An activity theory analysis of linear algebra teaching within university mathematics

This item was submitted to Loughborough University's Institutional Repository by the/an author.

Additional Information:

- A Doctoral Thesis. Submitted in partial fulfilment of the requirements for the award of Doctor of Philosophy of Loughborough University.

Metadata Record: [https://dspace.lboro.ac.uk/2134/9843](https://dspace.lboro.ac.uk/2134/9843)

Publisher: © Stephanie Thomas

Please cite the published version.
This item was submitted to Loughborough’s Institutional Repository (https://dspace.lboro.ac.uk/) by the author and is made available under the following Creative Commons Licence conditions.

For the full text of this licence, please go to:
http://creativecommons.org/licenses/by-nc-nd/2.5/
An Activity Theory Analysis of Linear Algebra Teaching within University Mathematics

by

Stephanie Thomas

A Doctoral Thesis

Submitted in partial fulfilment of the requirements for the award of Doctor of Philosophy of Loughborough University

in the Mathematics Education Centre
School of Mathematics

November, 2011

© by Stephanie Thomas, 2011
Declaration of Authorship

I, Stephanie Thomas, declare that this thesis titled, ‘An Activity Theory Analysis of Linear Algebra Teaching within University Mathematics’ and the work presented in it are my own. I confirm that:

■ This work was done wholly while in candidature for a research degree at this University.

■ Neither the thesis nor the original work contained therein has been submitted to this or any other institution for a degree.

■ Where I have consulted the published work of others, this is always clearly attributed.

■ Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work.

■ I have acknowledged all main sources of help.

Signed: 

Date:
Abstract

An Activity Theory Analysis of Linear Algebra Teaching within University Mathematics

by Stephanie Thomas

The focus of my research was to explore the teaching of linear algebra to a large group of mathematics undergraduates (> 200). With this thesis I present a characterisation of a university mathematics teaching practice in the context of linear algebra.

The study took place over a twelve week period, one academic semester, at a UK university with a strong tradition in engineering and design technology. Two researchers working closely with a mathematician, the lecturer of linear algebra, collected data in interviews with the lecturer, and in observations of his lectures (which were audio-recorded). Students’ views were sought via two questionnaires and focus group interviews. Data analysis was largely qualitative.

Linear algebra is an introductory module in most standard first year undergraduate degree courses in mathematics. Research shows that students find the highly conceptual nature of linear algebra very difficult and challenging. The lecturer, a research mathematician, had re-designed the linear algebra module based on his own experience of students’ difficulties with the topic in the previous year. He followed an inductive approach to teaching instead of a more traditional DTP (definition-theorem-proof) style. He based his teaching on informal reasoning about examples that were designed to engage students conceptually with the material.

Through this research I gained insight into the lecturer’s motivation, intentions and strategies in relation to his teaching. In applying an activity theory analysis alongside a traditional grounded theory approach to my research, I conceptualised the lecturer’s teaching practice and presented a model of the teaching process. This takes account of the lecturer’s didactical thinking in planning and delivering the linear algebra teaching.

Findings from the study give insight into the educational practice of a mathematician in his role as a teacher of university mathematics. I present some of the outcomes of the study in terms of mathematics (three linear algebra topics - subspace, linear independence and eigenvectors), in terms of the didactics of mathematics and in terms of the theoretical basis of Mathematics Education as a discipline.
Acknowledgements

There are many people who over the last four years have helped me to complete my studies. But there are some I wish to mention by name.

I thank my supervisor, Professor Barbara Jaworski, for her sustained and unfaltering guidance. I thank Antony Edwards for his patience in explaining to me all matters technical and computing. I thank also Somali Roy who with just a few words ensured that I continued when I thought of giving up. Most of all I thank the mathematician and lecturer of linear algebra for his openness, kindness and his time without which this study would not have been possible.

Finally I thank my family and friends for their patience and affections. I could not have done this without you.
For Anna
Contents

Declaration of Authorship i
Abstract ii
Acknowledgements iii
List of Figures viii
List of Tables x

1 Introduction and background to my study 1
   1.1 The University and the research team . . . . . . . . . . . . . . . . . . . . 3
   1.2 The linear algebra module . . . . . . . . . . . . . . . . . . . . . . . . . . 4

2 Literature Review 6
   2.1 The teaching and learning of linear algebra: A historical note . . . . . 7
   2.2 Students’ difficulties with linear algebra . . . . . . . . . . . . . . . . . . 8
   2.3 Models for teaching linear algebra . . . . . . . . . . . . . . . . . . . . . 10
      2.3.1 Change of contract and the meta-lever . . . . . . . . . . . . . . . . 11
      2.3.2 A matrix-based approach and how, and what questions to ask . . . 12
      2.3.3 Geometry: ‘Yes’ or ‘No’ . . . . . . . . . . . . . . . . . . . . . . . 13
   2.4 Research into teaching practice . . . . . . . . . . . . . . . . . . . . . . . 14
   2.5 The role of the lecture and of lecturing . . . . . . . . . . . . . . . . . . . 15

3 Methodology 18
   3.1 The development of the research questions . . . . . . . . . . . . . . . . . 18
   3.2 The design of my study . . . . . . . . . . . . . . . . . . . . . . . . . . . . 20
   3.3 The research paradigm . . . . . . . . . . . . . . . . . . . . . . . . . . . . 20
   3.4 The methods of data collection . . . . . . . . . . . . . . . . . . . . . . . 24
      3.4.1 Ethical considerations in data collection . . . . . . . . . . . . . . . 25
      3.4.2 The meetings data . . . . . . . . . . . . . . . . . . . . . . . . . . . 26
      3.4.3 The lecture observation data . . . . . . . . . . . . . . . . . . . . . 28
      3.4.4 The student interview data . . . . . . . . . . . . . . . . . . . . . . 28
      3.4.5 The student survey data . . . . . . . . . . . . . . . . . . . . . . . 29
   3.5 The methods of data analysis . . . . . . . . . . . . . . . . . . . . . . . . 31
      3.5.1 The analysis of the meetings data . . . . . . . . . . . . . . . . . . 31
      3.5.2 Ethical considerations in the coding analysis . . . . . . . . . . . 33
      3.5.3 The analysis of the lecture observation data . . . . . . . . . . . 34
3.5.4 The analysis of the student interview data .................................. 35
3.5.5 The analysis of the student survey data ........................................ 35
3.6 Addressing trustworthiness in my study .......................................... 36

4 Theoretical perspectives ................................................................. 39
4.1 Vygotskian perspectives as a foundation for activity theory .............. 40
4.2 Activity theory ............................................................................. 42
4.3 The structural elements of activity ................................................ 44

5 A structural analysis of teaching with respect to theory .................... 49
5.1 Course design and course structure ............................................... 49
5.2 Introduction to my analysis ........................................................... 50
5.3 Activity and motive ....................................................................... 52
5.3.1 The EAG approach ....................................................................... 55
5.4 Actions and goals ........................................................................... 60
5.4.1 The goals/intentions ................................................................. 61
5.4.2 The actions/strategies .............................................................. 63
5.4.3 Goal 1: Engagement with mathematics ...................................... 65
5.4.4 Goal 2: Intuitive understanding .................................................. 68
5.4.5 Goal 3: Acquisition of mathematical language ............................. 71
5.4.6 Goal 4: Conceptual understanding .............................................. 73
5.4.7 Goal 5: Mathematical competence ............................................. 76
5.4.8 ‘Verbalising of intentions’ .......................................................... 78
5.4.9 A brief summary of this section ................................................... 81
5.5 Operations and conditions .............................................................. 82

6 The Teaching of Linear Algebra Concepts ......................................... 86
6.1 Introduction to Chapter 6 ............................................................... 87
6.1.1 Analysing subspaces, linear independence and eigenvectors .......... 87
6.1.2 The four aspects of analysis ...................................................... 90
6.1.3 The reference textbooks and set of course notes ......................... 92
6.1.4 Notation ................................................................................. 94
6.1.5 Assumptions and points of reference ......................................... 97
6.2 The concept of a subspace ............................................................ 99
6.2.1 A mathematical account of vector spaces and subspaces .......... 99
6.2.2 The lecturer’s mathematical treatment of subspace in the course notes .................................................. 103
6.2.3 The lecturer’s didactical thinking in relation to the teaching of subspaces .................................................. 107
6.2.4 The lecturer’s teaching of subspace (to students) ......................... 114
6.3 The concept of linear independence ............................................. 119
6.3.1 A mathematical account of linear independence ....................... 119
6.3.2 The lecturer’s mathematical treatment of linear independence in the course notes .................................................. 123
6.3.3 The lecturer’s didactical thinking in relation to the teaching of linear independence .................................................. 127
6.3.4 The lecturer’s teaching of linear independence (to students) ....... 134
6.4 The concept of eigenvectors/values .............................................. 141
6.4.1 A mathematical account of eigenvectors/values . . . . . . . . . . . . 142
6.4.2 The lecturer’s mathematical treatment of eigenvectors/values in
the course notes . . . . . . . . . . . . . . . . . . . . . . . . . . . . 152
6.4.3 The lecturer’s didactical thinking in relation to the teaching of
eigenvectors . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 159
6.4.4 The lecturer’s teaching of eigenvectors/values (to students) . . . . 168
6.5 Didactical summary . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 172

7 Students’ views 177

8 Synthesis and Conclusions 184
8.1 On the teaching of linear algebra . . . . . . . . . . . . . . . . . . . . . . . 184
  8.1.1 The lecturer’s model for teaching linear algebra . . . . . . . . . . . . 185
  8.1.2 Some implications of the model . . . . . . . . . . . . . . . . . . . . . 186
  8.1.3 Cultural shifts and the devolution of learning . . . . . . . . . . . . . . 188
8.2 Implications for the learning of linear algebra . . . . . . . . . . . . . . . . 189
  8.2.1 Accounting for students’ views . . . . . . . . . . . . . . . . . . . . . 189
  8.2.2 Mathematical considerations . . . . . . . . . . . . . . . . . . . . . . . 190
8.3 The theoretical and methodological contribution of my research to the
field of mathematics education . . . . . . . . . . . . . . . . . . . . . . . . 192
  8.3.1 Conceptualising a university mathematics teaching practice from
an activity theory perspective . . . . . . . . . . . . . . . . . . . . . . . . 192
  8.3.2 The development of methodology in an interpretive paradigm . . . 193
8.4 Concluding remarks . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 194

A The Focus Group Interviews 197
B Student Questionnaires 201
C Extracts from Textbooks 206
D Examples from Coding Analysis 212
E The Linear Algebra Course Notes (Student Version) 218
F The Linear Algebra Course Notes (Complete Version) 233

Bibliography 248
# List of Figures

<table>
<thead>
<tr>
<th>Section</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.1</td>
<td>The Structural Elements of Activity</td>
<td>45</td>
</tr>
<tr>
<td>5.1</td>
<td>The solution of Example 3.14 in the course notes</td>
<td>58</td>
</tr>
<tr>
<td>5.2</td>
<td>Modelling the teaching process</td>
<td>61</td>
</tr>
<tr>
<td>5.3</td>
<td>The set of nested goals</td>
<td>62</td>
</tr>
<tr>
<td>5.4</td>
<td>Depicting four operations under the condition of ‘a mathematics lecture’</td>
<td>83</td>
</tr>
<tr>
<td>6.1</td>
<td>The diagonalisation procedure (Sproston, 1995)</td>
<td>150</td>
</tr>
<tr>
<td>A.1</td>
<td>Participant Information Sheet</td>
<td>198</td>
</tr>
<tr>
<td>A.2</td>
<td>Consent Form</td>
<td>199</td>
</tr>
<tr>
<td>A.3</td>
<td>Questions to students in the focus group interviews</td>
<td>200</td>
</tr>
<tr>
<td>B.1</td>
<td>Student Questionnaire 1, page 1</td>
<td>202</td>
</tr>
<tr>
<td>B.2</td>
<td>Student Questionnaire 1, page 2</td>
<td>203</td>
</tr>
<tr>
<td>B.3</td>
<td>Student Questionnaire 2, page 1</td>
<td>204</td>
</tr>
<tr>
<td>B.4</td>
<td>Student Questionnaire 2, page 2</td>
<td>205</td>
</tr>
<tr>
<td>C.1</td>
<td>Introducing the concept of a subspace in Hoffman &amp; Kunze (1961, p. 34)</td>
<td>207</td>
</tr>
<tr>
<td>C.2</td>
<td>Introducing the concept of a subspace in Hoffman &amp; Kunze (1961, p. 35)</td>
<td>208</td>
</tr>
<tr>
<td>C.3</td>
<td>Introducing the concept of a subspace in Greub (1967, p. 23)</td>
<td>209</td>
</tr>
<tr>
<td>C.4</td>
<td>Introducing the concept of a subspace in Poole (2006, p. 438)</td>
<td>210</td>
</tr>
<tr>
<td>C.5</td>
<td>Introducing the concept of a subspace in Sproston (1995, p. 4)</td>
<td>211</td>
</tr>
<tr>
<td>D.1</td>
<td>Coding analysis of document P5</td>
<td>213</td>
</tr>
<tr>
<td>D.2</td>
<td>Coding analysis of document P8</td>
<td>214</td>
</tr>
<tr>
<td>D.3</td>
<td>Coding analysis of document P12</td>
<td>215</td>
</tr>
<tr>
<td>D.4</td>
<td>Code list exported from atlas-ti</td>
<td>216</td>
</tr>
<tr>
<td>D.5</td>
<td>Code list exported from atlas-ti</td>
<td>217</td>
</tr>
<tr>
<td>E.1</td>
<td>Chapter 3, page 1: Subspaces (student course notes)</td>
<td>219</td>
</tr>
<tr>
<td>E.2</td>
<td>Chapter 3, page 2: Subspaces</td>
<td>220</td>
</tr>
<tr>
<td>E.3</td>
<td>Chapter 3, page 3: Subspaces</td>
<td>221</td>
</tr>
<tr>
<td>E.4</td>
<td>Chapter 3, page 7: Linear Independence (student course notes)</td>
<td>222</td>
</tr>
<tr>
<td>E.5</td>
<td>Chapter 3, page 8: Linear Independence</td>
<td>223</td>
</tr>
<tr>
<td>E.6</td>
<td>Chapter 3, page 9: Linear Independence</td>
<td>224</td>
</tr>
<tr>
<td>E.7</td>
<td>Chapter 3, page 12: Linear Independence</td>
<td>225</td>
</tr>
<tr>
<td>E.8</td>
<td>Chapter 3, page 13: Linear Independence</td>
<td>226</td>
</tr>
<tr>
<td>E.9</td>
<td>Chapter 3, page 14: Linear Independence</td>
<td>227</td>
</tr>
<tr>
<td>E.10</td>
<td>Chapter 4, page 1: Eigenvectors (student course notes)</td>
<td>228</td>
</tr>
<tr>
<td>Section</td>
<td>Chapter</td>
<td>Page</td>
</tr>
<tr>
<td>---------</td>
<td>---------</td>
<td>------</td>
</tr>
<tr>
<td>E.11</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>E.12</td>
<td>4</td>
<td>3</td>
</tr>
<tr>
<td>E.13</td>
<td>4b</td>
<td>4</td>
</tr>
<tr>
<td>E.14</td>
<td>4b</td>
<td>5</td>
</tr>
<tr>
<td>F.1</td>
<td>1</td>
<td>6</td>
</tr>
<tr>
<td>F.2</td>
<td>1</td>
<td>7</td>
</tr>
<tr>
<td>F.3</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>F.4</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>F.5</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>F.6</td>
<td>3</td>
<td>7</td>
</tr>
<tr>
<td>F.7</td>
<td>3</td>
<td>8</td>
</tr>
<tr>
<td>F.8</td>
<td>3</td>
<td>9</td>
</tr>
<tr>
<td>F.9</td>
<td>3</td>
<td>12</td>
</tr>
<tr>
<td>F.10</td>
<td>3</td>
<td>13</td>
</tr>
<tr>
<td>F.11</td>
<td>3</td>
<td>14</td>
</tr>
<tr>
<td>F.12</td>
<td>4</td>
<td>1</td>
</tr>
<tr>
<td>F.13</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>F.14</td>
<td>4</td>
<td>3</td>
</tr>
</tbody>
</table>
List of Tables

8.1 Aspects of Semester 1 and Semester 2 Teaching . . . . . . . . . . . . . . . 185
Chapter 1

Introduction and background to my study

In my thesis I describe a teaching practice of mathematics at university level. The research is based on one lecturer’s teaching of a linear algebra module to a large group of first year undergraduate students at a UK university, a single case study. By teaching practice I mean all that the lecturer does in connection with his teaching of linear algebra: the planning, preparing and thinking about teaching in terms of structuring of material content of the module. I include the preparation of resources such as course notes for students, examples and exercises, etc. Teaching practice includes also the face-to-face teaching of students.

