$$

Denote by $M_I(p)$ the set M_I (see (25)) with $t_i^\pm$ of form (26) and correspond to every p such that $m_g(M_I(p)) > 0$ a piecewise constant function $r_p(y) = p_i\delta$ when $y \in \Gamma_i$. Let $\mathcal{R}_\delta(M)$ be the collection of these functions. Then

$$Dist(\mathcal{R}_\delta(M), \mathcal{R}(M)) \leq 3\delta.$$

Taking $\delta \to 0$ we construct $\mathcal{R}(M)$.

6. Reconstruction of the lower-order terms.. Let $x_0 \in \mathrm{int} M$ and

$$M_I(\delta) \longrightarrow x_0 \quad \text{when} \quad \delta \to 0. \tag{27}$$

Consider a family $\mathcal{V}(\delta) \in \mathcal{F}_{\mathrm{ad}}(\mathcal{L}^0)$ such that

$$\mathrm{supp} V(\delta) \subset \mathrm{cl}(M_I(\delta)), \quad \mathcal{V} = \mathcal{F}_{\mathrm{ad}} V(\delta), \tag{28}$$

and for any $\mathcal{U} \in \mathcal{F}(\mathcal{L}^s), s < m/2 < s + 1$ there is a limit $\mathcal{W}^{x_0}(\mathcal{U})$;

$$\mathcal{W}^{x_0}(\mathcal{U}) = \lim_{\delta \to 0} (U, \mathcal{V}(\delta)),$$

where the inner product in the rhs of (28) is understood in Abel-Lidskii sense. Such families exist, indeed it is sufficient to take C_0^∞-approximations to $(\delta(\cdot - x_0), 0)^t$. On the other hand since

$$(\mathcal{U}, \mathcal{V}(\delta)) = (U, J^* V(\delta)),$$

the existence of the limit means that there is a limit $W^{x_0} \in [D'(M)]^2$ of $V(\delta)$. By (27) $\mathrm{supp} W^{x_0} \subset \{x_0\}$. Moreover as the limit exists for $U \in \mathcal{L}^s, s < m/2 < s + 1$, $W^{x_0} = (0, \kappa(x_0)\delta(\cdot - x_0))^t$.

Lemma 5. *Let BSD of an operator pencil (1), (2) be given and* (M, g) *satisfies the BLR-condition. Then it is possible to construct a map* $\mathbb{W} : M \longrightarrow \mathbb{C}^{\infty}$*;*

$$\mathbb{W}(x_0) = \mathcal{W}^{x_0}; \quad W^{x_0}_{j,k,l} = \overline{\mathcal{W}}^{x_0}(\mathcal{E}^{(j,k,l)}),$$

(where $\mathcal{E}^{(j,k,l)}$ *is the sequence with* 1 *at the* (j, k, l)*-place and* 0 *otherwise) such that*

$$\mathcal{W}(x_0)(\mathcal{U}) = \kappa(x_0)u^1(x_0), \quad \mathcal{U} \in \mathcal{F}(\mathcal{L}^s), \quad s < m/2 < s + 1;$$

$$\kappa \in C^{\infty}(M), \quad \kappa|_{\partial M} = 1, \quad \kappa \neq 0 \quad \text{on } M. \tag{29}$$

Proof. To prove Lemma it is sufficient to show the existence of $\mathcal{V}^{x_0}(\delta)$ such the their limits $\mathcal{W}^{x_0}$ satisfy the following conditions
i. $\mathcal{W}^{x_0} \neq 0$;
ii. $\mathcal{W}^{x_0}(\mathcal{U}) \in C^{\infty}(M)$ when $\mathcal{U} \in \mathcal{F}([C^{\infty}_0(M)]^2$;
iii. $\mathcal{W}^{x_0}(\mathcal{U}) = u^1(x_0)$ when $x_0 \in \partial M$; $\mathcal{U} \in \mathcal{F}(\mathcal{L}^s), s < m/2 < s + 1$.

