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Gelf’and Inverse Problem for a Quadratic Operator Pencil

Kurylev Y.* and Lassas M.**

* Department of Mathematical Sciences, Loughborough University
  Loughborough, LE11 3TU, UK

** Rolf Nevanlinna Institute, University of Helsinki
  Helsinki, PL4, FIN-00014, Finland

1. Introduction. Main result. In the paper we deal with an inverse problem for a quadratic operator pencil

\[ A(\lambda)u = a(x, D)u - ib_0 \lambda u - \lambda^2 u, \tag{1} \]

\[ Bu := \partial_\nu u - \sigma u|_{\partial M} = 0 \tag{2} \]

on a differentiable compact connected manifold \( M; \dim M = m \geq 1 \), with non-empty boundary \( \partial M \neq \emptyset \). Here \( a(x, D) \) is a uniformly elliptic symbol

\[ a(x, D) = -g^{-1/2}(\partial_j + b_j)g^{1/2}g^{jl}(\partial_l + b_l) + q, \]

where \( [g^{jl}]_{j,l=1}^m \) defines a \( C^\infty \)-smooth Riemannian metric and \( b = (b_1, ..., b_m) \) and \( q \) are, correspondingly, \( C^\infty \)-smooth complex-valued 1-form and function on \( M \). \( \sigma \) is a \( C^\infty \)-smooth complex-valued function on \( \partial M \) and \( \partial_\nu \) stands for the normal derivative.

Let \( R_\lambda \) be the resolvent of (1), (2) which is meromorphic for \( \lambda \in \mathbb{C} \) (see Sect. 3 and [1]) and let \( R_\lambda(x, y) \) be its Schwartz kernel. A natural analog of the Gel’fand inverse problem [2] is

Problem I. Let \( \partial M \) and \( R_\lambda(x, y); \lambda \in \mathbb{C}, x, y \in \partial M \) be given. Do these data (Gel’fand boundary spectral data, GBSD) determine \( (M, a(x, D), b_0, \sigma) \) uniquely?

Remark 1. Let \( \mathcal{G}_\lambda \) be the Neumann-to-Dirichlet map \( \mathcal{G}_\lambda f := u^f_\lambda|_{\partial M} \) where

\[ A(\lambda)u^f(\lambda) = 0, \quad Bu^f(\lambda) = f. \tag{3} \]

Then GBSD means that \( \mathcal{G}_\lambda \) are known for all \( \lambda \).

Remark 2. By Fourier transform, \( u(x, \lambda) \to u(x, t) \), Problem I is equivalent to the inverse boundary problem for the dissipative wave equation

\[ u^f_{tt} + b_0 u^f_t + a(x, D)u^f = 0, \tag{4} \]

\[ Bu^f = f|_{\partial M \times \mathbb{R}_+}; \quad u^f|_{t=0} = u^f_t|_{t=0} = 0, \tag{5} \]

where inverse data is given in the form of the response operator \( R^h; \)

\[ R^h(f) := u^f|_{\partial M \times \mathbb{R}_+}. \tag{6} \]

This hyperbolic inverse problem and its analogs were considered in [3-5a]. Paper [3] dealt with the inverse scattering problem, \( M = \mathbb{R}^n \), with \( g^{jl} = \delta^{jl} \). It was generalised in [4] onto the Gel’fand inverse boundary problem in a bounded domain.
in $\mathbb{R}^m; g^{jl} = \delta^{jl}$. In [5] the uniqueness of the reconstruction of the conformally euclidian metric in $M \in \mathbb{R}^m$ and the lower order terms (with some restrictions upon these terms) was proven for the geodesically regular domains $M$. At last a local variant of the problem with data prescribed on a part of the boundary was studied in [5a]. As for the case $b_0 = 0$ and self-adjoint studied in full generality in [6,7].

In the paper we give the answer to Problem I assuming some geometric conditions upon $(M, g)$. The main technique used is the boundary control (BC) method (see e.g. [8]) in the geometrical version [7].

**Definition 1.** $(M, g)$ satisfies Bardos-Lebeau-Rauch (BLR) condition if there is $t_* > 0$ and an open conic neighbourhood $O$ of the set of not-nondiffarctive points in $T^* (\partial M \times [0, t_*])$ such that any generalised bicharacteristic of the wave operator $\partial^2_t - \Delta_g$ passes through a point of $T^* (\partial M \times [0, t_*]) \setminus O$.

