On the existence of a symmetric acoustic mode in a quadratic solid wedge

This item was submitted to Loughborough University’s Institutional Repository by the/an author.

Citation: KRYLOV, V.V., 1991. On the existence of a symmetric acoustic mode in a quadratic solid wedge. Moscow University Physics Bulletin, 46 (1), pp. 45 - 49

Additional Information:

- This journal is currently distributed by Springer and can be found online at: http://springeronline.com/journal/11972

Metadata Record: https://dspace.lboro.ac.uk/2134/9698

Version: Published

Publisher: © Allerton Press

Please cite the published version.
This item was submitted to Loughborough’s Institutional Repository (https://dspace.lboro.ac.uk/) by the author and is made available under the following Creative Commons Licence conditions.

For the full text of this licence, please go to: http://creativecommons.org/licenses/by-nc-nd/2.5/
ON THE EXISTENCE OF A SYMMETRIC ACOUSTIC MODE IN A QUADRATIC SOLID WEDGE

V.V. Krylov

It is shown that in a solid truncated wedge whose local thickness depends quadratically on its height, along with the earlier predicted countable set of antisymmetric acoustic modes localized near the apex, there can also exist a single symmetric acoustic mode. This mode is localized in the vicinity of the wedge apex, and its characteristics are close to those of Rayleigh's surface acoustic mode. The phase velocity and the field structure of the mode are calculated. The lowest-frequency boundary for the region of its existence is determined.

Fig. 1
Quadratic solid wedge with $h(x) = \varepsilon x^2$.

Recently it has been shown [1] that along the apex of a solid truncated wedge whose local thickness is described by the expression $h(x) = \varepsilon x^2$ (Fig. 1), a countable set of antisymmetric (quasi-flexural) localized oscillation modes can propagate whose phase velocities can, in principle, be arbitrarily small. Below we shall show that in the wedge-shaped structure under consideration there can also exist a single localized symmetric mode whose characteristics are close to those of Rayleigh's surface acoustic wave.

It should be noted that, alongside the various applied aspects [2], the problem of existence of localized symmetric modes in acute-angle solid wedges is also of a great fundamental significance. There are rather few theoretical works devoted to this question, and their conclusions are often contradictory. For example, in [3] numerical methods were applied for studying the propagation of both antisymmetric and symmetric modes along the edge of an ideal solid wedge formed by intersection of two plane faces (for brevity, in what follows we shall call it a linear wedge). And it was shown, in particular, that in wedges with acute angles the existence of symmetric modes is impossible. In contrast to [3], the authors of [4] made the conclusion, also based on numerical calculations, that there can exist a symmetric mode in a linear acute-angle wedge.
In [5], among some other results, it was shown analytically that there can exist a symmetric quasi-Rayleigh mode in a truncated acute-angle linear wedge. This conclusion was drawn as a result of the solution of the corresponding boundary problem in the geometrical acoustics approximation valid under the condition \( k_p x_0 > 1 \), where \( x_0 \) is the height of the truncated wedge and \( k_p \) is the wave number of a longitudinal wave in a thin plate [6].

In contrast to the case of a linear wedge, symmetric modes in a quadratic solid wedge have not been analyzed up to now. However, as will be seen from what follows, a structure of this kind is of interest in view of the fact that it can be analyzed not only within the framework of the geometrical-acoustics approach but also based on the exact solution of the corresponding boundary problem for a thin plate of variable thickness \( h(x) = \varepsilon x^2 \). This makes it possible to establish the complete dispersion law for the localized symmetric mode and to determine the lowest-frequency boundary for its existence.

Assuming that the isotropic solid wedge is sufficiently acute, i.e., assuming the parameter \( \varepsilon x_0 \) to be small (the exact restriction on \( \varepsilon x_0 \) will be specified below), the displacements \( u_x \) and \( u_y \) of particles of the medium in the median surface of the wedge will be described by the equations of motion for a plane stressed state from [7] in which, however, the plate thickness \( h \) will be assumed to depend on \( x \):

\[
\begin{align*}
\frac{\partial}{\partial x} [h(x)\sigma_{xx}] + \frac{\partial}{\partial y} [h(x)\sigma_{xy}] + \omega^2 \rho h(x) u_x &= 0, \\
\frac{\partial}{\partial x} [h(x)\sigma_{yy}] + \frac{\partial}{\partial y} [h(x)\sigma_{xy}] + \omega^2 \rho h(x) u_y &= 0
\end{align*}
\]

(1)