To prove the existence of such $\mathcal{V}^{x_0}(\delta)$ we can take adjoint Fourier transforms of some smooth approximations to $(0, \delta(\cdot - x_0))^t$. On the other hand, conditions i-iii may be algorithmically verified due to Lemma 3, Corollary 3, Corollary 4, Lemma 4 and Lemma 1.

Corollary 7. *BSD of a pencil (1),(2) with* (M, g) *satisfying the BLR-condition determine the functions* $\kappa(x)\phi^1_{j,k,l}(x); j = 1, 2, ..., k = 1, ..., r_j, l = 1, ..., n_{j,k}$ *where* κ *satisfies relations (29).*

Proof. Since

$$\kappa(x_0)\phi^1_{j,k,l}(x_0) = \mathcal{W}^{x_0}_{j,k,l},$$

and $\Phi_{j,k,l} \in \mathcal{L}^s$ for any s the statement follows from Lemma 5.

The functions $\kappa\phi^1_{j,k,l}$ are the root functions for the pencil $A_\kappa(\lambda)$;

$$A_\kappa(\lambda_j)(\kappa\phi^1_{j,k,l}) := a_\kappa(x, D)(\kappa\phi^1_{j,k,l}) - i\lambda_j b_0(\kappa\phi^1_{j,k,l}) - \lambda^2_j(\kappa\phi^1_{j,k,l}) = \kappa\phi^1_{j,k,l-1}, \tag{30}$$

$$B_\kappa(\kappa\phi^1_{j,k,l}) := (\partial_\nu(\kappa\phi^1_{j,k,l}) - \sigma_\kappa(\kappa\phi^1_{j,k,l}))|_{\partial M} = 0, \tag{31}$$

where

$$a_\kappa(x, D) = \kappa a(x, D)\kappa^{-1}; \quad \sigma_\kappa = \sigma + \partial_\nu[\ln \kappa].$$

Lemma 6. *Functions* $\kappa\phi^1_{j,k,l}, j = 1, 2, ..., k = 1, ..., r_j, l = 1, ..., n_{j,k}$ *where* κ *satisfies (66) determine* $a_\kappa, \sigma_\kappa, b_0$.

Proof. By Lemma 1 finite linear combinations of $\kappa\Phi_{j,k,l} = (\kappa\phi^1_{j,k,l}, \lambda_j\kappa\phi^1_{j,k,l})^t$ are dense in $[C^N(\Omega)]^2$ for any $N \geq 0, \Omega \ll M$. In particular for $x_0 \in \text{int}M$ the vectors $(\kappa(x_0)\phi^1_{j,k,l}(x_0), \nabla(\kappa\phi^1_{j,k,l})(x_0), \lambda_j\kappa(x_0)\phi^1_{j,k,l}(x_0))^t \in \mathbb{C}^{m+2}$ span $\mathbb{C}^{m+2}$. Then equations (30) determine a_κ and b_0.

On the other hand for any $y \in \partial M$ there is $\phi^1_{j,k,l}$ such that $\phi^1_{j,k,l}(y) \neq 0$. Hence equations (31) determine σ_κ.

Theorem A is now a corollary of Lemma 6, Lemma 7 and properties (29) of κ.

Some remarks.

i. The BLR-condition is always satisfied for $M \subset \mathbb{R}^m$ with the metric $g^{j,l} = \delta^{j,l}$ or its C^1-small perturbations (see e.g. [14, 16]);

ii. In particular the results of the paper are always valid for $m = 1$ even when GBSD are prescribed at only one boundary point (see also [17]);

iii. Using the nonstationary variant of the BC-method (see e.g. [8, 18]) it is possible to prove an analog of Theorem A when the data is the response operator $R^h(t)$ of form (6) for the problem (4), (5) in the case when (M, g) satisfies the BLR-condition and $t > 2t_*$.

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