**Theorem A.** Let $(\partial M; G_\lambda, \lambda \in \mathbb{C})$ be GBSD for a quadratic operator pencil (1), (2). Assume that the corresponding Riemannian manifold $(M, g)$ satisfies the BLR-condition. Then these data determine $M$ and $b_0$ uniquely while $a(x, D)$ and $\sigma$ to within a gauge transformation

$$a(x, D) \longrightarrow \kappa a(x, D) \kappa^{-1}; \quad \kappa \in C^\infty(M; \mathbb{C}), \quad \kappa|_{\partial M} = 1, \quad \kappa \neq 0 \quad \text{on} \quad M.$$

2. **Auxiliary constructions.** In view of the gauge invariance we can assume that $\sigma = 0$. By $\lambda$-linearisation;

$$u \rightarrow U = (u, \lambda u)^t,$$

the pencil (1), (2) takes the form

$$\mathcal{A}U = \lambda U; \quad \mathcal{A} = A_0 + A_1;$$

$$A_0 = \begin{pmatrix} 0 & I \\ A_0 & 0 \end{pmatrix}; \quad A_1 = \begin{pmatrix} 0 & 0 \\ a_1(x, D) & -ib_0 \end{pmatrix}.$$

Here $A_0 = -\Delta_g$ is the Laplace operator with Neumann boundary condition;

$$\mathcal{D}(A_0) = H^2_\nu(M) := \{ u \in H^2(M) : \partial_\nu u|_{\partial M} = 0 \}$$

and $a_1(x, D) = a(x, D) + \Delta_g$. Operators $A_0, A$ with

$$\mathcal{D}(A_0) = \mathcal{D}(A) = H^2_\lambda(M) \times L^2(M)$$

are closed in $\mathcal{H} = [L^2(M)]^2$. By the transformation $\lambda \rightarrow \lambda + id; A_0 \rightarrow A_0 + d^2$ we get

$$||A_0^{-1}|| < 1; \quad ||a_1(x, D)A_0^{-3/4}|| < 1/2. \quad (7)$$

The adjoint operator, $A^*$ is then

$$A^* = \begin{pmatrix} 0 & A^* \\ I & ib_0 \end{pmatrix}, \quad \mathcal{D}(A^*) = L^2(M) \times \mathcal{D}(A^*);$$

$$\mathcal{D}(A^*) = H^2_{\nu,b} := \{ u \in H^2; \quad B^* u := \partial_\nu u - 2b_\nu u|_{\partial M} = 0 \},$$
where \( b_{\nu} = (\nu, b) \).

Using \( A^* \) instead of \( A \) we define operators \( A_{\text{ad}} \) and \( A_{\text{ad}}^* \):

\[
A_{\text{ad}} = \begin{pmatrix} 0 & I \\ A^* & i b_0 \end{pmatrix}, \quad D(A_{\text{ad}}) = H_{\nu, b}^2 \times L^2.
\]

Our goal is to use eigenfunction expansion corresponding to \( A, A^* \) and \( A_{\text{ad}}, A_{\text{ad}}^* \). To this end we introduce operators \( T_0, T = T_0 + T_1 \) where

\[
T_0 = \begin{pmatrix} 0 & A_0^{1/2} \\ A_0^{1/2} & 0 \end{pmatrix}, \quad T_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ A_0^{-1/4} & a_1 A_0^{-1/4} & -i A_0^{-1/4} b_0 A_0^{-1/4} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix};
\]

\[
D(T) = D(T_0) = [D(A_0^{1/2})]^2 = [H^1(M)]^2.
\]

By (7) \( T \) is bounded-invertible. We have

\[
T_0 U = L^{-1} A_0 L U; \quad T U = L^{-1} A L U \quad \text{for} \quad U \in D(A_0^{3/4}) \times D(A_0^{1/2});
\]

\[
L = \begin{pmatrix} A_0^{-1/4} & 0 \\ 0 & A_0^{1/4} \end{pmatrix}.
\]

3. **Abel-Lidskii expansion.** From (18) \( T_0^{-1} \in \Sigma_p, \quad p > m \) where \( \Sigma_p \) is the Schatten-von Neumann class (see e.g. [9]). As \( T_1 \) is bounded \( T = T_0 + T_1 \) is a weak perturbation of \( T_0 \). Due to the general theory of weak perturbations of self-adjoint operators (see e.g. [1, Sect.6.2-6.4]) the spectrum \( \sigma(T) \) of \( T \) is normal.

Let \( \beta > m \) be an even integer, \( \tau > 0 \) and \( \Gamma \) - a finite contour in \( \mathbb{C}, \Gamma \cap \sigma(T) = \emptyset \). Denote by \( P_{\Gamma, \tau}^\beta(T) \) the modified Riesz projector for \( T \);

\[
P_{\Gamma, \tau}^\beta(T) = -\frac{1}{2\pi i} \int_{\Gamma} e^{-\tau z^\beta}(T - z)^{-1} dz,
\]

and by \( P_{\Gamma, \tau}^\beta(T_0) \)-the analogous projector for \( T_0 \).