Here \( \sigma_{xx}, \sigma_{xy}, \) and \( \sigma_{yy} \) are the components of the elastic stress tensor in the \( xy \) plane; \( \rho \) is the density of the wedge material; and \( \omega \) is the circular frequency (the factor \( \exp(-i\omega t) \) is omitted). Equations of motion (1) must be supplemented with equations of state (Hooke's law for a thin plate) [7]:

\[
\sigma_{xx} = \frac{E}{1 - \sigma^2} (u_{xx} + \sigma u_{yy}), \\
\sigma_{yy} = \frac{E}{1 - \sigma^2} (u_{yy} + \sigma u_{xx}), \\
\sigma_{xy} = \frac{E}{1 + \sigma} u_{xy},
\]

(2)

where \( E \) and \( \sigma \) are, respectively, the Young modulus and the Poisson ratio of the material of the wedge (or, which is the same, of a plate with varying thickness), and \( u_{ij} = (1/2)(\delta u_i / \delta z_j + \delta u_j / \delta z_i) \) are the components of the linearized strain tensor (the subscripts \( i, j \) assume the values \( x, y \)). As is known, by formally substituting in Eq. (2)

\[
\sigma = \sigma^*, \quad E = \frac{E^*}{1 - \sigma^*},
\]

(3)

we obtain the formulas relating the stress and strain in the \( xy \) plane for an unbounded medium (for \( u_z = 0 \)) characterized by the elastic constants \( E^* \) and \( \sigma^* \) (the so-called plane strain state [7]).

The field of the symmetric localized waves we are interested in must satisfy not only relations (1) and (2) but also the boundary conditions expressing the absence of a normal stress on the truncated wedge apex (at \( x = x_0 \)):

\[
\sigma_{xx} = \sigma_{yy} = 0,
\]

(4)

and also the condition that the field vanishes as \( x \to \infty \).

Divide both sides of Eq. (1) by \( h(x_0) \). After that, substituting (2) into (3) and taking (3) into account, it can easily be seen that the boundary problem under consideration on the determination of a symmetric localized mode in a solid wedge reduces formally to the problem of finding a Rayleigh-type surface acoustic wave on the boundary of a vertically inhomogeneous solid half-space (Rayleigh's problem) whose equivalent elastic Lamé coefficients \( \lambda(x) \) and \( \mu(x) \) and also the density \( \rho(x) \) vary with depth according to the law

\[
\lambda(x) = \lambda^* \frac{h(x)}{h(x_0)}, \quad \mu(x) = \mu^* \frac{h(x)}{h(x_0)}, \quad \rho = \rho \frac{h(x)}{h(x_0)}.
\]

(5)
Note that the effective Lamé coefficients \( \lambda^* \) and \( \mu^* \) are related to their true values for the wedge material by the formulas

\[
\lambda^* = \frac{E^* \sigma^*}{(1 + \sigma^*)(1 - 2\sigma^*)} = \frac{\lambda - 2\sigma}{1 - \sigma}, \\
\mu^* = \frac{E^*}{2(1 + \sigma^*)} = \mu.
\]

It is well-known that in the case of arbitrary functions \( \bar{h}(x) \), \( \bar{u}(x) \), and \( \bar{p}(x) \), Rayleigh's problem has no exact analytical solution (see, e.g., [8]). Therefore, by analogy with [5], we first give a brief account of the results of the problem solution for the case \( h(x) = \varepsilon x^2 \) under study in the geometrical-acoustics approximation. In this case Eqs. (1) and (2) describe the propagation of two noninteracting longitudinal and transverse acoustic waves at an arbitrary angle to the surface (see also [8]). Here the role of the longitudinal wave is played by the symmetric (longitudinal) mode in a thin plate (i.e., the lowest symmetric Lamb mode) propagating with velocity \( c^p = (E/\rho(1 - \sigma^2)^{1/2} = [(\lambda^* + 2\mu^*)/\rho]^{1/2} \), and the role of the transverse (shear) wave is played by the lowest SH-mode of the plate propagating with velocity \( c_t = (\mu/\rho)^{1/2} \). The geometrical-acoustics expressions for the displacement amplitudes of the longitudinal and transverse wave, indicated in what follows by the indices \( p \) and \( t \), can be written as

\[
u^p(\nu^t) = \frac{A_1}{k_t z} \exp(ik_y - \nu^p z),
\]

\[
u^t(\nu^t) = \frac{B_1}{k_t z} \exp(i\nu_t - \nu^t z),
\]

