Teacher beliefs and the didactic contract on visualisation.

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In the last twenty years or so the debate about the potential contribution of visual representations to mathematical proof has intensified (e.g., Mancosu et al., 2005). Central to this debate is whether visual representation can be used not only as evidence and means of insight for a mathematical statement but also as part of its justification (Hanna & Sidoli, 2007). For example, Giaquinto (2007) argues that visual means are much more than a mere aid to understanding and can be resources for discovery and justification, even proof. Whether visual representations need to be treated as adjuncts to proofs, as an integral part of proof or as proofs themselves remains a point of contention.

Visualisation has gained analogous visibility within mathematics education. Its richness, the many different roles it can play in the learning and teaching of mathematics - as well as its limitations – are increasingly being written about (e.g., Arcavi, 2003). The foci of these works are as diverse as: curriculum development with an emphasis on visualisation (and often on related IT); mathematicians’ perceptions and use of visualisation; students’ seeming reluctance to engage (and difficulty) with visualisation; gender differences; links with embodied cognition; etc. – see Presmeg (2006) for a substantial review. Overall we still seem to be rather far from a consensus on the many roles visualisation can play in mathematical learning and teaching. So, while many works clearly recognise these roles, several also recommend caution with regard to ‘the ‘panacea’ view that mental imagery only benefits the learning process’ (Aspinwall, Shaw, & Presmeg, 1997).

Research into the relationship between teachers’ beliefs (epistemological and pedagogical) and pedagogical practice has attracted increasing attention by mathematics education researchers (e.g., Cooney, Shealy, & Arvold, 1998; Leatham, 2006; Leder, Pehknoen, & Torner, 2002; Thompson, 1992). With regard to teacher beliefs about mathematical reasoning and proof, as Harel and Sowder (2007) observe, ‘the emphasis that teachers place on justification and proof no doubt plays an important role in shaping students’ “proof schemes”’ (p. 827). The not very extensive research in this area shows that this emphasis is insufficient both in terms of extent and in terms of quality. Internationally, in most educational settings – even those with an official curricular emphasis on proof – little instructional time is dedicated to proof construction and appreciation. Furthermore, teachers’ own proof schemes are often predominantly empirical and teachers do not always seem to understand important roles of proof other than verification. For example, in Knuth’s (2002) study of practising secondary mathematics teachers, while all teachers acknowledged the verification role of proof, they rarely talked about its explanatory role. With regard to their proof schemes many of the interviewed teachers: felt compelled to check a statement on several examples even though they had just completed a formal proof; considered several of given non-proofs as proofs; accepted the proof of the converse of a statement as proof of the statement; and, found arguments based on examples or visual representations to be most convincing.

One of the aims of the study we draw on here is to explore the relationship between teachers’ beliefs about mathematical reasoning and pedagogical practice. Our particular focus is on teacher beliefs about the role of visualisation as evident in the reasoning and feedback they present to students. Even more specifically we aim: to explore teachers’ beliefs about the sufficiency of a visual argument, their views on the persuasiveness of a visual argument, their beliefs about the role of visual thinking in their students’ mathematical learning and their personal mathematical images; and, to discern the influence of these beliefs, views and images on the didactical contract (Brousseau, 1997) they offer their students with regard to the role of visualisation. One such potential influence originates in the ‘inherent sensibility’ of the teachers’ belief systems (Leatham, 2006).

The study and the Tangent Task
The study we draw on is currently in progress in Greece and in the UK. In this project we engage teachers with tasks that feature classroom scenarios that are hypothetical, but likely to occur in actual practice, and grounded on learning and teaching issues that previous research and experience have highlighted as seminal. The structure of the tasks is as follows: reflecting upon the learning objectives within a mathematical problem (and solving it); interpreting flawed (fictional) student solution(s); and, describing, in writing, feedback to the student(s). [1] In brief, in the terms of Adler and Davis’s (2006) classification, our tasks are of the MU’tU type: ‘a clear mathematical object that is primary … and a teaching object that is secondary…’ In both cases explicit reasoning of various solutions and pedagogical steps is required” (p. 286). Overall our tasks aim to help teachers develop their capacity to transform theoretical knowledge into theoretically-informed practice – see (Watson & Mason, 2007) – a transformation that has been described by concepts such as Hill and Ball’s (2004) mathematical knowledge for teaching and Shulman’s (1987) pedagogical content knowledge.