Let \( \Gamma \) be a contour in \( \mathbb{C} \) consisting of two segments \( \text{Im} z = \pm a, \text{Re} z \in [-b, b] \), and four semiaxes \( \text{Im} z = \pm c \text{Re} z \) (see Fig. 1).

![Fig. 1](image-url)

Parameters \( a, b, c \) are chosen so that

i) \( \sigma(T) \) lies inside \( \Gamma \);

ii) \( \text{Re} z^\beta \geq c_0 |z^\beta|, c_0 > 0 \) for \( |\text{Im} z| \leq c |\text{Re} z| \).
Theorem 1 (Abel-Lidskii convergence). There exist real numbers $\alpha_N > 0$, $N = 1, 2, \ldots$, which depend only upon $\sigma(T)$ such that

$$Y = \lim_{\tau \to +0} \lim_{N \to \infty} P_{N,\tau}^{\beta}(T)Y. \quad (10)$$

The convergence in (10) takes place in $[H^s]^2$, $s \in [-1/2, 1/2]$ when $Y \in [H^s]^2$ and in the graph norm of $T^n$ when $Y \in \mathcal{D}(T^n), n = 1, 2, \ldots$. Here $P_{N,\tau}^{\beta}(T)$ correspond to the contours $\Gamma_N$ obtained from $\Gamma$ by cutting it by vertical lines $Rez = \pm \alpha_N$ (see Fig.2).

**Proof.** Since $T_0 \in \Sigma_p$, $p > m$ and $T_1$ is bounded the results of [1, Sect. 6.2-6.4] (see also [10]) show the existence of $\alpha'_N$ which depend upon $\sigma(T_0), \sigma(T)$ such that

$$P_{N,\tau}^{\beta}(T) \xrightarrow{s} P_{\tau}^{\beta}(T).$$

The proof of the strong convergence is based upon exponential estimates for $(T - z)^{-1}, (T_0 - z)^{-1}$. However since $P_{N,\tau}^{\beta}(T)$ remains intact under small deviations of $\alpha'_N$ it is possible to choose $\alpha_N$ independent of $\sigma(T_0)$. Moreover the results of [1] show that

$$P_{\tau}^{\beta}(T) - P_{\tau}^{\beta}(T_0) = -\frac{1}{2\pi i} \int_{\Gamma} e^{-\tau z^p}(T - z)^{-1}T_1(T_0 - z)^{-1}dz; \quad (11)$$

$$||(T - z)^{-1}T_1(T_0 - z)^{-1}|| \leq c_s|z|^{-3/2}, \quad s \in [-1/2, 1/2], \quad z \text{ lies outside} \Gamma, \quad (12)$$

where $|| \cdot ||_s$ stands for the operator norm in $[H^s]^2$. As $s - \lim P_{\tau}^{\beta}(T_0) = I$ and the rhs of (11) tends to 0 when $\tau \to +0$ the statement follows for $Y \in [H^s]^2$.

The last part of Theorem follows from the case $s = 0$ since for $Y \in \mathcal{D}(T^n)$

$$T^n P_{N,\tau}^{\beta}(T)Y = P_{N,\tau}^{\beta}(T)T^nY.$$

Since $\mathcal{A}$ has only point spectrum and $\sigma_p(\mathcal{A}) = \sigma(T)$ equation (9) yields that $\mathcal{A}$ has normal spectrum.
Lemma 1. Let $U = (u^1, u^2)^t \in H^1(M) \times L^2(M)$ or $[C^\infty_0(M)]^2$. Then

$$U = \lim_{\tau \to 0} \lim_{N \to \infty} P^3_{N, \tau}(A)U,$$

where the convergence takes place in $H^1 \times L^2$ when $U$ lies in this space or in $C^N(\Omega)$ for any $N > 0, \Omega \leq M$ when $U \in [C^\infty_0(M)]^2$.

Proof. As $Y = L^{-1}U \in [H^{1/2}]^2$ when $U \in H^1 \times L^2$ Theorem 1, $s = 1/2$ proves the statement for this case. As $L^{-1}[C^\infty_0(M)]^2 \subset \mathcal{D}(T^n)$ for any $n > 0$ and $\mathcal{D}(T^n) \subset [H^n]^2$ this case also follows from Theorem 1 and the fact that $L$ is a pseudodifferential operator of the order 1/2.

Corollary 1. Let $U \in L^2(M) \times H^1(M)$ or $[C^\infty_0(M)]^2$. Then

$$U = \lim_{\tau \to 0} \lim_{N \to \infty} P^3_{N, \tau}(A^*)U,$$

where the convergence takes place in $L^2 \times H^1$ and $C^N(\Omega)$ for any $N > 0, \Omega \leq M$, respectively.