where \( k \) is the projection of the wave vectors of the propagating waves on the \( x \) axis; \( k_t = \omega/c_t \) is the wave number of the volume shear wave; \( \nu_{p,t} = (k^2 - k_{p,t}^2)^{1/2} \); \( A_1 \) and \( B_1 \) are constants determined from the boundary conditions; and the subscript \( t \) assumes the values \( x \) and \( y \). The presence of the factors \( x^{-1} \) in (7) and (8) reflects the energy conservation law for the two types of waves [6] in arbitrary sections \( z \) = const in the case of quadratic wedge under consideration. Note that in the geometrical-acoustics approximation the waves described by formulas (7) and (8) neither interact nor undergo refraction because, according to (5), the velocities \( c_p \) and \( c_t \) do not depend on \( z \). The interaction between waves (7) and (8) occurs only at the free boundary (the wedge edge) where boundary conditions (4) must be fulfilled. Substituting (7) and (9) into (4), using the well-known relations \( \text{div} \, u = 0 \) and \( \text{rot} \, u = 0 \) [7], and discarding the terms of the order of \( (1/k_t z)^2 \) and higher, we arrive, in accordance with the geometrical-acoustics method, at the following dispersion relation determining the velocity of the localized symmetric mode:

\[
(2k^2 - k_t^2)^2 - 4k^2 \nu_p \nu_t = 0.
\]

Eq. (9) coincides in its form with the well-known Rayleigh-type equation describing the propagation of a Rayleigh-type surface wave along the edge of a thin plate of constant thickness [2]. Thus, in the geometrical-acoustics approximation, the phase velocity of the symmetric localized mode of a quadratic solid wedge coincides with the velocity of a quasi-Rayleigh wave propagating along the edge of a thin plate. The relationships between the constants \( A_t \) and \( B_t \) also coincide with the corresponding expressions for Rayleigh waves (see, e.g., [7]). The only distinction consists in an additional amplitude drop proportional to \( (k_t z)^{-1} \) an the distance from the edge increases (see Eqs. (7) and (8)).

The above geometrical-acoustics solution which is valid for \( k_t z_0 \gg 1 \) or, which is the same, for \( k_p z_0 \gg 1 \), gives no information on the behavior of the symmetric mode under consideration at lower frequencies. However, with the specific Poisson ratio \( \sigma = 1/3 \) of the medium for the quadratic solid wedge under study it is possible to construct the exact solution to the boundary problem (1), (2), (4) (by virtue of relations (3) and (5)) based on the solution obtained in [6] for an elastic half-space with \( \lambda(x) = \mu(x) = \mu_0 (x/z_0)^2 \) and \( p(x) = p_0 (x/z_0)^2 \), i.e., for a medium with the Poisson ratio of 1/4 and the velocities of longitudinal and shear waves not depending on \( x \). The authors of [9] employed the substitution \( u_x = u_x' / \sqrt{\mu(x)} \), \( u_y = u_y' / \sqrt{\mu(x)} \) which enabled them to obtain the exact formulas for the amplitudes of \( u_x \) and \( u_y \) in the surface wave and also the dispersion relation determining the phase velocity of the wave.

For the sake of brevity, we shall not give here the formulas for the amplitudes of \( u_x \) and \( u_y \) obtained in [9], especially as, in the case under consideration, they coincide in form with the geometrical solution (7).
(8) (it should be taken into account that \( u_x = u_x^{(p)} + u_x^{(s)} \) and \( u_y = u_y^{(p)} + u_y^{(s)} \)). We give here only the corresponding dispersion relation for the case of a quadratic solid wedge, writing it in the following compact form:

\[
k^2 \hat{\omega}^2 F(\eta) - k_j x_0 f(\eta) - \varphi(\eta) = 0.
\]

Here \( \eta = c/c_1 \), where \( c \) is the velocity of the symmetric mode, \( c_1(\mu/\rho)^{1/2} \) is the velocity of the volume shear wave; \( F(\eta) = (2 - \eta^2)^2 - 4(1 - \eta^2)^{1/2}(1 - (1/3)\eta^2)^{1/2} \) is the so-called Rayleigh determinant for the Poisson ratio \( \sigma^* = 1/4 \) (this corresponds to the Poisson ratio of the wave material \( \sigma = 1/3 \)); \( f(\eta) = \eta^9[3(1 - (1/3)\eta^2)^{1/2} + (1 - \eta^2)^{1/2}] \) and \( \varphi(\eta) = 3\eta^2 - 3\eta^2(1 - \eta^2)^{1/2}(1 - (1/3)\eta^2)^{1/2} \).