My aim is to capture some of the complexity of a mathematics teaching practice. I draw on Leontiev’s development of activity theory as an analytical tool as well as a theoretical tool in analysis and interpretation. My focus is the teaching of linear algebra based on a characterisation in terms of motive, actions and goals of activity. My general research questions are

What does it mean to teach mathematics at university?

What does it mean to teach linear algebra at university?

My more specific research questions are

1. What are the strategies used by the lecturer in his teaching of linear algebra?

2. How and why does the lecturer use these strategies? What are his intentions for student learning?
3. In using these strategies,

   (a) what are the implications for students’ learning of linear algebra, and
   (b) what are the implications for the teaching of linear algebra?

In this introduction I present an overview of my thesis. I describe the background of my study, in terms of the university where the research took place, the mathematician whose teaching practice I describe and the linear algebra module that the lecturer taught and that formed the basis of my research. I introduce the research team which comprised of the lecturer/mathematician who taught the module and two researchers, my supervisor and I.

The structure of the thesis

In this first chapter, Chapter 1, I introduce my study. This includes my research questions and the university and mathematics department where the research took place.

In Chapter 2 I review the literature that informed and guided me in my research. In particular, I discuss the research literature into the teaching and the learning of linear algebra.

In Chapter 3 I discuss my methodological perspectives as well as the specific methods that I used in pursuit of my research questions.

In Chapter 4 I discuss my theoretical perspectives. I adopted a socio-cultural framework and Leontiev’s development of activity theory in particular in my research.

Chapter 5 is the first of three analysis chapters. In Chapter 5 I use activity theory as an analytical tool to analyse the lecturer’s didactical thinking and decision making. This is an analysis of the meetings data and relates to the lecturer’s actions and goals.

In Chapter 6 I report on my analysis of three topics in linear algebra: subspaces, linear independence, and eigenvectors and eigenvalues. In my analysis and interpretation I drew on the lecturer’s comments in research meetings as well as the lecturer’s face-to-face teaching. I included an analysis of the course notes that the lecturer provided for students and used in his teaching. I analysed also how four textbook authors introduced the three topics in their writing. The analysis is comprehensive and as a result this chapter is very long.

In Chapter 7 I report on my analysis of the student questionnaires and the student focus group interviews. This is a relatively short chapter where I describe students’ experience of the teaching based on the responses that students gave.
In the last chapter, Chapter 8, I draw my analyses and observations to a conclusion. I discuss possible implications of an inductive approach to teaching linear algebra by drawing on students’ responses in interviews. I discuss what we have learnt about the teaching of linear algebra at university, issues for students and issues for a teacher of linear algebra. I highlight the value of my research in applying activity theory to the teaching practice of linear algebra at university. I discuss the value of theory and methodology in a qualitative study.

1.1 The University and the research team

The research was carried out at a UK medium sized university, with a strong tradition in engineering and design technology. Mathematics is taught by a department comprising of more than 50 staff.

The lecturer in my study was in his mid-thirties and at the start of the study recently newly appointed. He had started to teach the linear algebra module in the previous year and had experience of lecturing for one year in the US prior to arriving in the UK. He was enthusiastic about his subject and about his teaching. At the time when he agreed to take part in the research he was interested in taking an approach to his teaching of the module that was different from the one he had taken previously. As a result he had re-written the whole of the first semester of the linear algebra module. He had worked closely with the lecturer in the second semester, a colleague in the department who was also a young male mathematician. As well as teaching linear algebra the lecturer also taught a second module, “Communicating Mathematics”. This was mainly a project based module for second and third year students. In interviews and conversations the lecturer often referred to this module and to his intentions in respect of teaching and to his students’ reactions. Hence he often expressed his views in respect of teaching more generally and not just in relation to his teaching of linear algebra.

The study had been set up by my supervisor. I joined at a later stage at the start of my PhD. The three of us formed a research team where we, the researchers had responsibility for the research process and the lecturer for the teaching of the module. During the data collection period with the lecturer (which was in Semester 1 only) all three of us met regularly for discussions and conversations. It became clear early in the study that the lecturer had thought deeply about his teaching and had clear ideas and a vision of what he wanted to achieve.

Both researchers attended the lectures and tutorials in the first semester. I, as PhD student took responsibility for cataloguing audio-recordings and copies of field notes,
both my own and those written by my supervisor. In meetings we all contributed, both researchers asked questions or offered interpretations, the lecturer responded to questions and often offered thoughts and insights without prompting. Thus over the period of teaching in the first semester (eleven weeks) the research process was a collaborative one between the researchers and the lecturer. In meetings my supervisor had a more dominant role in asking questions since I, as a new student in mathematics education, felt unsure about the process of conducting interviews with the lecturer.

In the second semester I alone continued to attend the linear algebra lectures and tutorials which were taught by a different lecturer. I took sole charge of organising and conducting interviews with students (also in the second semester) and began the process of analysis of the audio-recordings of the meetings and the lectures. As a result I am solely responsible for the interpretations that I offer in this thesis, for adopting an activity theory perspective and for the theoretical model of the teaching process that I present in Chapter 5. I continued my analysis of the lecturer’s teaching practice in respect of three topics in linear algebra (presented in Chapter 6). All this analytical work is my own.

As a result of the collaborations between the researchers and the lecturer we gave seminar presentations and published papers. Jaworski, Treffert-Thomas and Bartsch (2009) was one such publication that I refer to in later chapters. It came about at an early stage of analysis. It was the result of analysis conducted by my supervisor independently of me but with input by me and the lecturer during finalisation. Apart from this collaboration all analytical work in this thesis has been my own.

1.2 The linear algebra module

Research was carried out with a lecturer teaching a large group of first year undergraduate students (> 200). Students were either single or joint honours students. During their first year of study students took several modules from a prescribed list of modules of which some were compulsory and some were optional, depending on the students’ degree registration. For all students registered as mathematics students on a joint or single honours degree, linear algebra and calculus were compulsory modules. All students registered for a single honours degree in mathematics also took a geometry module in the first semester. Because of this arrangement the lecturer in my study took the decision not to include any geometric reasoning in the linear algebra module.

Each student had a personal tutor who was responsible for the individual student’s general welfare and the first port of call in respect of any personal or academic difficulties.
The personal tutor met his or her tutees on a weekly basis in what was called the SGT, the small group tutorial, each with six to eight students. Students attended the SGTs in the first year of their undergraduate studies only. These sessions were timetabled as a mathematics tutorial and led by the personal tutor. Most members of staff in the mathematics department were personal tutors. They gave help and support in respect of students’ difficulties with first year mathematics modules and were responsible for marking coursework for the compulsory modules of linear algebra and calculus.

Students attending the linear algebra module were predominantly young (18 to 21 years old) and had completed ‘A’-level study (or equivalent) just prior to coming to university.

Linear algebra was a one year long module at the university. Research took place over twelve weeks with the lecturer who taught the first semester. The lecturer had re-designed the module using an ‘inductive’ approach based on examples. He developed this approach in response to a more proof-based linear algebra module that he had taught the previous year. In his observations of students in lectures and tutorials and of his personal tutees in SGTs, he gained the impression (re-counted in research meetings) that students had great difficulties with the abstract nature of linear algebra. In re-thinking his teaching he developed an approach whereby he introduced (an) example(s) in order to motivate a definition or a theorem in linear algebra. He based his idea of using examples on an article by Frank Uhlig published in a mathematics research journal. Frank Uhlig, a research mathematician and teacher of linear algebra in the US, presented an account of his own teaching based on introducing an example and asking questions. The lecturer in my study had accessed this literature prior to any discussion about taking part in our research. This I took as an indication of the lecturer’s commitment to his teaching duties which also showed in the often very engaging conversations that we, the researchers, had with the lecturer in meetings. Discussions in meetings centred on mathematics (the lecturer frequently explained mathematical ideas and concepts to us, the researchers), on the teaching and learning of mathematics and on his personal experiences as a student and as a teacher and researcher. He often shared ideas about teaching with friends and colleagues in the department, most usually informally over lunch, as well as with one of his parents who was a teacher. I believe the lecturer was well-liked in the department and not representative of what one might associate with a ‘typical’ lecturer of mathematics at a UK university. His interest in education research was genuine which showed in his continual development of his teaching and his teaching resources after the research had ceased.
Chapter 2

Literature Review

In this chapter I present a review of the literature that informed my study. The literature into the teaching and learning of linear algebra was particularly relevant to my study as was the literature into the teaching of mathematics at tertiary level. A search of the first revealed literature dating back more than twenty years. Publications in both the research and professional literature dealt with curriculum development and reform, research into teaching and teaching practice, accounts of teachers’ reflections on their own practice and research into students’ difficulties with linear algebra. Research into teaching practice, that included face-to-face teaching of students, was relatively scarce. In relation to mathematics teaching at tertiary level I reviewed literature that was concerned with approaches to mathematics teaching as well as teaching issues more generally, in particular the role of the lecture and of lecturing, all in the context of mathematics teaching and lecturing.

Before I go further I need to clarify what I mean by research literature and professional literature. “Research is systematic enquiry made public” as Stenhouse (1984) wrote. Researching the teaching or learning of linear algebra involves the use of methodology and theory in the pursuit of rigorous inquiry and leads to research publications. There are also publications, often in relation to teaching, that are the result of an individual teacher’s reflections on his or her own practice. These often take the form of accounts of good or innovative practice and are part of what I call the professional literature. Publications in the professional literature are often not peer reviewed and not based in research activity that involves for example, systematic data collection and analysis.

I begin with a review of the literature in relation to linear algebra teaching and learning.
2.1 The teaching and learning of linear algebra: A historical note

In the US, in the 1960s and 70s the School Mathematics Study Group (SMSG), of which Guershon Harel was a member, considered curricular issues in the light of the reforms of the primary and secondary mathematics curriculum. They also considered the study of selected linear algebra topics by high school students. As part of their work the SMSG published numerous reports and materials for school use.

In January 1990, the Linear Algebra Curriculum Study Group (LACSG) was formed in response to a perceived decline in students’ ability to cope with the linear algebra curriculum in first year university courses. David Carlson was a member of the LACSG as were Charles Johnson, David Lay (who published an acclaimed linear algebra textbook) and A. Duane Porter, as well as Guershon Harel as member of a workshop panel. All were research mathematicians.

The task of the LACSG was to improve the undergraduate linear algebra curriculum (see Carlson, Johnson, Lay and Porter, 1993). They considered both curricular and teaching reforms, and published a list of recommendations as a result of their review which included a ‘core syllabus’ for teaching linear algebra. The LACSG welcomed responses to their proposals in order to encourage debate and further improvements. The influence of the LACSG on the development of, and subsequent research into the teaching and learning of linear algebra can be seen by the many references that were, and are still made to the group’s work (see, for example, Dorier and Sierpinska, 2001; Uhlig, 2003). Although I expected the list of topics that formed the ‘core syllabus’ to be dated it appeared in almost unaltered form as the content page in many introductory textbooks of linear algebra since. In addition, the core syllabus as formulated by the LACSG in 1993 contained the same list of concepts that the lecturer in my study highlighted as fundamental and most important.

Research into the teaching and learning of linear algebra has continued in the US, UK and France, in particular. In 2000 Jean-Luc Dorier published a textbook “On the teaching of linear algebra” (Dorier, 2000). This contained contributions by researchers active in the area of the teaching and learning of linear algebra. In 2001, the authors Jean-Luc Dorier and Anna Sierpinska published a joint article in D. Holton’s book “The teaching and learning of mathematics at university level” (Holton, 2001). Dorier and Sierpinska summarised the research in the area of the teaching and learning of linear algebra and included a detailed description and analysis of epistemologically based research. Their work is indicative of a change in focus to more didactical considerations.
In 2003 Frank Uhlig, a research mathematician, published his reflections on the teaching and learning of linear algebra (Uhlig, 2003). He made references to the work of the LACSG and to Jean-Luc Dorier and his colleagues. Frank Uhlig combined curricular and didactical considerations into a single framework for teaching linear algebra to first year undergraduates. However, his account was not research based.

### 2.2 Students’ difficulties with linear algebra

My review of the literature into students’ difficulties with linear algebra revealed publications dating back to at least 1989. Most authors appeared to agree that students perceived linear algebra to be a difficult subject. They also seem to agree on the nature and sources of these difficulties. The three most commonly quoted areas of students’ difficulties were: the overwhelming number of new concepts and definitions in a first year linear algebra course, the high level of abstraction required in mastering the linear algebra concepts, and the lack of connection with students’ prior knowledge and experience, particularly with mathematics at school level.

Harel (1989) represents an early publication that highlighted students’ difficulties with the abstract nature of linear algebra. Carlson (1993) referred to “Must the fog always roll in?” (p. 29) while Hillel and Sierpinska (1994) referred to “one persistent mistake in linear algebra” (p. 65) and Dorier, Robert, Robinet and Rogalski (2000b) wrote that “many students have the feeling of landing on a new planet and are not able to find their way in this new world” (p. 28). Dorier and Sierpinska (2001) gave a useful review of the research up to that date, and the book by Jean-Luc Dorier “On the Teaching of Linear Algebra” provided a comprehensive study of students’ difficulties (Dorier, 2000).

Students have particular difficulties with the highly conceptual nature of linear algebra and the many definitions that involve formal mathematical language. Rogalski (1996) cited students struggling with logic and set theory, particularly in relation to intersections of subspaces. Dubinsky (1997) highlighted students’ difficulties with quantifiers. Hillel (2000) referred to the different representations and modes of description in linear algebra. Dorier and Sierpenska (2001) highlighted the fact that teachers switched between different representations without alerting their students when they did, leading to student confusion and misinterpretations. Artigue (2001) made connections between students’ understanding in calculus and linear algebra and gave an overview of students’ cognitive difficulties. By far the most detailed analysis of students’ difficulties was presented by Dorier (2000). In his book he included a historical analysis of the development of linear algebra and the fundamental problems that mathematicians encountered in creating linear algebra as a unifying theory. Unlike other areas of mathematics, linear
algebra did not develop as a result of solving a mathematical ‘problem’. Linear algebra came about as a theory that unified the tools and methods that mathematicians used in solving problems; it became a framework that simplified dealing with problems stemming from linearity. Dorier et al. (2000b) wrote, that

\[ \ldots \text{the simplification is only visible to the specialist who can anticipate the advantage of generalization because they already know many contexts in which the new theory can be used. For the beginner, on the other hand, the simplification is not so clear} \ldots (\text{Dorier et al., 2000b, p. 28}). \]

Thus students encountered a theory without knowing of its value or power as a unifying method. Many problems that students encountered in an introductory linear algebra module such as solving systems of equations, students could solve with knowledge that they had acquired prior to coming to university, so that students failed to see a need in learning a new and cognitively demanding theory. In his work Dorier analysed in detail students’ difficulties with the linear algebra, for example with the concepts of vector spaces and subspaces, rank, bases and linear independence. In a recent review Britton and Henderson (2009) described students’ difficulties with the concept of closure and with functions as elements of a vector space.

In addition there was recent research by Stewart and Thomas (2007, 2009, 2010). They considered the role of geometry in teaching linear algebra in order to help students overcome their difficulties with linear algebra. They used APOS theory (Dubinsky, 1997) as theoretical perspective and in relation to analysis and interpretation of their data. While I acknowledge their contribution towards furthering our understanding of students’ difficulties, their research is not so relevant to my study. This is because the lecturer in my study did not use any geometry in his teaching. Nevertheless, I will refer to the work by Sepideh Stewart and Michael Thomas from time to time in my thesis.