Proof. As $||(T^* - \bar{z})^{-1} - (T_0 - \bar{z})^{-1}||_s = ||(T - z)^{-1} - (T_0 - z)^{-1}||_s$ estimate (12) remains valid for $T^*, T_0$ and $s = 1/2$ for $z$ outside $\Gamma$. The same arguments as in Theorem 1 show that

$$Y = \lim_{\tau \to +0} \lim_{N \to \infty} P^3_{N, \tau}(T^*)Y \in [H^{1/2}]^2.$$

As $Y = LU \in [H^{1/2}]^2$ when $U \in L^2 \times H^1$ (13) follows. As for the case $U \in [C^\infty_0(M)]^2$ the arguments are the same as in Lemma 1.

Using the representation

$$\mathcal{A}_{ad}^* = JAJ^{-1}; \quad \mathcal{A}^* = J^*\mathcal{A}_{ad}[J^*]^{-1};$$

$$J[(u^1, u^2)^t] = (u^2 + ib_0u^1, u^1)^t,$$

we come to

Corollary 2. The statement of Lemma 1 is valid for $\mathcal{A}_{ad}^*$. The statement of Corollary 2 is valid for $\mathcal{A}_{ad}$.

4. Root functions and boundary spectral data.. Let $\mu_j := \dim \mathcal{H}_j = \dim \mathcal{H}^*_j$ where $\mathcal{H} := P_{\lambda_j}(A)\mathcal{H}$; $\mathcal{H}^* := P_{\lambda_j}(A^*)\mathcal{H}$ and $r_j := \dim \ker(A - \lambda_j) = \dim \ker(A^* - \bar{\lambda}_j)$. Denote by $\Phi_{j,k,0} = (\phi_{j,k,0}^1, \phi_{j,k,0}^2)^t$, $\Psi_{j,k,0}, k = 1, ..., r_j$ the eigenvectors of $\mathcal{A}, \mathcal{A}^*$ at $\lambda_j, \bar{\lambda}_j$, correspondingly, and by $n_{j,k}, n_{j,1} \geq n_{j,2} \geq ... \geq n_{j,r_j}$, their partial null multiplicities; $\mu_j = n_{j,1} + ... + n_{j,r_j}$. Let $\Phi_{j,k,l}, \Psi_{j,k,l}, l = 1, ..., n_{j,k}$ be the root functions associated with $\Phi_{j,k,0}, \Psi_{j,k,0}$;

$$(\mathcal{A} - \lambda_j)\Phi_{j,k,l} = \Phi_{j,k,l-1}; \quad (\mathcal{A}^* - \bar{\lambda}_j)\Psi_{j,k,l} = \Psi_{j,k,l-1}.$$

It is possible to choose $\Phi_{j,k,l}, \Psi_{j,k,l}; j = 1, 2, ..., k = 1, ..., r_j, l = 1, ..., n_{j,k}$ so that

$$(\Phi_{j,k,l}, \Psi_{j',k',l'})_{\mathcal{H}} = \delta_{j,j'}\delta_{k,k'}\delta_{l,n_{j,k}-l'-1}$$

(16)
(see e.g. [11; Sect. 2] or [12; Sect. 1.2]). The choice of $\Phi_{j,k,l}, \Psi_{j,k,l}$ when $j$ is fixed is non-unique. The group of admissible transformations form a subgroup in $GL(\mu_j, \mathbb{C})$ defined by conditions (15), (16) (see e.g. [11; sect. 2]).

Let $U, V \in \mathcal{H}$. Denote by

$$\mathcal{F}(U) = U := \{U_{j,k,l}; U_{j,k,l} = (U, \Psi_{j,k,n_j,k-l-1})\};$$

$$\mathcal{F}^*(V) = V^* := \{V_{j,k,l}^*; V_{j,k,l}^* = (V, \Phi_{j,k,n_j,k-l-1})\}$$

their Fourier transforms with respect to $\mathcal{A}, \mathcal{A}^*$, correspondingly. Using Lemma 1 and Corollary 2 we obtain

**Corollary 3.** Let $U \in H^1 \times L^2, V \in L^2 \times H^1$. Then their Fourier transforms $\mathcal{U}, \mathcal{V}^*$ determine $(U, V)$ uniquely.

Due to the relations (14) the analogous results take place for $\mathcal{A}_{ad}, \mathcal{A}_{ad}^*$ with basis

$$\tilde{\Psi}_{j,k,l} = J \Phi_{j,k,l}; \quad \tilde{\Phi}_{j,k,l} = (J^*)^{-1} \Psi_{j,k,l}.$$  

(17)

The basis $\Phi_{j,k,l}, \Psi_{j,k,l}$ makes sense to the following

**Definition.** Boundary spectral data (BSD) of the pencil (1), (2) is the collection $(\partial M; \lambda_j, \phi_{j,k,l}^1|_{\partial M}, \psi_{j,k,l}^2|_{\partial M}, j = 1, 2, \ldots, k = 1, \ldots, r_j, l = 1, \ldots, n_{j,k})$.