In what follows we focus on one of the tasks (fig. 1) we
have used. The Task was one of the questions in a written examination taken by 91 candidates for a Masters in Mathematics Education programme. All were mathematics graduates with teaching experience ranging from a few to many years. Most had attended in-service training of about 80 hours.

The first level of analysis of the scripts consisted of entering in a spreadsheet summary descriptions of the teachers’ responses with regard to the following: perceptions of the aims of the mathematical exercise in the Task; mathematical correctness; interpretation/evaluation of the two student responses included in the Task; feedback to the two students. Adjacent to these columns there was a comments column for our observations regarding the approach the teacher used (verbal, algebraic, graphical) to convey their commentary and feedback to the students across the script. On the basis of this first-level analysis we selected 11 of the participating teachers for interviews. Their individual interview schedules were tailored to the analysis of their written responses and, mostly, on questions we had noted in the comments column of the spreadsheet. Interviews lasted approximately 45 minutes and were audio recorded.

The mathematical problem within the Task in figure 1 aims to investigate students’ understanding of the tangent line at a point of a function graph and its relationship with the derivative of the function at this point, particularly with regard to two issues that previous research (e.g., Biza, Christou, & Zachariades, 2008; Castela, 1995) has identified as critical:

• students often believe that having one common point is a necessary and sufficient condition for tangency; and,
• students often see a tangent as a line that keeps the entire curve in the same semi-plane.

The studies mentioned above attribute these beliefs partly to students’ earlier experience with tangents in the context of the circle, and some conic sections. For example, the tangent at a point of a circle has only one common point with the circle and keeps the entire circle in the same semi-plane.

Since the line in the problem is a tangent of the curve at the inflection point A the problem provides an opportunity to investigate the two beliefs about tangency mentioned above - similarly to the way Tsamir, Rasslan, and Dreyfus (2006) explore teachers’ images of derivative through asking them to evaluate the correctness of suggested solutions. Under the influence of the first belief Student A carries out the first step of a correct solution (finding the common point(s) between the line and the curve), accepts the line tangent to the curve and stops. The student thus misses the second, and crucial, step: calculating the derivative at the common point(s) and establishing whether the given line has slope equal to the value of the derivative at this/these point(s). Under the influence of both beliefs, and grounding their claim on the graphical representation of the situation, Student B rejects the line as tangent to the curve.

Figure 1. The Tangent Task
Evidence of teacher beliefs about the role of visualisation in the scripts

One of the themes that emerged from the comments recorded in the spreadsheet with regard to the means the teachers used (verbal, algebraic, graphical) to convey their feedback to the students concerned the beliefs (epistemological and pedagogical) of the teachers about the role of visualisation. In the course of this part of our analysis we noticed several influences on the teachers’ responses: for example, almost all teachers distinguished between (and often juxtaposed) Student A’s algebraic approach and Student B’s graphical approach. In many cases they referred explicitly to their beliefs about, for example, the sufficiency/acceptability of the graphical approach; or about the role visual thinking may play in their students’ learning. The teachers’ responses also appeared significantly influenced by the mathematical context of the problem within the Task; namely, by their own perceptions of tangents and their own views as to whether the line in the Task must be accepted as a tangent or not.

At this point of our analysis we were somewhat surprised by the fact that 43 of the 91 participating teachers supported Student B’s claim that the line in the Task is not a tangent line – explicitly (25/91) or implicitly (18/91). From the analysis of the data from the 25 teachers who explicitly supported Student B’s claim three interrelated themes regarding teachers’ perceptions of tangents and their beliefs about visualisation emerged: Mathematical views on whether the line is a tangent or not, beliefs about the sufficiency/acceptability of the visual argument used by Student B and beliefs about the role of visual thinking in their students’ mathematical learning. Below we take a closer look at the evidence from these 25 teachers. [3]