These studies all highlighted that students struggled with the conceptual nature of linear algebra. All authors seem to agree that students do have great difficulties with linear algebra when they first encounter linear algebra at university while at the same time acknowledging that conceptual understanding is necessary to advance to the level of mathematical abstraction required at university. However, there appears to be less agreement on the strategies that could be used to alleviate and overcome these difficulties.
2.3 Models for teaching linear algebra

Holton (2001) provided an overview of the teaching and learning of mathematics at university level. Derek Holton as the editor was reporting from an ICMI study into the teaching and learning of advanced mathematics. His book included an article by Jean-Luc Dorier and Anna Sierpinska on the teaching and learning of linear algebra, and by Joel Hillel on the use of computer algebra systems in the teaching and learning of linear algebra. Artigue, Batanero and Kent (2007) gave an overview of research into mathematical thinking and learning at university level. This included a discussion of David Tall’s work on ‘advanced mathematical thinking’ (Tall, 1991, 2004) as well as the work by Dorier et al. in what they called the ‘obstacle of formalism’ in the teaching and learning of linear algebra (Dorier et al., 2000b).

Some research studies have made suggestions as to what could be included in the teaching of linear algebra in order to ‘make it better’, for example use of technology (Berry et al., 2008; Sierpinska, 2005), use of geometry (Gueudet-Chartier, 2004; Sierpinska, 2005; Stewart and Thomas, 2009) or journal writing (Hamdan, 2005). However, there has been very little research into the practice of teaching linear algebra. One example of such research was carried out by Jean-Luc Dorier and his colleagues in France. Not only did they chart the progress of students, which led to important insights into students difficulties with the subject, they also carried out teaching experiments lasting several years. Findings included a discussion of a ‘change of contract’, and the use of a ‘meta-lever’ (Dorier, Robert, Robinet and Rogalski, 2000a) in helping students make the transition to conceptual thinking which I discuss below.

Apart from this research there were contributions to the professional literature. Carlson (1993) and Carlson et al. (1993) were among the first to discuss curricular issues surrounding the study of linear algebra at university, which culminated in the publication of a list of topics to be taught. More recently the work by Frank Uhlig (2002, 2003) centred on a description of teaching linear algebra based on a matrix approach. One difficulty with professional literature was the choice of language in describing findings which often took little account of the terminology in use in the educational research literature. As a result, some of the terminology used in the professional literature was at times confusing.

I discuss here two areas of research into the teaching of linear algebra that have influenced my thinking: the work by Jean-Luc Dorier and his colleagues, and the work by Frank Uhlig. I also acknowledge recent work by Sepideh Stewart and Michael Thomas (2009) into the teaching of linear algebra with geometry. Since the lecturer in my study did
not use any geometry or geometric insights in his teaching, this work is less relevant to my study.

2.3.1 Change of contract and the meta-lever

The contributions of Jean-Luc Dorier and his colleagues in research in the area of teaching and learning linear algebra were extensive. As early as 1990 they raised important issues in relation to students’ (mis)understandings in linear algebra, and addressed the question of how to improve the teaching of linear algebra.

Dorier and his colleagues identified ‘one massive obstacle’ that students seemed to have with linear algebra, which they called the ‘obstacle of formalism’. In brief, students were unable to abstract the formal theory from examples and problems posed because they could not recognise linear algebra as a unifying theory, that is a theory for all problems stemming from linearity. In students’ experience all problems that they solved in a first year linear algebra course could be solved without the need of formal theory. For example, solving linear equation systems did not require students to know matrix theory or Gaussian elimination. As a consequence Dorier and his colleagues suggested making a better connection between the elements of the formal theory and students’ prior knowledge and experience. To do this they introduced the notion of the ‘meta-lever’.

The meta-lever represented the use of information ABOUT mathematics. It denoted a ‘meta’ intervention by the teacher (often orally, and always deliberate) that would lead students to (actively) reflect on a problem posed, or a new concept. Students needed to engage with the activity, and not merely ‘carry out’ the activity. Thus students were asked, for example, to derive the axioms of a vector space for themselves. Using a meta-lever in these teaching situations required students to take a change of point of view. For example, students needed to be able to connect matrices to linear equation systems on the one hand, and to vectors on the other. Students needed to appreciate the change in the didactical contract. They needed to ‘see a point’ for abstracting and formalising. Hence any teaching sequence that Dorier and his colleagues built for students included the devolution of learning to the student. They generally acknowledged that a change in didactic contract took students time to adjust to. Certainly, this was an issue that Dorier and his colleagues recognised, and as result designed long term teaching experiments (with first year undergraduates) lasting several years.

The crucial point was the use of a ‘meta’-level activity or instruction that promoted students to ‘get into a reflective attitude’. These activities were introduced by the teacher and required a constant underlying questioning of new possibilities by the student.
2.3.2 A matrix-based approach and how, and what questions to ask

Uhlig (2003) writes about his approach to teaching linear algebra based on making use of the “inner workings and logic of linear algebra” (p. 147), and the “forces from within” (p. 151), to lead students to conceptual understanding. He suggested working with conceptual examples (and not concrete examples as suggested by the LACSG in 1993, see Carlson et al., 1993) and intuitive exploration in order to abstract the formal theory. Part of the exploration consisted of intuitive mathematical reasoning. This was promoted by the design of a sequence of questions to the students, to which Uhlig referred as the **WWHWT sequence**, namely,

- **What happens if?**
- **Why does it happen?**
- **How do different cases occur?**
- **What is true here?** (Uhlig, 2002, p. 338; underlined in original)

Then, based on the exploration of these questions students gained knowledge that could be assembled into “Theorems” (ibid, p. 338; underlined in original).

Uhlig stated that any topic in linear algebra was a suitable starting point but once decided, the order of topics was imposed by the logic of linear algebra. All starting points led to linear equation systems and Gaussian elimination techniques in a relatively short time. In Uhlig’s view this approach will ease students more gently into the formal ‘DTP’ (definition-theorem-proof) format of linear algebra as the former entailed no proofs but relied on “using salient point type arguments” (ibid, p. 338) as part of the exploration.

Jean-Luc Dorier and his colleagues commented on Frank Uhlig’s exposition in the research literature (Dorier, Robert and Rogalski, 2002). Although they agreed with Uhlig that students needed a good technical level of solving linear equation systems, they were critical of Uhlig’s extensive use of ‘REF’. The abbreviation ‘REF’ stood for ‘row echelon form’ which I viewed as equivalent to the method of Gaussian elimination. Dorier et al. stated that over-reliance on REF (and on pivot counting) was hiding important ideas in linear algebra. Whereas Uhlig claimed that it provided students with an intuitive basis (exemplified, for example, by checking for linear independence using pivot counting) Dorier et al. stated that it provided a technical basis (and not a conceptual basis).

Dorier et al. (2002) were not entirely critical of Uhlig’s approach. They compared Uhlig’s questioning that aimed at making students “step-aside-and-reflect” (ibid, p. 187), to their own use of the meta-lever (ibid, p. 188).
2.3.3 Geometry: ‘Yes’ or ‘No’

In this section I review research into the teaching of linear algebra using geometric reasoning or insights, and state my reasons for not including these in my considerations for analysis.

The role of geometry in the teaching and learning of linear algebra was discussed by Sierpinska (2005), Dorier and Sierpinska (2001), Gueudet-Chartier (2004), Berry et al. (2008) to name a few. Hillel (2000) conducted research which showed that the geometric language of linear algebra can lead to difficulties if students take it too literally. Gueudet-Chartier (2004) concluded that the use of geometry both helped and hindered students’ understanding while Sierpinska (2005) wrote that “the gap between successful and less successful students widened instead of shrinking” (p. 1). It appeared that weaker students were unable to make the transition from $\mathbb{R}^3$ to more abstract vector spaces whereas stronger students’ understanding deepened when geometric insights were provided. Gueudet-Chartier (2004) concluded that geometry “must be used very carefully in linear algebra courses” (p. 500) while Dorier and Sierpinska (2001) wrote that linear algebra concepts should not be built “as a generalization from geometry” (p. 271). Stewart and Thomas (2009) found that students benefited greatly from teaching that involved geometrical insights. The students involved in the study were (general) mathematics students. The context of the article seemed to suggest the study of linear algebra up to $\mathbb{R}^3$. It was unclear as to whether students were required to go beyond. This was a crucial question in relating the findings by Stewart and Thomas to previous research. In a private communication with Michael Thomas I have since learnt that students were not required to go beyond $\mathbb{R}^3$ in their study of vector spaces.

Research by Sinclair and Tabaghi (2009) focussed on mathematicians’ dynamical thinking. In an interview study they examined mathematicians’ gestures in explaining mathematical concepts (including the concept of eigenvectors). Their research showed that mathematicians had a very strong visual representation of mathematical concepts which I see as supporting the value of geometry in relation to linear algebra.

In my study I did not consider research into teaching that included the use of geometry since the lecturer in my study had not included any geometry in his teaching. The decision was one based on the overall design of the mathematics first year study programme at the university where my research took place. This meant that half of the students who took the linear algebra module also took a geometry module in the same semester. So while I acknowledge the ongoing research into the use of geometry (and visualisation) in teaching and learning linear algebra (and of interest in furthering our understanding of the issues these pose) I made the decision not to consider it further in my analysis.
2.4 Research into teaching practice

There was little research into teaching, that is face-to-face teaching of students in the context of linear algebra, or mathematics in general, at university level.

There was very little research that addressed the teaching practice of mathematics at university level. Speer, Smith III and Horvath (2010) defined teaching practice as what “teachers do and think daily, in class and out, as they perform their teaching work” (p. 1). It involved researching both the teacher’s theoretical point of view, his thinking and decision-making processes, and his actual teaching of students in the lecture hall. It was an (almost) un-examined area as Speer et al. (2010) wrote, and “virtually non-existent” (p. 1). It required “researchers to move into the classrooms and offices of collegiate teachers in order to collect data that can support analyses of practice” (p. 13). Thus research into the teaching practice of lecturers of mathematics involves agreement and collaboration with mathematicians. My study, at least partially, is aiming to fill this gap, in respect of linear algebra. Petropoulou, Potari and Zachariades (2011) is a study into the teaching practice of calculus which shares methodological perspectives and research aims with my study, namely a study of the lecturer’s teaching decisions and actions.

Research into teaching practice: With the exception of the work already mentioned, e.g. the teaching experiments in France (Rogalski, 2000), a search of the literature revealed very little collaborative ‘classroom’ research between mathematicians and mathematics educators. Barnard and Morgan (1996) was an example as was the work of Stewart and Thomas (2009). While Barnard and Morgan (1996) conducted research that involved just one lecture, it was research into the teaching of mathematics at university and involved a mathematician (Barnard) working with an educator (Morgan). Stewart and Thomas (2009) as mentioned already, were involved in creating more collaborative research studies at university level, and in the subject area of linear algebra. Weber (2004) worked with a mathematician by interviewing him and observing his teaching of an introductory analysis course (to a relatively small class of (16) students). Nardi, Jaworski and Hegedus (2005) worked with mathematics tutors teaching small groups of students within a university tutorial system.

At university level research with mathematicians has involved mainly interview studies (Burton, 2004; Hemmi, 2010). Here the focus was almost exclusively on mathematicians’ beliefs and intentions. Leone Burton’s interview study with 70 mathematicians provided valuable insights into research mathematicians’ thinking about mathematics, research, mathematics teaching and students’ learning of mathematics. Kirsti Hemmi (2010) explored mathematicians’ pedagogical convictions in relation to research into proof.
Little research existed that explored the relationship of mathematicians’ beliefs and intentions and their face-to-face teaching.

I defined teaching practice following Speer et al. (2010). In my research I defined teaching practice as all those activities and decisions the lecturer made as he prepared for and presented a lecture. This included planning and writing the course materials, thinking ahead, thinking through a lecture, choosing examples to present to students to work on either on a problem sheet or in lectures and tutorials, writing examinations, assessing students including marking examinations and coursework. Teaching practice related to the work of the lecturer as teacher, that is in the context of teaching, and not in the context of mathematical research work.

2.5 The role of the lecture and of lecturing

At university the lecture has remained as the primary medium of instruction. Research into the role of the lecture in the context of mathematics teaching was scarce, but there were many contributions to the professional literature, in particular mathematicians reflecting on their use of, and on the usefulness of the lecture (Millett, 2001; Pritchard, 2010; Wu, 1999). David Pritchard defined a lecture as one delivered to a class of more than 20, but in practice often more than 100 students, in which “the dominant direction of communication is from the lecturer to the student” (Pritchard, 2010, p. 2).

The role of the lecture was to encourage mathematical thinking (Holton, 1998; Mason, 2002), to engage (Millett, 2001), to enthuse (Pritchard, 2010), and to have students gain insight into mathematics (Wu, 1999) and/or the mathematicians’ world (Pritchard, 2010). These aspects were represented in what the lecturer in my study said in research meetings, and which had led him to re-design the first-year linear algebra course. A discussion about wanting to “make students think” led to considerations of conceptual versus procedural ways of learning, and to the cultural aspects present in the teaching of mathematics (for example, Holton, 1998).

The professional literature into the teaching of mathematics through the medium of the lecture is split between proponents of the traditional lecture (Wu, 1999), proponents with a desire to make changes to the lecture format or lecturing style (Krantz, 1993; Millett, 2001; Pritchard, 2010) and opponents (Uhl, 1999) who are in favour of abandoning the lecture format.

Broadly speaking, the discussions on how to teach mathematics at university could be separated into (more) teacher-centred and (more) student-centred approaches (see Sullivan, 2008, for a definition of teacher-centred and student-centred lessons). Whereas
Pritchard (2010) talked about students “standing in awe” which resonated with treating students as an audience in the lecture, Uhl (1999) took the opposite view. Adopting a student-centred approach to teaching he abandoned the lecture altogether in favour of lab-type sessions and drop-ins. Uhl’s main concern was students’ dissatisfaction with mathematics which he did not relate to the format of the instruction. The problem, as he saw it lay in communication. As a consequence he changed his style of interaction with students, adopting an essentially more personal contact. He talked about students not forming a passive audience but becoming the professor’s apprentices. I interpreted this as a more student-centred approach to teaching which emphasised students’ own generation of strategies and discussion.

Teaching styles

Research into lecturing and the role of the lecture often included analyses of teaching approaches to the teaching of mathematics. For example, Weber (2004) analysed the ‘DTP’ (definition-theorem-proof) format in working collaboratively with a mathematician. Alsina (2001) gave an overview of differing styles including the ‘bottom-up’ approach, the ‘deductive organisation’ and the ‘inductive style’. Barnard and Morgan (1996) identified a sequence of teaching acts which consisted of ‘computational-through-descriptive-to-deductive’ modes of reasoning for students.

Mason (2002) described the actions that defined lecturing based on a working group report (Mason, 2001). From seven case studies Mason drew out the principles and assumptions underlying differing teaching acts, and based on this analysis defined teaching. Teaching acts when viewed as an entity formed a teaching approach.

Underlying a ‘bottom-up’ approach are a number of intentions and strategies in relation to the structuring of content and of teaching. This approach included what some researchers described as informal, intuitive and/or inductive reasoning as a tool in bringing about student learning. (See Hemmi, 2010, for a description of formal versus intuitive approaches, drawing on Chin and Tall, 2000; Fischbein, 1987; Moore, 1994).

I define a ‘bottom-up’ approach as any approach that is contrary to a more traditional ‘top-down’ approach, often referred to as a DTP (definition-theorem-proof) or DLPTPC (definition-lemma-proof-theorem-proof-corollary) style of teaching. The term ‘bottom-up’ (approach) was used by the lecturer in my study in his own interpretation of his teaching style. As part of his ‘bottom-up’ design the lecturer included elements of explorative thinking and working, based on examples that could lead to an informal understanding or appreciation of the linear algebra concepts being studied. The course
notes for students and the structuring and format of the lecture were designed to support this approach.

Thus identifying the lecturer’s inductive approach as a core strategy led me to consider the role of examples more generally in structuring teaching and learning. Research in this area was extensive, including research carried out at university level. For example, Mason and Pimm (1984) categorised examples as specific, particular, generic and general. As an example from the professional literature, Uhlig (2003) used the terms ‘concrete’ and ‘conceptual’ to describe types of examples. However, in defining these terms the author, and other authors writing similar accounts of their own teaching in the professional literature, may be unaware of the educational research literature’s use of these terms. In presenting an example to introduce linear algebra concepts in lectures, the lecturer in my study stated explicitly that he meant the example that he had chosen to be representative of a wide class of examples. Research with students in this area suggested that students saw examples as merely illustrative and specific (see for example Bills, Dreyfus, Mason, Tsamir, Watson and Zaslavsky, 2006).