**Theorem 2.** GBSD determine BSD to within the group of transformations of the biorthogonal basis which preserve properties (15), (16).

**Proof.** Given $R_\lambda(x, y), x, y \in \partial M$ it is possible to find $u_\lambda^f|_{\partial M}$ where $u_\lambda^f$ is the solution to (3). Consider $U_\lambda^f = (u_\lambda^f, \lambda u_\lambda^f)^t$. Then

$$(a - \lambda)U_\lambda^f = 0,$$

where $a$ is an operator on $H^2 \times L^2$;

$$a = \begin{pmatrix} 0 & I \\ a(x, D) & -ib_0 \end{pmatrix}.$$

Let $e \in H^2, \partial_v e|_{\partial M} = f$ and $E = (e, 0)^t$. Then

$$U_\lambda^f = E - (A - \lambda)^{-1}(a - \lambda)E.$$

$U_\lambda^f$ is a meromorphic function of $\lambda$ with possible singularities only at $\lambda_j \in \sigma(A)$ and $U_\lambda^f - P_{\lambda_j}(A)U_\lambda^f$ is analytic at $\lambda_j$. But

$$[P_{\lambda_j}(A)U_\lambda^f]^1|_{\partial M} = \sum_{k=1}^{r_j} \sum_{l=0}^{n_{j,k}-1} U_{j,k,l}^f(\lambda)\phi_{j,k,l}^1|_{\partial M}.$$

By Green’s formula

$$(\lambda - \lambda_j)(U_\lambda^f, \Psi_{j,k,n_j,k-l-1}) = \int_{\partial M} f(\psi_{j,k,n_j,k-l-1}^2)|_{\partial M} dS$$

(18)
By means of equation (18) (with different $f$) it is possible to find all $\lambda_j \in \sigma(A) = \sigma(A(\lambda))$ as well as the boundary values $\phi_{j,k,l,1}^1|_{\partial M}$, $\psi_{j,k,l,1}^2|_{\partial M}$ to within a linear transformation preserving (15), (16) (for details see e.g. [11; Sect. 3]).

Let $u^f(x,t)$ be the solution to (4), (5) and $v^g(x,s)$ be the solution to the initial-boundary value problem

\begin{align}
  v_{ss}^g - b_0 v_s^g + a^*(x,D)v^g &= 0, \\
  B^*v|_{\partial M \times \mathbb{R}_+} = g, \quad v^g|_{s=0} = v_s^g|_{s=0} = 0,
\end{align}

which is associated with $A_{ad}$. Let

\begin{align*}
  U^f(t) = (u^f(t), iu^f_t(t))^t, \quad V^g(s) = (v^g(s), iv^g_s(s))^t.
\end{align*}

Then

\begin{align*}
  U^f_t + iAU^f = 0, \quad V^g_s + iA_{ad}V^g = 0.
\end{align*}

**Lemma 3.** For any $f, g \in L^2(\partial M \times \mathbb{R}_+)$ BSD $\{\lambda_j, \phi_{j,k,l,1}^1|_{\partial M}, \psi_{j,k,l,1}^2|_{\partial M}\}$ determine $\mathcal{F}U^f(t)$ and $\mathcal{F}_{ad}V^g(s) = V_{ad} = \{(V^g(s), \tilde{\Psi}_{j,k,n_j,n_k-1-1})\}$.

**Proof.** Part integration together with relation (15) for $\Psi$ yields that

\begin{align*}
  i\partial_t(U^f(t), \Psi_{j,k,n_j,n_k-1-1}) &= \lambda_j(U^f(t), \Psi_{j,k,n_j,n_k-1-1}) + (U^f(t), \Psi_{j,k,n_j,n_k-1-2}) + \\
  &\quad + \int_{\partial M} f(t)\psi_{j,k,n_j,n_k-1-1}^2|_{\partial M}dS.
\end{align*}

As $U^f|_{t=0} = 0$ this equation proves Lemma for $U^f(t)$. Taking into account (17) the same considerations prove Lemma for $V^g(s)$.

**Corollary 3.** Let $f, g \in L^2(\partial M \times \mathbb{R}_+)$. Given BSD and $t, s \geq 0$ it is possible to evaluate

\begin{align*}
  (U^f(t), J^*V^g(s)) = \int_M [u^f_t(x,t)\bar{v}^g(x,s) - u^f(t)\bar{v}^g_s(x,s) + b_0(x)u^f(x,t)\bar{v}^g(x,s)]dx.
\end{align*}

**Proof.** The statement is an immediate corollary of the fact that $U^f(t) \in H^1 \times L^2$, $J^*V^g(s) \in L^2 \times H^1$, Lemma 1, Corollary 1, definition (14), and Lemma 3.