**Fig. 2**

Dependence of the relative velocity of a symmetric mode in a quadratic solid wedge on the parameter \( k_j x_0 \) for a medium with the Poisson ratio \( \sigma = 1/3 \) for (1) \( c/c_1 = 0.9194 \) (corresponding to a quasi-Rayleigh mode propagating along the end face of a thin plate) and (2) \( c/c_1 = 0.9325 \) (corresponding to a Rayleigh wave in a half-space).

Since Eq. (10) was not analyzed in [9], we shall obtain its solution using the fact that it is quadratic in \( k_j x_0 \). On finding \( k_j x_0 \) from (10) as a function of \( \eta \) one can easily construct the inverse plot of \( \eta \) versus \( k_j x_0 \) (Fig. 2), which is also of interest. From this plot it follows that the phase velocity \( c \) is always greater than its geometrical-acoustics value (which, as was already mentioned, coincides with the velocity of a Rayleigh-type wave on the end face of a thin plate with \( \sigma = 1/3 \) corresponding to an infinite medium with \( \sigma^* = 1/4 \)) and tends to this value as \( k_j x_0 \to \infty \). Note that it is also easy to obtain directly from Eq. (10) an approximate analytical expression describing the behavior of \( \eta \) for \( k_j x_0 \gg 1 \):

\[
\eta = \eta_0 \left[ 1 + \frac{f(\eta_0)}{\eta_0 F'(\eta_0) k_j x_0} \right],
\]

Here \( \eta_0 \) is the value of \( \eta \) for \( k_j x_0 \to \infty \). However, it should be pointed out that the velocity of the symmetric mode in question cannot be greater than the velocity of Rayleigh's wave in a material with the Poisson ratio \( \sigma = 1/3 \), of which the wedge is made. Otherwise, the mode energy would have been partly emitted in the form of Rayleigh waves (Cerenkov-type radiation) and the mode would have been transformed into the so-called leak mode. It is the intersection of the horizontal line \( \eta = 0.9325 \) corresponding to the Rayleigh wave velocity for a half-space with \( \sigma = 1/3 \) and the constructed dispersion curve \( \eta = \eta(k_j x_0) \) that determines the lower boundary (with respect to \( k_j x_0 \)) for the region of existence of a symmetric mode in a solid quadratic wedge. According to Fig. 2, this intersection takes place at \( k_j x_0 \approx 40 \) or \( k_p x_0 \approx 23 \). Therefore, the nondecaying symmetric mode exists for \( k_j x_0 > 40 \) or, which is the same, for \( k_p x_0 > 23 \).

Finally, we determine the constraint on the parameter \( x_0 \) in the structure under consideration. This constraint is related to the condition under which Eqs. (1), (2) for a thin plate hold and is implied by
the requirement that the local wedge thickness $h(x)$ should be small as compared to the shear wave length $\lambda_t = 2\pi/k_t$ for the values of $z$ corresponding to the characteristic penetration depth $l = \lambda_t = 2\pi/k_t$ of the mode under study.

$$\varepsilon(z_0 + 2\pi/k_t^2) < 2\pi/k_t.$$  \hspace{1cm} (12)

Since, according to what has been said, the propagation of the nondecaying mode takes place at $k_t z_0 > 40$, the term $2\pi/k$ on the left-hand side of inequality (12) can be neglected. Then formula (12) implies that $40 < k_t z_0 < 2\pi/\varepsilon z_0$, i.e., the wedge must be sufficiently acute, namely $\varepsilon z_0 \ll 1$.

Thus, in a quadratic solid wedge there can exist a single symmetric acoustic mode whose phase velocity is close to but somewhat greater than that of a Rayleigh-type wave on the end face of a homogeneous thin plate. For the Poisson ratio $\sigma = 1/3$ of the wedge material, the lowest-frequency boundary for the existence of the mode is determined from the condition $k_t z_0 > 40$. One should expect that for other values of the Poisson ratio the constraint on $k_t z_0$ has the same order of magnitude. A similar restriction from below for $k_t z_0$ is likely to take place in the case of a truncated acute-angle linear wedge [5]. Note that this assumption agrees in a natural way with the conclusion drawn in [3] that nondecaying symmetric modes are absent in an ideal (nontruncated) structure of this type, i.e., in the case of $k_t z_0 = 0$.

REFERENCES


16 May 1990