The lecturer’s intention in using informal reasoning while working within an examples-based approach was to lead students first to an intuitive understanding of the fundamental concepts of linear algebra, and then to a more conceptual understanding. Whenever concepts are referred to I used as my point of reference the recommendations of the Linear Algebra Curriculum Study Group (LACSG, Carlson et al., 1993). I follow Harel (1997) in defining conceptual understanding “as knowing both what to do and why” (p. 109). Harel’s definition relates conceptual understanding to the ability to solve problems which resonates with statements made by the lecturer in research meetings.

In my analysis I focussed on the lecturer’s strategies and intentions in teaching linear algebra, and which related to data obtained in interviews and in observations of lectures.

All the areas of literature that I have mentioned contributed to my developing an understanding of the issues in relation to the teaching and learning of linear algebra. As I will show in the next chapter I came to my research from a grounded approach to data collection and analysis. However, all that I had read and continued to read continually informed and developed my understanding and knowledge of the field as well as my methods and my methodological understandings. This generated insights that ultimately contributed to my findings and the conclusion of my study.
Chapter 3

Methodology

In this chapter I discuss the methodological choices that I have made in conducting my research. Methodology is more than a description of the methods that I have applied in my study. It concerns itself with the underlying reasons, the values and the beliefs that I, the researcher, have brought to the research process. As Leone Burton wrote, in distinguishing methodology from methods,

...out of all the methods that could have been used, what influenced the researcher to choose to do the research in the manner described. (Burton, 2002, p. 1)

Hence, in this part of my thesis I give details of how I came to use certain research methods as well as describing the methods themselves. In particular, I discuss how my theoretical and methodological perspectives became interlinked during data analysis. This led to a refinement of my research questions, to further analysis, and to the development of the research process as a whole.

3.1 The development of the research questions

The purpose of my study was to explore the teaching of mathematics (and of linear algebra in particular) at the university level of education. I aimed to gain a better understanding of the issues and concerns that revolve around the teaching (and learning) of mathematics at university. To this purpose I had formulated the general research question(s):

- What does it mean to teach mathematics at university?
What does it mean to teach linear algebra at university?

I proceeded to collect data while working closely with my supervisor. Based on the general research questions I also began to analyse data, initially using a grounded approach. As a novice researcher I was also working largely inductively, that is, I was conducting fieldwork and analysis before recognising the knowledge, beliefs and assumptions that I was bringing to the research process. In particular, I did not apply a theoretical framework until fairly late in the research process, and until all data had been collected and analysis was well under way. By adopting an activity theory perspective, and by applying Leontiev’s structural elements of activity theory to my data, my analysis developed leading to new insights.

As a result I refined my research questions. The general research question remained as:

What does it mean to teach linear algebra at university?

The more focussed and specific research questions were as follows:

1. What are the strategies used by the lecturer in his teaching of linear algebra?
2. How and why does the lecturer use these strategies? What are his intentions for student learning?
3. In using these strategies,
   (a) what are the implications for students’ learning of linear algebra, and
   (b) what are the implications for the teaching of linear algebra?

Adopting activity theory and Leontiev’s structural elements of activity theory, in particular, organised my data and invigorated my data analysis. Thus activity theory was an analytical tool in my research (Simon, 2009). The three hierarchical levels of activity-motive, actions-goals and operations-conditions were suitable for ‘framing’, and explaining the lecturer’s teaching practice. My methodological choice of using activity theory as an analytical tool arose out of my data analysis, that is my observations and interpretations of the data sources. I give a rigorous account of this analysis in Chapter 5.

I used activity theory as a theoretical lens (Simon, 2009) in furthering the research process. I analysed the teaching of three particular mathematical concepts, namely subspaces, linear independence, and eigenvalues and eigenvectors (see Chapter 6). Findings from this analysis were used to answer the research question 3 above, and contributed to my synthesis in Chapter 8.
3.2 The design of my study

A research question drives the research and the inquiry process (Bassey, 1999). A research design, on the other hand, is a plan for getting from here to there, from a set of initial questions to a set of conclusions (Yin, 2003).

The purpose of conducting my research was to explore and to understand, and, following analyses, to be able to explain and, if applicable, raise issues in relation to the teaching (and learning) of linear algebra. To pursue my research I had formulated research questions that were of an explorative nature. To gain understandings of issues I, in collaboration with my supervisor, worked closely with a lecturer of mathematics, gathering data in interviews and in conversations with the lecturer and observing his teaching. The context and natural setting of the teaching and learning environment were important as well as maintaining a prolonged contact with the lecturer.

Thus my methodology could be described in three ways. Firstly, my research was a case study as I studied a single case, that is, one lecturer’s teaching of linear algebra to first year undergraduates. Secondly, my research was a naturalistic inquiry. In the tradition of a naturalistic inquiry, researchers are ‘doing what comes naturally’ (Lincoln and Guba, 1985, p. 187). To explore the lecturer’s didactical thinking involved interviewing as a means to gain ‘access to his thinking’ by my interpreting of what he said. To explore the lecturer’s face-to-face teaching involved entering and observing teaching in its natural setting or environment, that is the university mathematics lecture.

Thirdly, my study was ethnographic in style. In an ethnographic study researchers seek to gain understandings “by entering into close and relatively prolonged interaction with people (…) in their everyday lives, …” (Hammersley, 1992). Methods commonly used in ethnographic studies are interviewing and participant observation (Eisenhart, 1988; Tedlock, 2000). As I said above we, the researchers, worked closely with the lecturer and had prolonged contact over one whole academic semester, a twelve week period. We used interviews and lecture observations as our main methods of data collection.

3.3 The research paradigm

To specify my research paradigm means to align myself with a community of researchers who share a common attitude towards research and the kind of knowledge that the research produces (see Carr and Kemmis, 1986, p. 72-75). A research paradigm also represents common assumptions that researchers hold in respect of the nature of the world that they investigate (see Romberg, 1992, p. 54).
I situated myself in an **interpretive paradigm**. An interpretive paradigm is compatible with an exploratory case study design and ethnographic and naturalistic research methodologies. As with all paradigms the interpretive paradigm has a set of underlying assumptions concerning the **nature** of knowledge (ontology) and the **origin** of knowledge (epistemology). Denzin (1978) described the components of an interpretive paradigm as follows:

The social world of human beings is not made up of objects that have intrinsic meaning. The meaning of objects lies in the actions that human beings take towards them. ...Social reality as it is sensed, known, and understood is a social production. Interacting individuals produce and define their own definitions of situations [and] the process of defining situations is everchanging. ...Second, humans are ... capable of ... shaping and guiding their own behaviour and that of others [intentionally and unintentionally, and] humans learn ... the definitions they attach to social objects through interactions with others. (Denzin, 1978, p. 7, cited in Eisenhart, 1988)

Thus in the interpretive paradigm the meaning that humans assign to their actions is the focus of research. Human actions are seen as social in origin and changing, as the context in which an action is carried out changes. In interactions meanings are negotiated, meanings that are liable to change and develop as the social context in which the interactions took place change and develop.

*Making interpretations* of the actions of others (and one’s own) is at the heart of research within an interpretive paradigm. This raises the issue as Smetherham (1978) wrote, “that any particular set of meaning structures ... will not necessarily be symmetrically shared either as between the researcher and respondent, nor yet between their words and actions” (*ibid.*, p. 98).

As an interpretive researcher my aim was to *explore* and to *observe*, in order to *explain* and to *understand*. I was not motivated by a search for a “truth” that could be applied to other cases or situations. The notion of “truth” implies an *understanding* that reflects an account of a ‘single’ reality independent of the interpretations of the researcher or the participants in the study. In an interpretive paradigm multiple realities are taken to exist since the very basis of interpretive research is to explore the meanings that individuals assign to their actions. The context is implicated in the interpretations that are offered by the participant, that is in my study by the lecturer of linear algebra, as well as the interpretations that I, the researcher, made in relation to what I observed, recorded, ‘heard’ and analysed. Hence social aspects and context are contributory factors in the interpretations that both the lecturer and I made. This aspect is reflected in the notion
of **intersubjectivity** (see Eisenhart, 1988). Intersubjective meanings are implicit in the practices we engage in. We can come to understand each other's interpretations only if there is a sense of shared practice, a consensus, and an acceptance of rules of engagement, for example. As Jaworski (1997) wrote,

> The term intersubjectivity has been used ... to capture a sense of common knowledge arising from group negotiation in sharing and comparing interpretations. Shared meanings ... [are] central to an epistemology of the social construction of ... knowledge. (Jaworski, 1997, p. 118)

In my study we, the researchers and the lecturer of linear algebra, engaged in negotiating meanings arising from conversations in research meetings and observations in lectures. The shared practice may be regarded as the mathematics lecture embedded in the culture of a university mathematics department to which the researchers, the lecturer and the students belonged.

Because, as Eisenhart (1988) wrote, intersubjective meanings are implicit, "the ways in which beliefs and actions make sense may only be accessible to insiders" (ibid., p. 103). Engaging in meaning-making and sharing practices implied that I, the researcher, could not stay outside the research process, and take the role of neutral observer. I was involved in communication and interactions with the lecturer, making sense of actions and hence party to the construction of knowledge. Maxwell wrote,

> As observers and interpreters of the world, we are inextricably part of it; we cannot step outside our own experience to obtain some observer-independent account of what we experience. (Maxwell, 1992, p. 283)

Contexts change as a result of interactions, and so do meanings. Hence within an interpretive paradigm a study cannot be repeated and the same result obtained twice. The notion of replicability as a means of validation does not apply. That results cannot be generalised or replicated is one of the main criticisms of research within this paradigm. A second criticism relates to the assertion that results based on subjective interpretations are 'no more than opinion'. These are criticisms levied at those engaged in exploratory case studies, ethnographic and naturalistic inquiries, and qualitative research in general.

However, the question of validity is a ‘valid’ one. As Stenhouse (1984) wrote,

> I define research as systematic enquiry made public. (Stenhouse, 1984, p. 77)
That is, research has to be made available, and accessible, in order to be approved by the ‘critical community’ (i.e. other academics in the field) and/or confirmed by the participants in the research (Carr and Kemmis, 1986, p. 91).

Lincoln and Guba (1985) argued for alternative criteria to address the ‘validity’ in qualitative research, criteria that could take account of the nature of a qualitative study. Bryman (2004, p. 273) cites Lincoln and Guba (1985) and Guba and Lincoln (1994) as proposing the criteria of trustworthiness and authenticity in place of generalisability and validity (where generalisability and validity are criteria that are commonly applied in empirical studies).

**Trustworthiness** refers to providing ‘enough evidence’ for the reader to have ‘trust’ or ‘confidence’ in the inquiry, its methods, interpretations and findings. It requires the researcher to provide a detailed account of all sources that informed on analyses and interpretations, any limitations or difficulties experienced in data collection and analysis, say. The aim is for the reader to be able to build a complete picture of the research process, and to judge and ‘trust’ its findings. In respect of trustworthiness Lincoln and Guba (1985) proposed four elements that make up the criteria of trustworthiness: credibility, dependability, confirmability and transfer.

**Authenticity** refers to authentic (i.e. truthful) and rigorous inquiry which includes the notion of fairness whereby all contextual factors and all views expressed by participants should be made apparent. Authenticity also refers to the wider impact of research in relation to findings, its ‘merit’ or “truth value” (see Lincoln and Guba, 1985; Miles and Huberman, 1994). Since my study includes descriptive elements, others need to be sure that the account I give is authentic, and includes all aspects of the context that framed the teaching practice. Questions to be answered relate to, ‘Do findings make sense and give a fair and accurate (in the sense of truthful and authentic) account of the overall study?’

Next I present my methods and methodological choices in respect of data collection and analysis. I present a detailed account with the aim of increasing confidence in my data collection and analysis. This is not to say that another researcher would have done the same as me or arrived at the same conclusions that I have, for the other person would have come with different perspectives, different questions and made different interpretations.
3.4 The methods of data collection

I collected mainly qualitative data. This consisted of audio-recordings of interviews (with the lecturer and with students), audio-recordings of lectures, and the taking of field notes (during interviewing and while attending lectures). In addition I administered two questionnaires to students which resulted in both quantitative and qualitative data.

Data was collected over twelve weeks, in the first semester of a 1-year Linear Algebra course, and consisted of:

- **Audio-recordings of meetings with the lecturer (meetings data):** Interviews with the lecturer were semi-structured, lasted between half an hour to one hour and took place in the lecturer’s office or over lunch at one of the university’s catering outlets.

- **Audio-recordings of lectures and tutorials (lecture observation data):** All lectures and tutorials (each fifty minutes long) held by the lecturer in Semester 1 (the length of the study) were audio-recorded. In addition, the researchers took field notes (in long hand) when attending the lectures or tutorials.

- **Two surveys of students’ views (student survey data):** Questionnaires were handed out to all students attending Lecture 11 and Lecture 30, held in Weeks 4 and 11 of Semester 1. 172 and 107 students, respectively, completed the questionnaires during normal lecture time (taking ten minutes that the lecturer had set aside at the end of each lecture). The majority of questions were in closed form for ease of analysing (using a computer software package). Some questions were stated in open form.

- **Audio-recordings of student focus group interviews (student interview data):** This data was collected in the second semester when all teaching by the first lecturer had ceased. Focus group interviews took place with students in groups of two or three and lasted approximately for one hour (each). There were 6 groups/14 students in total.

The lecturer participating in the research taught in the first semester only. Another lecturer took over the module after Semester 1 but no research with the (second) lecturer took place as part of my study. However, one researcher attended the linear algebra lectures in the second semester in order to gain a sense of the mathematical content that students encountered after Semester 1. As a result informal observations of the lectures took place and, from time to time, brief conversations with the (second) lecturer about the mathematics that he was teaching.
3.4.1 Ethical considerations in data collection

Nearly all the data collected was qualitative and in the form of audio-recordings of interviews and conversations with the lecturer, on the one hand, and of his lectures, on the other hand. This raised a number of ethical issues. Conducting interviews, in general, raises ethical issues in relation to the selection of participants, questions of anonymity, and of dealing with sensitive information, for example.

Most data was in the form of interviews and lecture observations obtained with the cooperation of just one lecturer. The lecturer in the study was approached by one of the researchers and invited to take part in the research. Once the lecturer’s cooperation had been secured no further members of staff were approached. Here the overarching consideration was to carry out a study of teaching using a naturalistic inquiry approach. This included observing a lecturer in the natural setting of his teaching and meeting with the lecturer before and after lectures. Integral to the research was the lecturer’s willingness to freely share ideas and thoughts with the researchers, a pre-requisite for a successful conclusion to the study. Thus obtaining the commitment from one person was paramount and we, the researchers, felt lucky to have such a willing collaborator who seemed to share our enthusiasm with regard to the aims of the research, namely a deeper understanding of teaching issues in relation to university mathematics in general, and linear algebra in particular.

The second set of interview data came from six student focus groups. The students who participated in the focus group interviews did so by self-selection and were paid £10 for taking part. Initially, I approached students at the end of a linear algebra lecture in Semester 2 Week 5, in the same lecture that I presented the results of the second questionnaire to the students. I asked for volunteers for the student group interviews at the end of the lecture and again via an email to all students one week later. This stated that £10 was offered for taking the time to take part in the interviews. A number of students came forward and indicated that they wished to participate. I assembled the groups based on availability of students at certain times and dates. Four students volunteered and indicated that they wished to be interviewed together as a friendship group to which I agreed. (In the end one of the students from the friendship group was unable to attend.)

Self-selection error, or volunteering error, writes Oppenheim (1992), introduces an element of bias. Although I felt that this was a disadvantage, I also thought it was offset by a need for openness on the part of the students. As was the case during the interviews
with the lecturer, students’ willingness to share their ideas and thoughts with the researcher was paramount to the research. Having students who volunteered for this task was more likely to result in the kind of rich data that I hoped to obtain for my research.