Mathematical views

Of the twenty-five teachers who explicitly accepted Student B’s claim, ten rejected the line as a tangent without stating an argument (phrasing their responses as if this was obvious). The other fifteen stated that the line intersects with the curve without being its tangent either because point A is an intersection point but not a tangency point; or because it ‘cuts across’ the graph as student B argued. Three of these fifteen based the rejection on the fact that the line does not keep the entire curve in the same semi-plane. For example, one teacher claimed that ‘it is not sufficient that the tangent line has only one common point, but it must keep the graph on the same side’ and offered the graph in figure 2. Taking a local perspective on Student B’s ‘cutting across’ argument the teacher also offered figure 3 and said: ‘the tangent could cut across the curve … the line is a tangent at \( x_0 \) [in figure 3] although it cuts across the curve [at another point, our addition].’

Sufficiency of the visual argument used by Student B

Ten of these twenty-five teachers did not dispute the sufficiency of the visual argument used by Student B. Of these ten, eight made no reference to an algebraic argument. One teacher made some reference to both the ‘algebraic and graphical methods’ implying that she accepted the validity of both. Another set out with some reference to an algebraic argument that involved accepting the line. As she proceeded to a consideration of Student B’s solution she deleted the algebraic argument and concluded her response with agreeing with Student B.

The other fifteen teachers, while basing their inference on the graph and supporting Student B’s claim, stated the need for supporting and verifying the claim algebraically (11 explicitly and 4 implicitly). These teachers, although they hinted at the algebraic argument for the justification of the answer, did not employ it in the argument they offered the students. As a result they did not confront the inconsistency between rejecting the line and the algebraic argument.

The teachers proposed the algebraic argument either in order to support (e.g., ‘more feasible’, offering a ‘more rounded view of the problems’, etc.) or to validate mathematically the graphical one (e.g., ‘the graph not necessarily constituting a valid complete proof’). So, the teachers, while appreciating their students’ employment of visualisation to reach a conclusion, were keen to stress that ultimately students are expected to demonstrate their capacity to complete the Task algebraically. It is therefore possible that these teachers’ embrace of the visual approach evident in Student B’s solution is driven more by their belief in the gradual enculturation (Sierpinska, 1994) of the students into formal mathematical practice and their belief in the assistance that visualisation can provide towards reaching a conclusion (rather than a belief in the completeness of a graph-based argument).

Views on the role of visual thinking in student learning

Nine of these twenty-five teachers declared overtly their view of the graphical approach employed by Student B as evidence of ‘conceptual’ understanding (e.g., ‘understands to a good degree what mathematical thinking is about’, ‘imaginative’, ‘quick and ready’, ‘clever’, ‘not wasting any time’, ‘natural’, ‘real’; as opposed to the algebraic method being ‘traditional’, ‘mechanical’, etc.). In this vein, several teachers, while expressing their appreciation for the graphical approach, stressed that students are not always at ease with it and are often reluctant to use visualisation – they therefore saw its use by Student B as meriting appreciation because of its rarity.

With regard to how these twenty-five teachers responded to Student A – particularly given that student’s exclusive use of the algebraic method – many attempted to balance their feedback (‘the teacher must encourage students to work in both ways’). They therefore encourage Student A to also work graphically (‘this would help him’) and Student B to also ‘justify their answer algebraically’.

Probing further – particularly through the interviews – into these teachers’ commitment to accepting visual arguments, such as the one used by Student B, revealed a more ambiguous landscape than some of their statements in the written
responses had suggested. We exemplify the complexity of
this landscape through the case of one of the teachers, Nikos
(see also Biza, Nardi, & Zachariades, 2009).

**Variation and relativity of teachers’ beliefs
about the role of visualisation – the case of Nikos**

Nikos initially claimed that, generally, he would accept a
proof based on a graphical argument. Then he added that ‘it depends on the case’ and he offered a case in which the line
is a tangent without splitting the curve (e.g., a tangent at a
point of the graph of \( y = x^2 + 1 \)). ‘For this case’, he said, ‘I
would like the algebraic solution’ because ‘when you need to
prove something … [the graph] is not enough … the
graph could be inaccurate’. When asked ‘why he accepted
Student B’s graphical response’, he said that he believed
that this line was not a tangent and ‘if you make a graph in
which the line is not a tangent, I will accept it … that means,
from the graph I will accept it’ and he added that ‘if I
believed that the line is a tangent … I would not accept only
the graphical [solution]’.