**5. Reconstruction of** $(M,g)$. Denote by $\mathcal{L}^s$, $s \in \mathbb{R}$ the subspace in $H^{s+1} \times H^s$ of the functions which satisfy natural compatibility conditions for the hyperbolic problem (4), (5) (see e.g. [13]) and by $\mathcal{L}^s_{ad}$ the analogous subspace for (19), (20).

**Theorem 2** [14]. Let $(M,g)$ satisfies the BLR-condition. Then

\begin{align*}
  \{U^f(T); f \in H^s_0(\partial M,[0,T])\} = \mathcal{L}^s, \quad T > t_*, s \geq -1/2.
\end{align*}
Corollary 4. Let $(M, g)$ satisfies the BLR-condition. Then BSD determine $\mathcal{F}(\mathcal{L}^s)$, $\mathcal{F}_{\text{ad}}(\mathcal{L}^s_{\text{ad}})$, $s \geq -1/2$.

Proof. The statement follows from Lemma 3 and Theorem 2.

Let $\Gamma \subset M$ be open, $t \geq 0$. Denote

$$M(\Gamma, t) = \{x \in M : d(x, \Gamma) \leq t\}.$$

Lemma 4. Let $\mathcal{U} \in \mathcal{F}(\mathcal{L}^s)$, $s \geq 0$, $U = \mathcal{F}U$. Then for any $\Gamma \subset \partial M, t_0 \geq 0$ BSD determine whether $m_g(\text{supp}U \cap M(\Gamma, t)) = 0$ or not. Analogous statement takes place for $\mathcal{V}_{\text{ad}}$.

Here $m_g$ is the measure on $(M, g)$.

Proof. Consider $\mathcal{U}(t) = \{U_{j,k,l}(t)\}$ where

$$\frac{d}{dt}U_{j,k,l}(t) + i\lambda_j U_{j,k,l}(t) + iU_{j,k,l+1}(t) = 0, \quad t \in \mathbb{R}, \quad (21)$$

$$U_{j,k,l}(0) = U_{0;j,k,l}, \quad (22)$$

where $\{U_{0;j,k,l}\} = \mathcal{U} \in \mathcal{F}(\mathcal{L}^s)$. Then $\mathcal{U}(t) \in \mathcal{F}(\mathcal{L}^s)$ for all $t$ and $\mathcal{U}(t) = \mathcal{F}U(t)$ where

$$U_i(t) + iA U(t) = 0, \quad U(0) = U_0.$$

As $s \geq 0$ Lemma 1 and Sobolev embedding theorem show that

$$u^1(t)|_{\partial M} = \lim_{\tau \to 0} \lim_{N \to \infty} [P^\beta_\tau(A)U(t)], \quad (23)$$

where the convergence takes place in $L^2(\partial M)$. In view of the Homgren-John theorem [15] the fact that $m_g(\text{supp}U \cap M(\Gamma, t)) = 0$ is equivalent to the fact that

$$\text{suppu}^1|_{\partial M \times \mathbb{R} \cap (\Gamma \times [-t_0, t_0])} = \emptyset. \quad (24)$$

However $\phi^1_{j,k,l}|_{\partial M}$ are known so that the statement follows from (21), (22) and (23), (24).

Corollary 5. Let $\Gamma \subset \partial M, t_0 \geq 0$ and $s \geq 0$. Then BSD determine subspaces $\mathcal{F}(\mathcal{L}^s(\Gamma, t_0)), \mathcal{F}([\mathcal{L}^s(\Gamma, t_0)]^c)$, and $\mathcal{F}_{\text{ad}}(\mathcal{L}^s_{\text{ad}}(\Gamma, t_0)), \mathcal{F}_{\text{ad}}([\mathcal{L}^s_{\text{ad}}(\Gamma, t_0)]^c)$, where

$$\mathcal{L}^s(\Gamma, t_0) = \{U \in \mathcal{L}^s : \text{supp}U \subset \text{cl}(M(\Gamma, t_0))\};$$

$$[\mathcal{L}^s(\Gamma, t_0)]^c = \{U \in \mathcal{L}^s : \text{supp}U \subset \text{cl}(M \setminus M(\Gamma, t_0))\}$$

and analogous definitions are valid for $\mathcal{L}^s_{\text{ad}}(\Gamma, t_0), [\mathcal{L}^s_{\text{ad}}(\Gamma, t_0)]^c$.