Based on my observations of lectures and tutorials I believe that several of the students that I interviewed were very committed to their studies. At least 5 of the 14 students attended lectures frequently and regularly, and were still present in the last few weeks of the course when attendance fell sharply.

As regards the second point, the question of anonymity, in many situations researchers can guarantee anonymity to participants in a research study. Given that my research took place in a Mathematics department of average size the lecturer’s identity was generally known to other members in the department. In addition, the researchers (with the lecturer’s agreement) announced to the students attending the linear algebra module that research was taking place with their lecturer and that lectures and tutorials were going to be audio-recorded. I could not guarantee anonymity for the lecturer in this study on this level. However, as far as publications from the research are concerned, I am in a position to disguise his identity (by not identifying him by name, for example). So far, the lecturer has waived his right to anonymity. He has contributed to journal articles in relation to the research which resulted in his being listed as a co-author. He has also been involved as a co-presenter in seminars given at the home institution as well as at other universities on the findings from the research.

In conducting the student focus group interviews I adhered closely to the University’s ethical code. Permission was sought to audio-record the sessions, and to use all data for research purposes (using the University’s standard consent forms). Students’ identities were known only to the two researchers in the study. They were not disclosed to the lecturer, nor anyone else at the university.

Although we, the researchers, did not anticipate that this study would highlight any particular sensitive issues during the data collection period, we remained alert to the possibility. However, there were no instances during the interviews with the lecturer or with the students of any information becoming available that was of a sensitive, offensive or indiscreet nature.

3.4.2 The meetings data

The main area of data collection was based on meetings with the lecturer. The meetings took place on a weekly basis (approximately), sometimes in the more formal setting of the lecturer’s office and sometimes more informally over lunch (in the University’s
dining hall, for example). With very few exceptions, we, the researchers, did not use any prepared questions at the start, or at any time during the interview. We simply invited the lecturer to reflect on ‘last week’s teaching’, and to feel free to talk about anything he wished in connection with his teaching of linear algebra. The interviews were exploratory and relatively informal. They could be classed as conversations. However, they were conversations with the overt aim of obtaining information. As Oppenheim (1992) wrote,

An interview is not an ordinary conversation, although the exploratory interview appears similar in some respects. ... The purpose of all research interviews is to obtain information of certain kinds. (Oppenheim, 1992, p. 65)

With the aim of gathering data in respect of the lecturer’s didactical thinking and decision-making, the interviews followed the type of an exploratory, depth or free-style interview (Oppenheim, 1992).

During the meetings the researchers asked follow-up questions and probed more deeply if they considered it necessary to follow up a line of thought for the purpose of the research. There were also times when the lecturer talked with little interruption or probing by the researchers. The main focus of the interviews was the lecturer’s reflection on his teaching. This included his reasons for what, how and why to teach a particular linear algebra topic as well as mathematics in general at a university. The lecturer also made comments of an evaluative nature in relation to his own teaching, that is, the teaching that he performed in the previous week, for example.

All interviews were audio-recorded. This enabled the researchers to analyse the data in detail, and provided a possibility for inspection by other researchers, for example, or outsider evaluation if considered desirable at some point in the future.

All aspects in relation to the aims of the research were communicated to the lecturer in advance. In my evaluation the study initially followed a data-extraction agreement (Wagner, 1997). However, as the study progressed and a kind of working mode was established, there was evidence of the lecturer becoming more involved in the research process. For example, without any probing or encouragement by the researchers, the lecturer made suggestions regarding the content of the second questionnaire. Developing a sense of a shared understanding and purpose is a feature of a clinical partnership (Wagner, 1997). In a clinical partnership the roles of researcher and participant are seen as complementary rather than separate, each having their own area of expertise yet jointly engaging in a research enterprise. Thus in my study we, the researchers and the
lecturer jointly engaged into inquiring about the teaching and learning of linear algebra while contributing different sets of expertise, the researchers in mathematics didactics and the lecturer in mathematics. I believe this is an appropriate way to describe the researcher-practitioner relationship in my study.

3.4.3 The lecture observation data

A second area of data collection consisted of the audio-recordings of all linear algebra lectures and tutorials in Semester 1. These recordings were made in the same weeks as the meetings took place. Every lecture and tutorial was attended by at least one of the researchers (and most usually by both). The researchers (independently) took field notes (in long hand) in addition to the audio-recordings. Field notes in the form of long hand notes, taken during observations in the “field”, are an important primary data source (Stenhouse, 1978), and were the only primary data source in many areas of social research before the advent of recording devices.

Taking field notes meant sitting in lectures, among the body of students listening to the mathematics being taught. Field notes were made in relation to the mathematics being produced or discussed by the lecturer, or by the students during time spent on exercises. Other aspects of the teaching setting were also noted from time to time such as perceived student concentration levels on the tasks given to them, room layout, student behaviours, attendance, etc. Although we attended lectures as researchers, I felt our presence did not interfere with the normal working relationships between students and lecturer in the lecture hall. I agree with the view that observing classroom interactions in the role of neutral observer (which is usually taken to mean ‘non-participant’ observation) is impossible to achieve in a school setting where the mere presence of an additional adult affects the working relationship between pupils and teacher (see Jaworski, 1994). Given the size of the linear algebra class (more than 240 students, with an attendance of 150 to 180 on average) and the relative anonymity of students, I felt that our presence was largely neither noted, nor noticed. However, no research data was collected to verify or refute this claim.

3.4.4 The student interview data

Many of the discussion points in Section 3.4.2 applied to the focus group interviews with the students. However, there were also some differences. Interviews with the students were one-off, one hour long interviews. There was no continuity. The building of a relationship was limited as interviews took place under time constraints. I saw the students for a relatively brief period, and at that one time only.
Interviews took place during the daytime hours in a meeting room at the university and were conducted by just one researcher, myself. Students had some time at the start of the interview to read and sign a consent form in accordance with University guidelines and the Data Protection Act. Students were also handed a written set of questions that acted as a prompt for the interview. I include a copy of both the consent form and the interview questions in Appendix A.

The group interviews were audio-recorded. I did not take any additional field notes as my time and energy were focused on managing the interviews. This was a new experience for me and I was keen that the interviews went smoothly and contained input from each participant.

Group interviews are sometimes conducted because they are less time-consuming. However, the main aim of conducting group interviews in my study was the intention that a discussion should develop among the participants. This is considered a good way of raising issues as participants support, influence, complement, agree and disagree with each other (see Cohen, Manion and Morrison, 2008). There may be a problem with domineering individuals; but this was an issue that arose just once, in one interview with one individual for a short period of time only.

Students were approached for their views in Semester 2, and the first group of students was interviewed on 23 March (Week 7). By that time a different lecturer had taken over the teaching of linear algebra which was the way that the department had organised the one year long module. This meant that I was asking students retrospectively for their views on the teaching that they experienced in the previous semester. Interviews were spread out across the teaching weeks, starting in Week 7 of Semester 2 and ending in Week 11. Interviewing students in Semester 2 rather than earlier, in Semester 1, may be seen as a limitation to this part of the study and the responses that students gave. In particular, the last student focus group interview that took place in Week 11, was very close to the end of the module. Views expressed in this session indicated that students had developed an overall perspective of the course and of linear algebra as a mathematical topic. Although this may have limited the possibility of capturing students’ initial responses to the teaching in Semester 1, it did provide a sense of students’ progression as they neared the end of the module.

3.4.5 The student survey data

In Semester 1 students’ views were sought via two questionnaires. These were given out during the teaching of the linear algebra module, at the end of two lectures. One of the aims was to gain factual information about students’ background and learning
preferences. A second aim was to highlight and raise issues in relation to the teaching and learning processes as seen from the students’ point of view, and in advance of the interviews that were planned for Semester 2. Questions were designed to allow for a relatively simple and quick response as time for completion of the questionnaires was limited (to approximately ten minutes).

It is accepted practice in research that all questionnaires, that is, all question and answer formats, should be piloted, and if changes were made to any part of the questionnaire, piloted again (Cohen et al., 2008; Oppenheim, 1992). However, no piloting of the questionnaires used in my research took place due to time constraints. I considered it more important to issue a questionnaire quickly as the need arose, and capture the data. I did not see a lack of piloting as an impediment to the results. I had had prior and recent experience in conducting a survey that included an extensive period of piloting. Based on this information I drew up a questionnaire that contained questions in both ‘closed’ and ‘open’ form. The questions and format of the questionnaire were discussed in a meeting between the researchers and the lecturer. All parties provided input, and the final design was used for the survey.

For the majority of the questions I had designed a rating scale, ranging from ‘strongly agree’ to ‘strongly disagree’. Some questions were of the ‘Yes/No’ type, and a few questions were in ‘open’ form which provided students with an opportunity to write an answer using their own choice of words.

An important consideration in conducting surveys is the question of sampling, that is, sample size and sampling procedures. The target group in my study was a particular group of students, namely the 236 students registered for the Linear Algebra module. The number of students was thought not to be too large by the researchers (not in 1000s, say) to consider a sampling procedure. In particular, a computer package was going to be used for part of the analysis, and handing out questionnaires to all students was considered possible and manageable during the lecture. Also, the number of students was not too small (less than 30, say) for statistical analysis and any resulting inferences or generalisations to be considered invalid. Thus, all 236 students were deemed eligible to complete the questionnaire(s).

For practical reasons and in order to avoid a low response rate, questionnaires were handed out to students towards the end of a lecture. There were two questionnaires, handed out in Weeks 4 and 11 of Semester 1, respectively, and chosen to coincide with the end of a chapter in the course notes in the case of the first questionnaire, and to maximise the student response rate towards the end of term in the case of the second questionnaire. Copies of the questionnaires can be found in Appendix B.
3.5 The methods of data analysis

My initial research questions,

- What does it mean to teach mathematics at university?
- What does it mean to teach linear algebra at university?

were formulated in deliberately broad terms. I took a holistic view and aimed to explore in the first instance, and then to explain and understand. Having a broad initial perspective made following a grounded approach to data collection and analysis an appropriate choice for proceeding. Later on in data analysis I focussed on the lecturer’s intentions and strategies (resulting in a set of revised research questions). Adopting an activity theory perspective crystallised the analysis process (see Chapter 5) and led to a conclusion of the study.

3.5.1 The analysis of the meetings data

I listened to all audio-recordings of the meetings with the lecturer in full soon after the meeting. Within one week (usually) I made a data-reduction of the meeting. This consisted of a one-page summary of the interview, a kind of resumé, which I used to highlight some of the key topics that we had discussed. The data reductions constituted a first level analysis. I identified passages and ideas that I, the researcher, considered significant. These passages were also transcribed at that point. This was my interpretation of the data guided by my research question “What does it mean to teach...?” It provided me with insights into the data that I collected and an awareness of emerging issues. In producing data reductions I aimed to capture the content of the meetings. I assigned a ‘theme’ to various ‘blocks of conversations’ even if I considered the content relatively unimportant at that point in time. I wanted to stay ‘close’ to the data, a stance consistent with a grounded approach. I was also aware that a previously ‘insignificant’ passage of conversation could gain more significance as the analysis progressed and my research questions evolved. Having an ethnographic/naturalistic inquiry approach meant that all aspects of the data were important as they all formed part of the context and the overall picture of what it meant to be a lecturer of mathematics at a university.

After all the meetings data were collected I made full transcriptions of the interviews in order to code using the computer software package Atlas-ti. I used an open coding procedure for the coding of the transcripts. Open coding implies that the researcher
“starts with no preconceived codes – he remains entirely open” (Glaser, 1992, p. 38). I believe this is not a realistic view to take in any human activity including in research. As researchers, we make all kind of choices in conducting research, whether that is in the process of a quantitative or a qualitative studies. Burton stated,

I do not believe that there is ever a case where the researcher’s beliefs, attitudes, and values have not influenced a study, ... (Burton, 2002, p. 3)

We all bring our own experiences of the world and the life that we have lived to the research process. Furthermore, all researchers have research questions to guide them, and for which they purposefully and actively seek an answer. Both these aspects lead to a narrowing of focus so that it is not humanly possible to have “no preconceptions at all”. Becoming alert to this fact, I tried to bring as few preconceived ideas as I could to the research analysis. I was influenced by my knowledge of teaching as a former teacher of mathematics at a secondary school. In particular, I was aware of literature into teacher intentions, and teachers’ knowledge and beliefs.

Open coding involves breaking down a script into sentences or passages (called incidents), and assigning a code that characterises that sentence or passage (called a category). This is what Glaser (1992) called a substantive code. During coding I was guided by my research question “What does it mean to teach . . .?” I, (a) attached categories to a sentence or passage that addressed this central question in some way, and (b) chose names for the categories that captured the meaning of the sentence or passage (a conceptual name).

Open coding results in a proliferation of categories as there is a tendency to ‘overanalyse’, that is to generate as many categories as possible to different incidents, as I did in my analysis. I have included examples from my coding analysis in Appendix D.

Whilst coding, I continually compared an incident, and the category that I had assigned to it, with other incidents in my data. I also compared the categories that I was creating with each other. This was what Glaser (1992, p. 14) described as the constant comparison method. The aim is to look out for patterns that can simplify the data (resulting in a smaller set of categories) and integrate it into a framework (via properties of categories, called dimensions, and theoretical codes).

In addition I created many memos. These captured ideas for analysis and possible routes of explanations seen ‘in the moment’. They were personal reflections on, and interpretations of the data, or of the categories, or both. Memoing, in particular, brought
increased focus to the study. It gave me a very concrete sense of the exploratory nature of the research, and aided my coding and interpretation.

I generated over 200 codes while analysing the first eight meeting transcripts, nearly half of the total number of transcripts (18). Coding increased my awareness of issues (issues for me and issues for the lecturer, in both cases as I saw it) and brought focus to my study. Using the constant comparison method I reduced the initial 200 codes to a more manageable number of (approximately) 80 codes. Throughout this process I was guided by my research question “What does it mean to teach mathematics at university?” and “What does it mean to teach linear algebra at university?” Working with a reduced number of categories I continued coding by taking a sample of the remaining transcripts. This was a time-saving device and justified, I felt, as the saturation point for coding had almost been reached. The coding of each transcript took less time, now that I was dealing with a (more or less) pre-defined set of codes. I focussed on comparing current incidents with the categories assigned previously. This constituted theoretical sampling whereby I was constantly comparing the current set of codes against uncoded data in order to find agreement or to highlight discrepancies (Glaser and Strauss, 1967). I began to increasingly focus on the lecturer’s intentions and the strategies that he aimed to use to accomplish his intentions. His intentions (“what he will do”) were underpinned by reasons (“why”), both expressed explicitly in the meetings data.

To summarise, my grounded approach to the analysis of the meetings data consisted of: listening to the tapes and making data reductions soon after a meeting; coding the transcripts using an open coding approach; memoing; and theoretical sampling. In addition, I also listened to all tapes in chronological order. Since meetings with the lecturer took place (approximately) every week and in between lectures, I decided to listen to the recordings in order, two or three lectures followed by a meeting followed by two or three further lectures, and so on. The aim was to create a reference point between the comments that the lecturer made in research meetings and his face-to-face teaching of students.

3.5.2 Ethical considerations in the coding analysis

Using Atlas-ti, I decided to code while reading the transcripts (the text file) and simul-
taneously listening to the tape (the audio file). This decision was based on experiencing difficulties (on two occasions) in assigning a category to a particular passage when using the text file only. I became aware (by checking with the tape) that the tone of voice affected the category that I assigned.
On the second occasion placing a full stop in a long passage of a transcript seemed to result in a different meaning. Checking with the tape clarified the situation. Other potentially problematic instances may have slipped my attention.

This highlighted a difficulty of coding audio data held in a visual (text) format only. It alerted me to the problematic nature of analysing qualitative data more generally, whereby researchers frequently have to make decisions on the way that they handle a specific piece of data. It requires a flexibility in dealing with data that should not, but often is confused with bias. It is an unavoidable aspect of qualitative data analyses and part of what it means to work as an interpretative researcher.

All data, that is the audio-recordings and the data reductions, were made available to the lecturer during the research. The aim was to be open about the study and to aid its trustworthiness (see Sections 3.3 and 3.6). It provided the lecturer (as a participant in the research) with an opportunity to be 'close' to the analysis and the interpretation of the data.