It appears that Nikos is ready to accept the visual argu-
ment without any algebraic verification as the information
in the image is, to him, a clear and convincing proof that the
line is not tangent. Furthermore, he made clear that to prove
that a line is a tangent an algebraic argument is necessary.
What is not clear from the written response and the inter-
view is whether Nikos intended to make this distinction clear
to his students. Keeping this distinction covert is likely to
have serious ramifications for the didactical contract a
teacher offers their students with regard to mathematical rea-
soning and proof. Below we draw on the evidence from two
teachers, Spyros and Anna, in order to discuss briefly some
of these ramifications. We note that Spyros is one of the 38
(out of 91) teachers who rejected Student B’s claim that the
line is not a tangent. Anna is one of the 25 teachers who
agreed with Student B’s claim specifically. [3]

**Covert variation of teacher beliefs and the
didactical contract – the cases of Spyros and Anna**

While discussing Student B’s response we asked Spyros to
elaborate on whether he would accept an argument based
on a graph. His answer was firm: ‘No, first of all it is not an
adequate answer in exams’. (We note that in the Year 12
examination, which is also a university admission exam,
there is a requirement for formal proof). We asked him to
let aside the examination requirements for a moment and
consider whether an argument based on a graph would be
adequate mathematically. He replied: ‘Mathematically, in
the classroom, I would welcome it at lesson-level and I
would analyse it and praise it, but not in a test’. Asked to
elaborate he said: ‘Through [the graph-based argument] I
would try to lead the discussion towards a normal proof …with the definition, the slope, the derivative, etc.’. Asked
to justify he said: ‘This is what we, mathematicians, have learnt so far. To ask for precision. … we have this
axiomatic principle in our minds. … And this is what is
required in the exams. And we are supposed to prepare the
students for the exams.’

In the above, Spyros’s statement is clear: while he cannot
accept a graph-based argument as proof, he recognises graph-
based argumentation as part of the learning trajectory towards
the construction of proof. He seems to approach visual argu-
mentation from three different and interconnected perspectives: the *restrictions of the current educational set-
ting*, in this case the Year 12 examination; the *epistemological constraints* with regard to what makes an argument a proof
within the mathematical community; and, finally, the *pedag-
gogical role of visual argumentation as a means towards the
construction of formal mathematical knowledge*.

These three perspectives reflect three roles that a mathe-
matics teacher needs to balance: *educator* (responsible for
facilitating students’ mathematical learning), *mathematician*
(accountable for introducing the normal practices of the
mathematical community) and *professional* (responsible
for preparing candidates for one of the most important
examinations of their student career). Spyros’ awareness of
these roles, and their delicate interplay, is evidence of the
multi-layered didactical contract he appears to be able to
offer to his students.

Anna’s case is rather different. Her beliefs about the
acceptability or not of a visual argument appear less clear.
She appears ready to accept a visual argument without any
algebraic justification if the information in the image con-
istutes, for her, convincing support for a claim. In her script
she regarded the image in the Task as sufficient evidence
for determining that the line is not a tangent - also drawing
on her belief that a tangent cannot intersect the graph. How-
ever she stated clearly that to prove that the line is a tangent
an algebraic argument was necessary. Later, she stated that
she could accept a *correct* statement based on the graph.

When we shook her faith in the graph she declared the alge-
braic solution necessary. While initially she did not speak
of validation of the visual statement through reference to
mathematical theory, she asked for such validation when
she realised that the image could be misleading.

Many times in her interview she returned to her appreci-
ation of visual representation and argumentation as evidence
of a student’s in-depth understanding and as an important
means towards students’ construction of mathematical
knowledge. She did not specify whether she meant *formal*
mathematical knowledge (for example, proof). Furthermore,
hers views with regard to the sufficiency and acceptability
of a visual argument appeared rather ambivalent and heav-
ily dependent on the specific images involved in the
discussion. In this sense the didactical contract she appears
to be able to offer to her students seems less informed by
the multitude of issues that underlies Spyros’ approach.