Proof. By Lemma 4 BSD determine $[\mathcal{L}^s(\Gamma, t_0)]^c, [\mathcal{L}^s_{\text{ad}}(\Gamma, t_0)]^c$. As $U \in \mathcal{L}^s(\Gamma, t_0)$ is equivalent to the fact that $(U, J^*V) = 0$ for all $V \in [\mathcal{L}^s_{\text{ad}}(\Gamma, t_0)]^c$ the remaining part of Corollary 5 follows from Corollary 3.
Corollary 6. Let $\Gamma_i \subset \partial M$, $t^+_i > t^-_i \geq 0$; $i = 1, ..., I$. Denote by $M_I$ the set

$$M_I = \cap_{i=1}^I (M(\Gamma, t^+_i) \setminus M(\Gamma, t^-_i)).$$  \hfill (25)

Then BSD determine whether $m_g(M_I) = 0$ or not.

Corollary 6 is the basic analytic tool in the reconstruction of $(M, g)$. For this end introduce $R : M \to L^\infty(\partial M)$;

$$R(x) = r_x(y) = d(x, y), \quad y \in \partial M.$$  

It is shown in [7] that $R(M) \subset L^\infty(\partial M)$ has a natural structure of a Riemannian manifold such that $R : M \to R(M)$ is an isometry.

Theorem 3. BSD of the operator pencil (1), (2) which satisfies the BLR-condition determine $(M, g)$ uniquely.

Proof. In view of the above remark about isometry between $(M, g)$ and $R(M)$ it is sufficient to show that BSD determine $R(M)$. Choose $\delta > 0$ and a collection of $\Gamma_i, i = 1, ..., I(\delta)$ such that $\text{diam}(\Gamma_i) \leq \delta, \cup \Gamma_i = \partial M$. Let

$$p = (p_1, ..., p_I(\delta)), \quad p_i \in \mathbb{N}, \quad t^+_i = (p_i + 1)\delta; \quad t^-_i = (p_i - 1)\delta.$$  \hfill (26)

Denote by $M_I(p)$ the set $M_I$ (see (25)) with $t^+_i$ of form (26) and correspond to every $p$ such that $m_g(M_I(p)) > 0$ a piecewise constant function $r_p(y) = p_i\delta$ when $y \in \Gamma_i$. Let $R_\delta(M)$ be the collection of these functions. Then

$$\text{Dist}(R_\delta(M), R(M)) \leq 3\delta.$$  

Taking $\delta \to 0$ we construct $R(M)$.

6. Reconstruction of the lower-order terms.. Let $x_0 \in \text{int}M$ and

$$M_I(\delta) \to x_0 \quad \text{when} \quad \delta \to 0.$$  \hfill (27)

Consider a family $V(\delta) \in \mathcal{F}_{ad}(\mathcal{L}^0)$ such that

$$\text{supp}V(\delta) \subset \text{cl}(M_I(\delta)), \quad V = \mathcal{F}_{ad}V(\delta),$$  \hfill (28)

and for any $U \in \mathcal{F}(\mathcal{L}^s), s < m/2 < s + 1$ there is a limit $W^{x_0}(U)$;

$$W^{x_0}(U) = \lim_{\delta \to 0}(U, V(\delta)),$$

where the inner product in the rhs of (28) is understood in Abel-Lidskii sense. Such families exist, indeed it is sufficient to take $C^\infty_0$-approximations to $(\delta(\cdot - x_0), 0)^t$.

On the other hand since

$$(U, V(\delta)) = (U, J^*V(\delta)),$$

the existence of the limit means that there is a limit $W^{x_0} \in [D'(M)]^2$ of $V(\delta)$. By (27) $\text{supp}W^{x_0} \subset \{x_0\}$. Moreover as the limit exists for $U \in \mathcal{L}^s, s < m/2 < s + 1$, $W^{x_0} = (0, \kappa(x_0)\delta(\cdot - x_0))^t$. 

Lemma 5. Let BSD of an operator pencil (1), (2) be given and \((M,g)\) satisfies the BLR-condition. Then it is possible to construct a map \(\mathbb{W}: M \rightarrow \mathbb{C}^\infty\):

\[
\mathbb{W}(x_0) = \mathbb{W}^{x_0}; \quad W^{x_0}_{j,k,l} = \mathbb{W}^{x_0}(\mathcal{E}(j,k,l)),
\]

(where \(\mathcal{E}(j,k,l)\) is the sequence with 1 at the \((j,k,l)\)-place and 0 otherwise) such that

\[
\mathbb{W}(x_0)(U) = \kappa(x_0)u^1(x_0), \quad U \in \mathcal{F}(\mathcal{L}^s), \quad s < m/2 < s + 1;
\]

\[
\kappa \in C_\infty(M), \quad \kappa|_{\partial M} = 1, \quad \kappa \neq 0 \quad \text{on } M. \tag{29}
\]