3.5.3 The analysis of the lecture observation data

All lectures and tutorials were audio-recorded by placing a recording device at the front of the lecture theatre. I listened to the recordings of lectures at least twice during the early stages of the analysis. The analysis of the lecture observation data had the aim of aiding my interpretation of the meetings data, in particular, in relation to the lecturer’s intentions and strategies. The audio-recordings of the lectures, therefore, were not analysed independently; I used a set of pre-determined categories.

Initially, I analysed four consecutive lectures (covering two weeks and excluding the tutorials) selected from the middle section of the course. Analysis of two more lectures from the beginning and two nearer the end of the course were planned for a later stage. This amounted to one-quarter of all lecture recordings. I decided to take lectures ‘in a block’ in order to gain a sense of continuity of teaching across a topic.

I listened to the recordings and made partial transcripts, that is, I transcribed sections where the lecturer addressed the class while explaining the mathematical concepts, or what students should focus on. I did not transcribe sections where the lecturer talked through a calculation as it appeared in the course notes. I had narrowed my focus, concentrating on those aspects that related to the lecturer’s intentions and strategies. In particular, I analysed how he communicated his intentions for students’ learning of linear algebra to his students, that is, in the natural setting of teaching, the lecture hall.
Both the analysis of the meetings data and the analysis of the lecture observation data stalled until I formulated my theoretical perspective in activity theory. In particular Leontiev’s (1981) structural elements of activity-motive, actions-goals, and operations-conditions provided me with an analytical tool to organise my data, to provide structure and hence advance my analysis. This is the focus of Chapter 5 where I apply, as (Mills, 1993) described, “...theory as an analytical and interpretive framework that helps the researcher make sense of ‘what is going on in the social setting being studied’ ” (Mills, 1993, p. 103, cited in Anfara and Mertz, 2006). In Chapter 5 I describe in detail my analysis of the lecturer’s intentions and strategies, and his reasons for choosing a ‘bottom-up’ or ‘inductive’ teaching approach. I relate my data analysis to the conceptual framework of activity theory. I then use activity theory as a theoretical lens in further analysis. This is the focus of Chapter 6.

3.5.4 The analysis of the student interview data

The student focus group interviews were audio-recorded, and I listened to all recordings in full within one week of the interviews. For piloting, one interview with students was data-reduced and coded. The data reduction was cumbersome to perform as there were several participants, and an analysis based on the data reduction did not seem to capture adequately, in my view, the content of the interviews. I, therefore, decided to make partial transcriptions. In particular, I wanted to know (a) how students responded in lectures to working on the exercises that the lecturer set, (b) whether the inductive approach had led to an increased focus on the concepts of linear algebra and (c) if, as a result of the teaching design students were coming to view mathematics as a creative, stimulating subject.

The results from this analysis are presented in Chapter 7. The reason for including an analysis of the student data was to present the student dimension of the study and include it in my synthesis of the results from all analyses in Chapter 8.

3.5.5 The analysis of the student survey data

The student survey data consisted of two questionnaires that were answered by 172 and 107 students respectively. There were two types of questions: Closed questions (which were used most frequently) that required students to choose and then mark (or circle) their answer from a list of predefined categories; and open questions where students could write answers using their own words. Answers provided in the form of a closed question were analysed using the software SPSS. Results from this analysis were displayed in frequency tables and bar charts which I saved electronically for later use.
Answers provided in respect of the open questions were analysed qualitatively. For each question this involved listing the responses and analysing by combining ‘similar’ responses in an overarching category.

The results from the analysis of the questionnaires are presented together with the results from the analysis of the student interview data in Chapter 7.

3.6 Addressing trustworthiness in my study

Criticisms of interpretive research centre on two main areas namely, that research results and findings are based on ‘subjective’ accounts and interpretations and secondly, that findings cannot be replicated or generalised. To answer these criticisms Lincoln and Guba (1985) proposed alternative criteria for assessing the ‘value’ of qualitative studies that take account of the nature of the research conducted in this paradigm. In this section I use the criteria of trustworthiness and authenticity to address ‘the value’ of my study.

In the preceding sections I gave details of my methods of data collection and analysis. The purpose of providing such details was to establish trustworthiness in the data that I collected and analysed, and confidence in the findings and conclusions that I drew as a result. I did this by a variety of means, embedded to a large extent in the traditions of a naturalistic inquiry. In addressing the trustworthiness of my research I refer also to the notions of credibility, dependability, confirmability and transfer (see Lincoln and Guba, 1985).

I kept detailed records of my data collection, in manual and electronic format, which are accessible for inspection, for example, in order to ‘verify’ quotations that I have used to support interpretations. For reporting on my data analysis I recalled my methods and procedures based on the principles of a grounded approach, and the use of theory as an analytical tool in furthering analysis. I reported on working closely with my supervisor and the lecturer of linear algebra. This included sharing ideas and interpretations with the lecturer as they emerged in research and making all records available to the lecturer. I reported on some of the issues that arose in data collection and analysis, including ethical issues and issues arising in interpretations. By providing such details in my thesis I aim to increase confidence in the methods and procedures that I used, and ultimately credibility in the findings and conclusions that I present in later chapters.

I used triangulation to increase confidence in the interpretations that I have made. I drew on the description by Denzin (1978) who listed four different modes of triangulation
(Denzin, 1978, cited in Lincoln and Guba, 1985, p. 305) all of which applied in my study and for which I give details: triangulation of sources, of methods, of investigators and of theories.

In the analysis chapters that follow I report on (and quote from) the different sources that I used for making interpretations. These include the research data such as the meetings data, lecture observations and the student interview data, as well as the linear algebra course notes and (other) published linear algebra textbooks, for example. I report on how I related comments made in one format to data held in an alternative format, all aimed at justifying interpretations. I combined the methods of grounded theory analysis with applying the theoretical constructs of an established theoretical framework (Activity Theory) to code, categorise and model, what I refer to as the ‘teaching process’. The aim of these methods was to increase confidence in the interpretations that I made, and ultimately credibility in the findings and conclusions.

Throughout the research process I provided the lecturer with ongoing feedback in respect of my analysis and interpretations. The lecturer took part in seminar presentations and co-authored papers. In addition, towards the end of the research, I approached three lecturers of mathematics, colleagues of the lecturer who participated in the research, to check sections of the thesis (parts of the second analysis chapter) for mathematical accuracy. This related to the mathematical formulations (of a definition or theorem in linear algebra) that I either reproduced or created, and to the mathematical discussion of the topics that I offered in that chapter. This chapter was made available also to the lecturer in my study. Thus I deliberately involved the lecturer and members of the department to which the lecturer belonged in my analysis and interpretation. It provided the lecturer as well as individuals not connected with the research with an opportunity to comment, to highlight inaccuracies and to offer alternative interpretations. Proceeding in this way and providing the lecturer with ongoing feedback as mentioned above represented a case of member checking (Bryman (2004) refers to respondent or member validation) aimed at confirming the interpretations and results that I offer and at increasing credibility in findings and conclusions.

One criticism that is frequently raised in connection with a case study in particular, and a qualitative study in general, centres on the question of how results can be generalised or replicated. In general, within an interpretive paradigm, a case study cannot be replicated since the context of the study is implicated in the outcomes of the study. Repeating the study would inevitably involve different contexts and lead to different outcomes. This is not to say that findings from case studies cannot be transferred to other situations or contexts. But transferability depends on the individual case to which it is to be applied. The possibility for transferability is increased by providing a rich and detailed
description of the research which could allow others to make a judgement as to whether a transfer is possible (Lincoln & Guba, 1985). Thus transferability has to be addressed, and assessed uniquely in each case, and by the person who seeks to make the transfer.

The criteria of confidence in the methods and procedures, the criteria of confidence in the interpretations made as well as their confirmability through the method of member checking, and the criteria of transferability of findings were all aimed at increasing credibility in the findings and conclusions that I offer in Chapter 8.

In summary, in this chapter I discussed the methodological perspectives underpinning my study and gave details of my methods of data collection and data analysis. In the next chapter, Chapter 4, I discuss my theoretical perspectives in activity theory. In Chapter 5 I report on my analysis of the interviews with the lecturer (the meetings data). I draw on my theoretical framework and give concrete meaning to activity theoretical concepts and terminology in terms of an educational setting.
Chapter 4

Theoretical perspectives

In the last chapter I gave details of how my research developed and how my methodological and theoretical perspectives became intertwined in data analysis. Working in an interpretive paradigm carried with it certain assumptions in respect of the origin and nature of knowledge. These assumptions are shared also in the theoretical framework that frames my study.

Activity theory presupposes that knowledge is social in origin and produced in social interactions. This echoes the assumptions of my methodological choice within an interpretive paradigm. Hence activity theory, as one of the socio-cultural theories of the mind, is an appropriate theoretical perspective for my study and fits well with the aims, methods and nature of my research.

Activity theory is rooted in the work of Vygotsky. I cannot adequately describe activity theory without acknowledging the founding contribution that Vygotsky made to the development of activity theory. Hence, before I discuss activity theory I elaborate on those elements of Vygotsky’s socio-cultural theory of the mind that contributed to the development of activity theory.

Since Leontiev’s formulation of activity theory there has been an ‘explosion’ of (third generation) activity theoretical points of view. While acknowledging that these exist, I propose to go back to the roots of both a socio-cultural theory of the mind (Vygotsky’s founding contribution to activity theory) and Leontiev’s (second generation) work on activity theory as theoretical foundation for my study.
4.1 Vygotskian perspectives as a foundation for activity theory

Wertsch (1991) points to three themes that run through Vygotsky’s writing and that provided the basis for Vygotsky’s socio-cultural theory of the mind. (1) Vygotsky emphasised a genetic, or developmental analysis of the mind. He claimed that (2) all higher mental functioning is social in origin, and that (3) all human actions are mediated by tools and signs (see also Daniels, 2008; Wertsch and Stone, 1985). I discuss each aspect in turn but note that none of the three themes is more important than the other two and that all three are in fact interlinked and presuppose each other.

Vygotsky focussed on an analysis of the development (or genesis) of the human mind. That is, Vygotsky shifted his attention from studying the mind as an object to studying the processes through which the human mind developed, that is how the individual came to acquire knowledge of the world. Second, Vygotsky proposed that the origin of all human mental functioning lay in the social life of the individual. This is most clearly expressed in Vygotsky’s “general genetic law of cultural development” which states:

Any function in the child’s cultural development appears twice, or on two planes. First it appears on the social plane, and then on the psychological plane. First it appears between people as an interpsychological category, and then within the child as an intrapsychological category. This is equally true with regard to voluntary attention, logical memory, the formation of concepts, and the development of volition . . . [I]t goes without saying that internalization transforms the process itself and changes its structure and functions. Social relations or relations among people genetically underlie all higher functions and their relationships. (Vygotsky, 1981a, p. 163)

The developmental aspect is related to the origin of cultural development and the formation of higher mental functions. The social origin of higher mental functioning is related to the expression “first appearing on the social plane” and “between people”, and the notion of mediation to “between people as an interpsychological category”.

Vygotsky studied mental development as the result of an individual’s (deliberate or conscious) interactions in the social domain. He defined mental development that was the result of conscious interaction as cultural development. Cultural development referred to more than the kind of development that occurred naturally or spontaneously in an individual’s everyday lived-in experience. Vygotsky also related cultural development
to higher mental functions and the acquisition of scientific concepts. In contrast, the lower or elementary functions were related to the acquisition of spontaneous or everyday concepts. Cole (1978) explained the difference in terms of a Vygoskian perspective and Kozulin (1986) in terms of an activity theory perspective as follows.

Cole wrote that Vygotsky made the distinction between “elementary functions, involuntarily applied, and higher functions that incorporate planning elements in a deliberate manner” (Cole, 1978, p. xvii). In terms of activity theory, everyday concepts are the result of the individual engaged in spontaneous activity while scientific concepts are the result of an individual engaged in more structured and deliberate activity (Kozulin, 1986). This led Kozulin to refer to purposeful activity (Kozulin, 1986), and Wertsch, in interpreting Vygotsky, to goal-directed action in place of purposeful activity (see Wertsch, 1991).

The notion of cultural development on the interpsychological plane places emphasis on the social element of development and on mediation. Mediation is a key concept in Vygosky’s socio-cultural theory. Vygotsky proposed mediational means as the intermediate link between subject and object expressed in Vygotsky’s (famous) mediational triangle. Mediation includes the use of tools. For Vygotsky, sign systems and language, in particular, were the main mediational means. He listed as examples of sign systems, language; various systems for counting; mnemonic techniques; algebraic symbol systems; works of art; writing; schemes, diagrams maps and mechanical drawings; all sorts of conventional signs; etc. (Vygotsky, 1981b, p. 137).

Vygostky referred to these as psychological tools. Psychological tools are part of cultural heritage. They are artificial and designed in order to affect a change in mental states. As Wertsch wrote, “mediational means were created with the express intent of shaping individual action” (Wertsch, 1991, p. 33). While language, signs and symbols are psychological tools, material tools, on the other hand, have physical form and are designed to affect a (physical) change in the object. As an example from my own research, computers and textbooks are material tools. Language as a psychological tool has a special status. Language “serves not only as a means of social interaction but also as a carrier of the socially elaborated meanings that are embedded therein” (Leontiev, 1981, p. 56). Vygotsky himself referred to language as “the tool of tools”. As an example from my own research, mathematical notation, and the formulation of a mathematical problem are psychological tools.

Mediation of elementary functions through the use of psychological tools leads to the development of the higher mental functions. The higher mental functions develop out of
mediated actions on the social plane. Wertsch (1991) wrote that “a defining property of higher mental functioning, . . . is the fact that it is mediated by tools and by sign systems such as natural language” (ibid, p. 21). Internalisation refers to the (gradual) transformation of the interpersonal processes into intrapersonal processes, from the social plane to the individual, internal plane (Vygotsky, 1978).

While Vygotsky focussed on the role of interaction in the development of individual mental processes, in activity theory, the individual is included in activity and an integral part of it. Minick (1997) referred to “socially organized goal-oriented actions” (Minick, 1997, p. 117). In activity theory, activity mediates cultural development. Human activity presupposes the existence of tools in activity. Thus activity includes the use of both material and psychological tools in social interactions. As Kozulin (1998) noted, psychological tools “transform the unmediated interaction of the human being with the world into mediated interaction” (ibid, p. 4).

4.2 Activity theory

In formulating activity theory Leontiev drew on two ideas, the unity of individuals and their social environment (expressed in activity as unit of analysis), and the social nature of the mind (the foundation of a socio-cultural perspective). Activity theory has its origin in the work of Vygotsky who was the founder of a socio-cultural theory of the mind. In building on Vygotsky’s work Leontiev developed a comprehensive activity-theoretical framework that many scholars refer to, and use when adopting an activity-theoretical approach.