**Towards a multi-layered and explicit didacti-
cal contract regarding visualisation**

The Task in figure 1 invited the teachers to offer feedback
to two students, one of which had used (incompletely) the
algebraic method for deciding whether the line is a tangent
and the other had used (incorrectly) a graphical representa-
tion of the problem. On this occasion the graph contained
information that conjures up images that may lead to the
rejection of the line as a tangent. About half of the teachers
in our study appeared to get ‘carried away’ by this informa-
is unequivocally accepted by the whole class, the class still of their claim. Even when a visually based (correct) claim approach to convince the proposer about the incorrectness claim is proposed, the class employs the algebraic, formal classroom discussion where a visually-based (incorrect) complexity that characterises mathematical activity. However, ation and multi-layeredness are welcome reflections of the when a visual argument is acceptable and when it is not? Vari-

cations a little further.

Spyros’ clear insistence on the class’ collective arrival at a formal proof as closure to the lesson is distinctly different from Anna’s fluctuation between cases where she would and would not accept a visual argument. Her willingness to rely, occasion-

ally, on imagery in order to support a claim is a ‘practice that may mislead students into thinking that such are accept-

able mathematical “proofs” and reinforcing the acceptability of their empirical proof schemes’ (Harel & Sowder, 2007, p. 829). Furthermore, her own criteria about what makes a visual argument acceptable appeared very personal and rather fluid. Within the unstable didactical contract that this vagueness might imply, how would her students distinguish between when a visual argument is acceptable and when it is not? Vari-

ation and multi-layeredness are welcome reflections of the complexity that characterises mathematical activity. However, in the already compounded didactical contract of school math-

ematics, such vagueness can be detrimental.

A less vague, subtler contract could be as follows: in a classroom discussion where a visually-based (incorrect) claim is proposed, the class employs the algebraic, formal approach to convince the proposer about the incorrectness of their claim. Even when a visually based (correct) claim is unequivocally accepted by the whole class, the class still employs the algebraic approach to establish the validity of the claim formally. In both cases visualisation emerges as a path to insight, and proof as the way to collectively estab-

lish the validity of insight. In both cases there is a pedagog-

ical opportunity for linking imagery with algebra and for embedding the algebra in the immediately graspable mean-

ing in the image. The above suggest a role for proof in the mathematics classroom that is not disjoint from the creative parts of visually-based classroom activity and that reflects an essential intellectual need, ‘an understanding of proof that is consistent with that shared and practised by the mathematicians of today’ (Harel & Sowder, 2007, p. 836).

The identification of subtle issues regarding the teachers’ pedagogical and epistemological beliefs (such as the above, about visualisation) – and the raising of teacher awareness of these issues – was made possible, we believe, by the combina-

tion ‘task engagement-followed by-interview’. In general (Biza et al., 2007), we propose that these tasks can be employed in a teacher education context as follows:

… as tools for the identification and exploration of mathematically, didactically and pedagogically spec-

ific issues regarding teacher knowledge (that purely theoretical questions on pedagogy or mathematics could not have identified); and, as triggers for teacher reflection on these issues. We also note that engage-

ment with these tasks can function as a preliminary, preparatory, smoother transitory phase for pre-service teachers prior to their exposure to real classroom situ-

ations. (p. 309)

Notes

[1] For an elaborated description of the theoretical origins of this type of task, see Biza, Nardi, and Zachariades (2007).
[2] For details about the Greek curricular context of the task, see Biza, Nardi, and Zachariades (2008).
[3] For more detail on these views, see Biza, Nardi, and Zachariades (2008).

References


ematics Teacher Education 10, pp. 301–309.


Here are responses given by three different learners in a Grade 8 class to the task, “Draw a triangle with 2 obtuse angles.” How would you evaluate these explanations?

1) It is impossible. An obtuse angle is more than 90 degrees, so two obtuse angles gives you more than 180 degrees. The angles of a triangle sum to 180 degrees.

2) It is impossible because you get a quadrilateral, not a triangle.

3) It is impossible, because if you draw an angle close to 90 degrees, say 89 degrees, and you stretch it, the other two angles will shrink so you won’t get another obtuse angle.