Proof. To prove Lemma it is sufficient to show the existence of \(\mathbb{W}^{x_0}(\delta)\) such that their limits \(\mathbb{W}^{x_0}\) satisfy the following conditions

i. \(\mathbb{W}^{x_0} \neq 0\);

ii. \(\mathbb{W}^{x_0}(U) \in C_\infty(M)\) when \(U \in \mathcal{F}([C_\infty^0(M)]^2)\);

iii. \(\mathbb{W}^{x_0}(U) = u^1(x_0)\) when \(x_0 \in \partial M; U \in \mathcal{F}(\mathcal{L}^s), s < m/2 < s + 1\).

To prove the existence of such \(\mathbb{W}^{x_0}(\delta)\) we can take adjoint Fourier transforms of some smooth approximations to \((0, \delta(-x_0))^t\). On the other hand, conditions i-iii may be algorithmically verified due to Lemma 3, Corollary 3, Corollary 4, Lemma 4 and Lemma 1.

Corollary 7. BSD of a pencil (1), (2) with \((M,g)\) satisfying the BLR-condition determine the functions \(\kappa(x)\phi^1_{j,k,l}(x); j = 1, 2, \ldots, k = 1, \ldots, r_j, l = 1, \ldots, n_j,k\) where \(\kappa\) satisfies relations (29).

Proof. Since

\[
\kappa(x_0)\phi^1_{j,k,l}(x_0) = \mathbb{W}^{x_0}_{j,k,l},
\]

and \(\Phi_{j,k,l} \in \mathcal{L}^s\) for any \(s\) the statement follows from Lemma 5.

The functions \(\kappa\phi^1_{j,k,l}\) are the root functions for the pencil \(A_\kappa(\lambda)\);

\[
A_\kappa(\lambda_j)(\kappa\phi^1_{j,k,l}) := a_n(x,D)(\kappa\phi^1_{j,k,l}) - i\lambda_j b_0(\kappa\phi^1_{j,k,l}) - \lambda_j^2(\kappa\phi^1_{j,k,l}) = \kappa\phi^1_{j,k,l-1}, \tag{30}
\]

\[
B_\kappa(\kappa\phi^1_{j,k,l}) := (\partial_\nu(\kappa\phi^1_{j,k,l}) - \sigma_\kappa(\kappa\phi^1_{j,k,l}))[x_0 M] = 0, \tag{31}
\]

where

\[
a_\kappa(x,D) = \kappa a(x,D)\kappa^{-1}; \quad \sigma_\kappa = \sigma + \partial_\nu[\ln \kappa].
\]

Lemma 6. Functions \(\kappa\phi^1_{j,k,l}; j = 1, 2, \ldots, k = 1, \ldots, r_j, l = 1, \ldots, n_j,k\) where \(\kappa\) satisfies (66) determine \(a_\kappa, \sigma_\kappa, b_0\).

Proof. By Lemma 1 finite linear combinations of \(\kappa\Phi_{j,k,l} = (\kappa\phi^1_{j,k,l}, \lambda_j\kappa\phi^1_{j,k,l})^t\) are dense in \([C^N(\Omega)]^2\) for any \(N \geq 0, \Omega \ll M\). In particular for \(x_0 \in \text{int}M\) the vectors \((\kappa(x_0)\phi^1_{j,k,l}(x_0), \nabla(\kappa\phi^1_{j,k,l})(x_0), \lambda_j\kappa(x_0)\phi^1_{j,k,l}(x_0))^t\in \mathbb{C}^{m+2}\) span \(\mathbb{C}^{m+2}\). Then equations (30) determine \(a_\kappa\) and \(b_0\).

On the other hand for any \(y \in \partial M\) there is \(\phi^1_{j,k,l}\) such that \(\phi^1_{j,k,l}(y) \neq 0\). Hence equations (31) determine \(\sigma_\kappa\).

Theorem A is now a corollary of Lemma 6, Lemma 7 and properties (29) of \(\kappa\).
Some remarks.
i. The BLR-condition is always satisfied for $M \subset \mathbb{R}^m$ with the metric $g^{i,l} = \delta^{i,l}$ or its $C^1$-small perturbations (see e.g. [14, 16]);

ii. In particular the results of the paper are always valid for $m = 1$ even when GBSD are prescribed at only one boundary point (see also [17]);

iii. Using the nonstationary variant of the BC-method (see e.g. [8, 18]) it is possible to prove an analog of Theorem A when the data is the response operator $R^h(t)$ of form (6) for the problem (4), (5) in the case when $(M, g)$ satisfies the BLR-condition and $t > 2t_+$.

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