Leontiev formulated the notion of human activity as the non-additive, molar unit of life (Leontiev, 1981, p. 46), and activity as the unit of analysis. He considered activity as central to all human mental functioning. That is, through engaging in activity, including practical and social activity, the individual acquired knowledge about the world. As Wertsch (1981b) put it,

Leontiev’s main point is . . . that our knowledge of the world is mediated by our interaction with it. (Wertsch, 1981b, p. 38)

This statement implies that individuals acquire knowledge actively, through engaging in practical activity and in social interaction with others. Knowledge is not passively received, say, by reacting to stimuli in the environment and appropriating that what is on offer. The individual comes to know the world through social mediation. In an
activity-theoretical perspective the individual is an active subject in activity and hence in ‘acquiring’ knowledge. At the same time the individual in engaging in activity *changes* the activity and the kind of knowledge held within. Wertsch, in interpreting Leontiev’s work, said,

…neither the external world nor the human organism is solely responsible for developing knowledge about the world. (Wertsch, 1981b, p. 38)

In this sense knowledge acquires a different nature and status when compared with a Vygotskian perspective. In a Vygotskian perspective the social domain has primacy over the individual domain, that is, the individual (internal) plane of consciousness is formed by the social (external) plane. In expressing mental development in terms of a dialectical relationship between individual and culture, one influencing the other, Leontiev developed a theory which extended Vygotsky’s theory of cultural development. For Vygotsky, culture and society determined individual mental development which he expressed in his (famous) ‘general genetic law of cultural development’ (Vygotsky, 1981a, p. 163). That is, the individual mind is a result of interactions with the world and is shaped by it. In activity theory the emphasis on the social origin of mind is retained through the concept of activity. But, as Leontiev said, individuals do not enter ‘a world without activity’. Activity pre-exists the individual and is an expression of societies and culture. Activity defines the relationship of the individual with culture and of culture with the individual. Leontiev wrote,

…, in a society, humans do not simply find external conditions to which they must adapt their activity. Rather these social conditions bear with them the motives and goals of their activity, its means and modes. In a word, society produces the activity of the individuals it forms. (Leontiev, 1981, p. 48)

The changing nature of activity due to individuals actively engaging in and with their environment is captured in the activity-theoretical perspective. Both the individual and the activity change as a result of the individual’s interactions in the world. All interactions between the individual and the environment, whether mental or physical, take the form of practical contact with the world, through human actions. Thus external practical activity (activity that can be ‘readily observed’) is closely linked to mental development. It is in this context that Leontiev explained mental development in terms of internalisation processes. For Leontiev, individuals engage in practical activity which is internalised. Internalisation is a gradual process that transforms external activities into internal activities. Leontiev wrote,
Internal activity, which has arisen out of external, practical activity, is not separate from it and does not rise above it; rather, it retains its fundamental and two-way connection with it. (Leontiev, 1981, p. 58)

Thus, in activity theory, *activity* is central to human mental functioning and the *mediating link* between the individual and the world, and between the individual and the individual’s *knowledge* of the world. Leontiev referred to “a process of reciprocal transformations between subject and object poles” (Leontiev, 1981, p. 46). As a result of these transformations, Leontiev wrote, the individual, the environment and the activity itself will *change*. Leontiev argued further that the *structure of human thinking* will also change (see Wertsch, 1981b, p. 39).

Because of the dialectical relationship between individual and culture, in an activity theory perspective ontogenic and phylogenetic development are closely related. Activity, also a culturally, historically and socially determined entity, mediates this development. The social, cultural and historical factors contributing to the theoretical basis of activity theory has led some scholars to refer to activity theory as cultural-historical activity theory (or CHAT), and to place a greater emphasis on cultural, historical and social factors. Proponents of CHAT have taken account of these factors by enlarging Vygotsky’s mediational triangle into an expanded form (see Engeström, Miettinin and Punamaeki, 1999). The expanded triangle reflects the relationships between object and subject poles and the new poles of *rules, community* and *division of labor*.

### 4.3 The structural elements of activity

I have introduced the concept of activity in the previous section in terms of an overall theoretical perspective. In Soviet Psychology the concept of activity was developed and used both in a ‘broad’ sense and in a ‘narrower’ sense (Davidov and Markova, 1983). In the broad sense, activity was regarded as a theoretical concept, and in the narrower sense as a device giving structure to human activity. In the broad sense, it was used “in connection with the principle of the unity of mind and activity” (*ibid*, p. 54), as a principle for a theoretical foundation. In the narrower and more specific sense, it was used to describe internal activity by distinguishing “two sets of structural characteristics: activity-action-operation, and motive-goal-constraint” (*ibid*, p. 55) which related to Leontiev’s work on activity. I now discuss the concept of activity in its ‘narrower’ sense, as an analytical device.
Leontiev (1981) defined the *components* of activity on three levels. At the top level the subject’s *activity* is related to a *motive*, at the second level *actions* are related to *goals*, and at the third level the *operations* are related to the *conditions*, or *constraints*, underpinning an activity. This provides, to some extent, a hierarchical structure that is suitable for analysis and categorisation. However, the elements at each level relate to each other in different ways which results in three *qualitatively different* levels of analysis.

I have reproduced a diagram from Goodchild (1997, p. 28) that shows the elements of activity (see Figure 4.1)\(^1\). The diagram shows the relationship between motive and activity. Activity is fundamentally related to motive and consists of, but is not limited to actions and goals. Actions realise and give form to activity. Actions in turn consist of operations which are constrained by conditions. These provide the elements of human activity. However, human activity is not limited to the sum of these elements. It includes also, for example, the relationships between the elements.

At the top level of analysis *activity* is characterised by its motive, or objective. Activity *comprises* of actions. Leontiev wrote,

\(^1\)Simon Goodchild published a book from his thesis (Goodchild, 2001) that also includes this diagram.
...actions are not special “parts” that constitute activity. Human activity exists only in the form of an action or a chain of actions. (Leontiev, 1981, p. 61)

The actions are subordinated to goals. Whereas goals are consciously held, the individual engaged in activity may be less aware of the motive of activity. Motives underpin activity, and there cannot be activity without a motive. Leontiev explained:

There can be no activity without a motive. “Unmotivated” activity is not activity devoid of a motive: it is activity with a motive that is subjectively and objectively concealed. (Leontiev, 1981, p. 59)

Thus the motive may not be ‘obvious’ or ‘transparent’ to the subject, or to other participants engaged in activity, or the motive may be hidden or obscured by the context or situation. If the subject engaged in activity is, or becomes conscious of the motive then the motive becomes a motive-goal.

An action, on the other hand, is determined by its goal orientation.

We call a process an action when it is subordinated to the idea of achieving a result, i.e., a process that is subordinated to a conscious goal. (Leontiev, 1981, p. 59; underlined in original)

Due to their link with actions, goals are consciously held and for the most part achievable, while the motive of the activity may not be.

Thus the motive is the energising function for activity, whereas the goals and actions, at the second level of analysis, take over a directive function (Leontiev, 1981, p. 60). It is sometimes difficult to distinguish between the two levels, activity and action. Leontiev offered the following explanation which I found helpful:

When a concrete process is taking place before us, external or internal, then from the point of its relation to motive, it appears as human activity, but when it is subordinated to purpose, then it appears as an action or accumulation of a chain of actions. (Leontiev, 1978, p. 64; italics my emphasis; cited in Kozulin, 1986)

I used this explanation (in my analysis in Chapter 5) to distinguish activity from the actions that contributed to activity.
But operations and actions are also often difficult to distinguish from one another. Human actions have both an *intentional aspect* (*what* must be done) and an *operational aspect* (*how* it can be done). The means by which an action is carried out are the *operations*, the ‘*how* it can be done’.

The individual involved in an action must perform operations to carry out the action (where the action is in pursuit of a goal). Goals are given under certain conditions, or constraints which are related to factors in the social or cultural environment of human activity. Operations, on the other hand, are dependent on, and *limited* by these very *conditions*. Hence I summarise the three levels of activity by writing activity *correspondent* to a motive, actions *correspondent* to goals and operations *dependent* on conditions (see below). The elements at each level of activity are related to each other in different ways. In applying Leontiev’s framework to my data I obtained three *qualitatively different* levels of analysis.

In following Leontiev, Zinchenko and Gordon (1981) proposed ‘mobility’ between the elements of activity and in the relationships between elements, reflecting the changing nature of human activity.²

> The relationships among the components of activity are mobile and changeable. What is the goal of an activity can become its means under other conditions; conversely, the means of an activity can become actions. The mobility of these relationships can also be seen in the fact that one and the same goal can be attained by various means just as one and the same set of means can be used to reach different goals. The interrelationship between goals and motives can change in an analogous way. (Zinchenko and Gordon, 1981, p. 74)

Intermediate goals may also be identified from a more global goal which may result in dividing an action into separate successive actions. Or, on the other hand, intermediate goals may merge into an overarching goal and become less conscious to the individual in activity. Leontiev expressed ‘mobility’ between the elements of activity (the “units”) in terms of “division” or, conversely, “consolidation”. He said,

> The mobility of the various “units” of the system of activity is expressed by the fact that each of them can become fractional or, conversely, can embrace units that formerly were relatively independent. (Leontiev, 1981, p. 65)

²In the quotation that follows, Zinchenko used ‘means’ instead of ‘operations’ as Wertsch explained in Wertsch (1981a).
In relation to Leontiev’s structural elements of activity theory, the work by P. I. Zinchenko provides a theoretical view of learning in an activity theory framework. Leontiev provided the example of driving a car where the action of shifting gears was first a goal-directed action that became an operation as the learner became more proficient in driving (see Leontiev, 1981, p. 64). Zinchenko presented an empirical study with children that supported this claim. Zinchenko wrote,

Leontiev’s (1945) research showed that any complex intellectual operation always starts out as an independent, goal-directed action and then develops into an operation. (Zinchenko, 1981, p. 309)

Zinchenko’s study into “incidental learning”, that is learning ‘by the way’, not the result of a deliberate strategy, led to the claim that information closely connected with the goal of an action is better remembered than information concerned with the operation(s) of an action.

In summary, in this section, I have presented Leontiev’s development of activity theory. In its ‘narrower’ sense, activity comprises three structural elements: activity correspondent to a motive, actions correspondent to goals and operations dependent on conditions.

For my study I have adopted Leontiev’s formulation and use of the concept of activity. Leontiev’s framework involves three (different) levels of analysis and theory building. In the next chapter, Chapter 5, I exemplify my interpretation of these three levels with examples drawn from my data.
Chapter 5

A structural analysis of teaching with respect to theory

This is the first of two chapters in which I present the analysis of my data. In this chapter I present a theoretical model of the teaching process based on my analysis of the interviews with the lecturer (the meetings data). In developing the model I drew on Activity theory (Leontiev, 1981), and I relate the structural elements of Activity, that are the activity-motive, the actions-goals and the operations-conditions, to my coding analysis of the meetings data. For example, I relate what I referred to as (the lecturer’s) intentions and strategies (in my coding and analysis of the meetings data) to the theoretical concepts of actions and goals in Activity theory.

Before presenting the results of my data analysis, I discuss the linear algebra module where the research took place. I give an overview of the structure and design of the module summarising my discussion in Chapter 1. I present an account of the lecturer’s reasons and motivation in changing the design of the linear algebra module, in particular in adopting an ‘inductive’ style for presenting the mathematical content, as opposed to the more ‘traditional’ (DTP) style, as is often used in university mathematics teaching. I based my interpretation on the data collected in interviews with the lecturer (the meetings data).

5.1 Course design and course structure

The linear algebra module was a one year long module where the first semester was taught by one lecturer and a different lecturer took over at the start of the second semester. Research took place with one of the lecturers only, in the first semester. The
lecturer in Semester 1 (with the agreement of, and working closely with the lecturer who taught in Semester 2) had structured the material of the course so that all the linear algebra concepts were introduced in the first semester in an informal way, with the more formal treatment of the same material by the second lecturer in Semester 2. In this chapter I present my analysis of the teaching approach, based in the main on the meetings data, that is on the comments made by the lecturer in conversation and interview with the researchers, but drawing also on the observational data from lectures and tutorials.

Meetings with the lecturer were transcribed and labelled chronologically using the abbreviations M1, M2, etc. Similarly, lectures and tutorials were (partially) transcribed and labelled L1, L2, etc. I labelled the lectures and tutorials chronologically as they were given so that L5, for example, related to a tutorial. Whenever I reproduce a comment that the lecturer made in a meeting or in a lecture I provide a reference in the form (M15, 13:45), for example. This means that the lecturer made this comment in meeting M15, and 13 minutes and 45 seconds after I had set the voice recorder to record.

As stated in Chapter 1 the lecturer re-structured the content of the module in order to present a more inductive approach to the teaching of linear algebra. He produced course notes that accompanied the module in two versions: one set of notes for students (the student version) that contained ‘gaps’, and a second, full or complete set of notes (no ‘gaps’). The student version of the course notes contained blank areas where students could write the solution to examples presented in lecture. The formulation of the example was printed in full, as were any definitions and observations or remarks that the lecturer wanted to refer to in lecture. Thus, in general, only the solutions to examples were ‘missing’. The second, full or complete set of notes contained all that the student version had; in addition it also contained the worked solutions to the examples. This latter set was not made available to students until the end of the module. The student version was made available for students ahead of the lectures on LEARN, the university’s virtual learning environment. The lecturer asked students to print out a copy and bring to the lecture. At times I refer to the student version of the notes as ‘notes-with-gaps’ or ‘gappy notes’ in short, as this is what they were commonly called (Burn and Wood, 1995).

5.2 Introduction to my analysis

In the sections that follow I report on my analysis of the meetings data, that is the interviews with the lecturer. As described in Chapter 3, I initially used a grounded approach to analysis guided by the research questions:
• What does it mean to teach Linear Algebra at University?

• What are the strategies used by the lecturer in his teaching of Linear Algebra?

• How and why does the lecturer use these strategies? What are his intentions for student learning?

I used activity theory to organise and model the teaching of linear algebra. In Sections 5.3 to 5.5 I address the three levels of analysis in accordance with the structural elements of Activity Theory that Leontiev proposed (Leontiev, 1981). These are activity-motive at the top level, actions-goals at the second level, and at the third level the operations-conditions.

At the top level of analysis, in Section 5.3, I discuss the structural elements of activity and motive in relation to the lecturer’s teaching of this module. I enter into a discussion of the lecturer’s motivation for teaching and the knowledge and beliefs that the lecturer brought to his teaching. As a result of wanting to change students’ view of mathematics, the lecturer had decided on a more ‘inductive’ style for teaching this module. As part of my discussions in this section I include a detailed description of the lecturer’s overall approach which I termed EAG, and which stands for ‘Example - Argument - Generalisation’. I include an example to demonstrate how I interpreted the process of students gaining a different view of mathematics based on the EAG approach. Gaining a different view included gaining a more conceptual understanding of mathematics, and of the important concepts in linear algebra, in particular.

Activity comprises actions that have one or more goals. If there are no actions, then there is no Activity (Leontiev, 1981). At the second level of analysis, in Section 5.4, I used the theoretical ideas of actions and goals to relate the lecturer’s strategies and intentions as expressed in research meetings (and as a result of a partial analysis of the lecture observation data). This level of analysis relates most closely to the practicalities of teaching, the designing and planning of materials and sequences of teaching acts, and the behaviours associated with being a teacher.

At the third level, in Section 5.5, I report on my analysis in relation to the operations, that is, all those processes that the lecturer performed in order to carry out the actions. These operations were restricted, or constrained by conditions, for example by the fact that he was teaching a course to a large group of students for three hours a week.

Associated with each level of analysis I have a unit of analysis. My unit of analysis is the motive at the top level, the goal at the second level and the condition (or
constraint) at the third level. However, in any activity system activity is the all-encompassing notion and taken as the unit of analysis with which to explain human behaviour and actions in a given context or setting.

5.3 Activity and motive

At the top level of analysis I defined the teaching and learning of linear algebra inductively as Activity. I focused on the lecturer’s motive, or motivation for teaching linear algebra using an ‘inductive’ or ‘bottom-up’ approach. At this level the motive was the focus of my analysis, the single unit that I could not break into smaller units without losing the essential character of motive. Hence, at the top level of analysis the motive was my unit of analysis.

In one of the research meetings, quite early on in the study (at the start of Week 4), the lecturer said that he wanted to change the way that students viewed mathematics and what it meant to do mathematics. He said that he re-structured the first year module in linear algebra based on his knowledge of students and the difficulties that he perceived students to have with linear algebra. As a result of my analysis, I concluded that the lecturer’s decision to make changes to the linear algebra module was based on his personal knowledge and experience, that is, on his knowledge of students, and on his knowledge and beliefs of the nature of mathematics, and the nature of mathematics teaching and learning. In a research meeting he said,

I mean, problem solving is one thing, and my impression from our incoming students is that there hasn’t been a lot of emphasis on that, because rules and algorithms is what they’re good at. Anything that goes beyond that is very difficult, and my impression is many don’t have the idea even that they could try and approach a problem if the lecturer hasn’t shown them how to do it. (M3, 08:17)

In another meeting, recalling an encounter with a student after a lecture, he said,

Actually, today after the session a student came up to me asking me, he’s seen so many proofs, and he’s also seen so many proofs in Calculus, and does he really need them, because he knows how to do the calculations. I explained to him that ‘yes’, he needs them because, what I’ve said a couple of times this year, if you understand where a technique comes from you can
modify it, if you are faced with an application that isn’t quite what you’re used to. So that’s how I try to sell my students why it’s important to, not only to be able to do the calculations, but to understand exactly why they work the way they work. (M6, 03:08)

Thus the lecturer’s motive to change students’ view of mathematics related to designing teaching that was aimed at students gaining a more conceptual view and understanding of mathematics, and a less algorithmic or computational view.

In the quotations above, the lecturer referred to his knowledge of students and, based on his teaching experience in a previous year, the difficulties that students had with the module. As a result he re-structured the first year linear algebra module. He made changes to the material, putting more emphasis on the concepts in linear algebra and less emphasis on calculations and computational aspects generally. He adopted a ‘bottom-up’ or ‘inductive’ style of teaching, with a greater emphasis on informal reasoning. This was an overall approach and provided the rationale for his teaching rather than a specific strategy. I labelled this approach ‘EAG’. A ‘bottom-up’ or ‘inductive’ approach to teaching contrasts with a more traditional approach using a ‘deductive’ style (based on the formal logic of mathematics). I give a detailed description of the ‘EAG’ approach, based on my analysis of the meetings data, in Section 5.3.1.

All the changes that he made to the module were aimed at changing how students viewed mathematics, and hence how they learnt mathematics. He was aiming to engage students more conceptually with linear algebra topics, and to make students think about mathematics and mathematical problem solving like “a mature mathematician would” (see quotation M8, 18:18 below). In the lecturer’s view, studying mathematics involved developing ways of thinking and behaving mathematically. This represented his view of students and his view of the nature of mathematics and of mathematical work. His views were often most clearly expressed when he was talking about the Communicating Mathematics module, a second module that he was teaching at the same time as the linear algebra module. For example, when one researcher asked “Now, are you teaching mathematics or are you teaching mathematics education?” he replied,

I’m teaching communicating mathematics. And, . . . , I think it goes straight to the heart of what a mathematics programme is about, because there are, of course, the general things, how to structure writing, . . . , but at the same time I think this is about how you think about mathematics, and how you think about a mathematical problem. And in that sense I am asking students to think about a mathematical topic the way a mature mathematician would, and to demonstrate by presenting that they think about the topic in that
way. And because of that, I think that that module goes straight to the heart of what it means to study mathematics, and I’m really happy that we’ve got that module. (M8, 18:18; *italics my emphasis*)

Thus changing students’ point of view entailed changing their view of the nature of mathematics and how mathematicians engaged with mathematics. *Enculturation* was a term that was first introduced in a meeting by one of the researchers. The lecturer had stated that he wanted students to ‘grow into the community of professional mathematicians’, and what it meant to do mathematics. When the term ‘enculturation’ was suggested, the lecturer seized upon it saying,

> I like that word. I probably wouldn’t have, no, *certainly* wouldn’t have come up with that word. But I like it because that’s exactly what I’m after. I’m hoping students are going to change the way they think about mathematics as they change [from school] to university. . . . And so, to that extent that’s exactly what I’m hoping for. (M5, 56:18; *italics my emphasis*)

Thus, in the meetings, the lecturer stated *explicitly* that he wanted to *change* students’ way of thinking.

The motive for Activity is its *energising function*. The lecturer’s motive was to *enculturate students into mathematical practice*. That is, through teaching (and learning) linear algebra ‘inductively’ (as Activity) he wanted to introduce students to the way mathematicians work and think when engaging with a mathematical problem. The motive *drives* Activity, and for the lecturer in this study this involved designing material content for lectures, teaching sequences, etc. that mirrored the way mathematicians work when they approach a problem.

In summary, the lecturer’s knowledge of students, and what he perceived to be difficult for them in terms of mathematical learning and understanding led him to re-structure the linear algebra module at his university. He decided to teach the concepts of linear algebra that are so fundamental and crucial for students’ further study of mathematics, in an *inductive* rather than a *deductive* style (as is more traditional at university). I now describe the lecturer’s inductive style of teaching, which I termed ‘EAG’, in more detail.
5.3.1 The EAG approach

Based on my analysis, I have used the EAG approach as an overall design for developing and explaining the teaching process. The approach was closely linked to the motive of the activity. I could discuss the EAG approach at the action-goal level of analysis if it were a strategy in pursuit of a (achievable) goal. However, this was not the case. Activity in my study related to the teaching of linear algebra ‘inductively’. Thus the EAG approach was inherent in Activity, and not separate from it. It represented a framework for the didactical decisions that the lecturer made in relation to his planning and designing of the linear algebra module. I discuss the individual strategies that the lecturer designed in support of an ‘inductive’ approach at the action-goal level analysis in Section 5.4. Based on this analysis I describe the process of teaching linear algebra concepts based on the presentation of examples, and develop a theoretical model of the lecturer’s design for teaching linear algebra.

The lecturer based his approach on ‘presenting an example first, followed by a definition or theorem’. In Section 5.3 I had termed this approach ‘EAG’ where the initials stand for ‘Example - Argument - Generalisation’, and describe the process of:

- we introduce an Example,
- we make an Argument on the example, and then
- we Generalise to an observation (a definition or theorem).

The lecturer’s aim was to present an example, and then to make an argument on the example in order to derive a rule or an observation. The observation could highlight a generality in what was observed in the example. This is my interpretation of comments that the lecturer made in research meetings. For example, the lecturer made his approach explicit (in a research meeting), when he said,

Generally speaking, I decided that I would focus on doing the development of the argument on examples, and then trying to abstract a general fact from the example, as I have done in most cases so far. And so then, what I am doing is go[ing] through the example, and then highlight[ing] the important facts on the example, and then condens[ing] them into a general observation.\(^2\)

\(^1\)The motive drives Activity. The motive also gives rise to goals (forming a dialectical, i.e. mutually constitutive relationship) which are realised in and through actions.

\(^2\)This terminology agreed with the use of the term ‘observation’ in the course notes. The lecturer used the term ‘observation’ instead of ‘theorem’ in the course notes. The observation that is referred to here is a theorem, but unlike the theorem, the observation is not proved (see M12, 20:48).
And I have several times mentioned to students that this is what we’re doing, and that it’s a good idea to see an example not as an isolated example but rather as a representative of a big class. (M9, 11:58)

and

...And the way that the observations work is that, most of the time, we go over an example first and then we extract a general statement from the example we have gone through. And the entire thing is based on the observation that the same argument we have just used in the example will also apply in a very large class of other examples, which makes the step from the specific to the general. And I have discussed that step with students in class many times. (Slight pause.) And again, I don’t know how much of that actually sticks. I repeat it because I think it’s important, and then I’m hoping that students go away and start thinking about things the way I demonstrate in class. (M15, 56:10)

The lecturer made his approach explicit also to his students in lecture. For example, in Week 9 he said to his students,

We haven’t seen very many proofs in this lecture. Most of the time we have derived our observations from examples. And that’s a good thing to do. It’s a good thing always to have a look at the examples, and see if there are any general statements that we can derive from them, anything that we can learn from the examples. (L25, 43:05)

All the lecturer’s intentions and strategies that I list at the action-goal level in Section 5.4, I interpreted as describing the EAG approach as indicated above.

I demonstrate the EAG approach with an example that the lecturer presented to students in Week 5 (Example 3.14, in lecture L14). The outline of the example (e.g. in this case the vectors and the questions posed) were written out in full in the student version of the course notes. After each question (a) to (d) there was a blank space where students could enter the solution to the example. The lecturer presented the solution in the lecture. With this example the lecturer sought to demonstrate, or derive, the theorem “The range of a matrix is a subspace”. I have reproduced Example 3.14 below without the solution. (A copy of the full solution of Example 3.14 is reproduced on page 58.)
Example 3.14. Consider an unknown $2 \times 3$ matrix $A$. We know that $A$ satisfies $A\mathbf{x}_1 = \mathbf{b}_1$ and $A\mathbf{x}_2 = \mathbf{b}_2$, where

$$b_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \quad b_2 = \begin{pmatrix} -1 \\ 5 \end{pmatrix}, \quad x_1 = \begin{pmatrix} 1 \\ 3 \\ -7 \end{pmatrix}, \quad x_2 = \begin{pmatrix} 3 \\ -3 \\ 2 \end{pmatrix}.$$

(a) Is $b_1$ in the range of $A$? Is $b_2$ in the range of $A$?

(b) Is $b_1 + b_2 = \begin{pmatrix} 1 \\ 8 \end{pmatrix}$ in the range of $A$?

(c) Take the number $\lambda = 3$. Is $\lambda \mathbf{b}_1 = \begin{pmatrix} 6 \\ 9 \end{pmatrix}$ in the range of $A$?

(d) Is the zero vector $\mathbf{0}$ in the range of $A$?

Earlier in the course the lecturer had introduced the null space of a matrix $A$, that is the solution set of the homogeneous equation system $Ax = 0$. He had shown that the null space has similar properties to the set of all $n$-component vectors: It is closed under addition and scalar multiplication and contains the zero vector. With this observation the lecturer introduced the definition of the subspace. (See Sections 6.2.2 and 6.2.4 for a detailed discussion of this example.)

In relation to Example 3.14, I have interpreted the EAG approach as follows:

- We introduce an Example:

  (i) With Example 3.14 the lecturer posed four questions for students to answer. The formulation of the example included an unknown matrix $A$ and four vectors for which the lecturer gave concrete, numerical values.

- We make an Argument on the example:

  (ii) The four questions (a) to (d) were designed to lead students to recognise the correspondence between the answers to the questions and the definition of a subspace.

- We Generalise to an observation (a definition or a theorem):

  (iii) As a result of (ii), students were to arrive at, and recognise that the range of a matrix is a subspace. This was then summarised in what the lecturer called ‘Observation 3.15’. (See page 58 for the full solution of this example.)
Example 3.14. Consider an unknown $2 \times 3$ matrix $A$. We know that $A$ satisfies $Ax_1 = b_1$ and $Ax_2 = b_2$, where

$$b_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \quad b_2 = \begin{pmatrix} -1 \\ 5 \end{pmatrix}, \quad x_1 = \begin{pmatrix} 1 \\ 3 \\ -7 \end{pmatrix}, \quad x_2 = \begin{pmatrix} 3 \\ -3 \\ 2 \end{pmatrix}.$$

(a) Is $b_1$ in the range of $A$? Is $b_2$ in the range of $A$?

Solution:

$b_1 \in \text{range } A$ because the equation system $Ax = b_1$ is solvable ($x_1$ is a solution).

$b_2 \in \text{range } A$ because the equation system $Ax = b_2$ is solvable ($x_2$ is a solution).

(b) Is $b_1 + b_2 = \begin{pmatrix} 1 \\ 8 \end{pmatrix}$ in the range of $A$?

Solution: Yes. The equation system $Ax = b_1 + b_2$ is solvable, and $x_1 + x_2 = \begin{pmatrix} 4 \\ 0 \\ -5 \end{pmatrix}$ is a solution because

$$A(x_1 + x_2) = Ax_1 + Ax_2 = b_1 + b_2.$$

(c) Take the number $\lambda = 3$. Is $\lambda b_1 = \begin{pmatrix} 6 \\ 9 \end{pmatrix}$ in the range of $A$?

Solution: Yes. The equation system $Ax = \lambda b_1$ is solvable, and $\lambda x_1 = \begin{pmatrix} 3 \\ 4 \\ -21 \end{pmatrix}$ is a solution because

$$A(\lambda x_1) = \lambda Ax_1 = \lambda b_1.$$

(d) Is the zero vector $0$ in the range of $A$?

Solution: Yes. The equation system $Ax = 0$ is solvable, and $x = 0$ is a solution because $A0 = 0$.

In this example, we have verified that the range of a matrix has the three properties of Observation 3.5. We can therefore conclude:

**Observation 3.15.** The range of a matrix is a subspace.
'Observation 3.15' is the theorem “The range of a matrix is a subspace”. As stated in Section 5.1 the lecturer chose the terminology of ‘Observation’ (rather than a ‘Theorem’) because he did not give a formal proof as this point in the course. In general, proofs were provided in the second semester, when results were re-visited in the context of abstract vector space theory.

By proceeding in this manner the lecturer sought to provide students with a gentle introduction to mathematical reasoning. In a research meeting he said,

... And because I know that many of our students are unfamiliar with the way how mathematical arguments are usually phrased, and how the structure of a mathematical exposition in definitions, theorems and examples, for most of our students is something entirely new and something they find difficult. So I decided to go for an almost exclusively example-based development and go for these observations as an indication of that, which in my first semester linear algebra, I find very easy to do because I know there’s going to be a second semester which is going to make a second tour through more or less the same material in the context of abstract vector spaces, and on a much more formal level. (M9, 15:36)

In the quotation above, the lecturer refers to his knowledge of students and, based on his teaching experience in a previous year, the difficulties that students had with the module. I interpreted going “for an almost exclusively example-based development” and going “for these observations” as indicating a ‘bottom-up’ or ‘inductive’ style of teaching. This was an overall approach and contributed to the rationale of his teaching. A ‘bottom-up’ or ‘inductive’ style contrasts with a more traditional ‘DTP’ (definition-theorem-proof) style.

In this section I have discussed the lecturer’s ‘inductive’ teaching style which I termed the ‘EAG’ approach. It was an examples-based approach where theorems and definitions were stated after students had worked on an example first. In contrast, in the more traditional ‘DTP’ style, theorems and definitions are usually stated first, followed by examples that explain or exemplify the theorem (or definition), or test students’ understanding of the theorem (or definition).

I now present the second level of analysis. This is my interpretation of the meetings data in respect of the actions and the goals of Activity. I develop a theoretical model of the teaching process that links goals with associated actions.
5.4 Actions and goals

In theoretical terms there cannot be Activity without actions. Actions realise Activity. In my study the lecturer’s actions (which I labelled strategies during coding) realised and gave form to the teaching and learning of linear algebra ‘inductively’. Actions are subordinated to one or more goals. Whereas the motive energises Activity, actions are directed towards achieving a goal.

In this section I report on the second level of my analysis and interpretation of the meetings data, in respect of the actions and the goals of Activity, where the teaching and learning of linear algebra ‘inductively’ defines Activity in my study. My unit of analysis was the goal. In using activity theory I identified the lecturer’s goals and actions. Based on my analysis I associated each goal with one or more actions and developed a theoretical model of the teaching process.

During the coding stage of analysis I labelled the lecturer’s goals either as an intention, as a goal or as an aim. This indicated a hierarchy that I perceived when analysing the data, a hierarchy that became subsumed in the theoretical model that I developed subsequently. I labelled the lecturer’s actions as strategies during coding and categorised them further in terms of their ‘function’ or ‘intent’. For example, I labelled one strategy as “strategy: hands-on”. This indicated that I had perceived a comment made by the lecturer as referring to a strategy that was aimed at providing students with ‘hands-on’ experience in learning about linear algebra concepts.

For each goal I list a set of associated actions based on my initial coding analysis. I develop a theoretical model of the teaching process (see Figure 5.2). This model encapsulates and exemplifies, in my view, the EAG approach that the lecturer adopted in planning and designing his teaching.

In my analysis I identified a number of goals. I analysed how the goals related to each other, and to the actions that the lecturer said he wanted to use in the pursuit of a goal, or goals. This has been a difficult task as many goals (and actions) were interrelated giving rise to what appeared to be complex relationships. Furthermore, Activity is liable to change and develop, a fact that Leontiev (1981) pointed to when referring to Educational Activity as human Activity. Therefore, goals and actions are also liable to change and develop in changing Activity.

With reference to the diagram (Figure 5.2) the goals are arranged on the horizontal axis, and the actions are arranged on the diagonal line. The goals are arranged in a hierarchical order (from lower to higher in reading from the left to the right), whereby a higher goal cannot be achieved unless a lower one has been achieved first. For example,
the goal of engagement is a necessary first step in attaining the goal of conceptual understanding. This is an important aspect of my model which I discuss in more detail in Section 5.4.1. I provide an example of how a (lower) goal contributes to the pursuit of a next (higher) goal. In labelling the goals and the actions I stayed close to the codes that I assigned during my coding analysis (see Chapter 3).

I now discuss the model and explain each goal and the associated actions as displayed on the diagram. I include examples from my coding analysis to support the relationship(s) that I interpreted as a result, and that I show in my model.

### 5.4.1 The goals/intentions

I identified five goals based on my analysis of the meetings data. The first (in a sense ‘lower’) goal was student **engagement with mathematics**, where I interpreted engagement as *physical and mental participation in lecture*. The goal of engagement was followed by two intermediate goals: the goal of an **intuitive understanding** and the goal of the **acquisition of mathematical language**. The goal of an intuitive understanding was supported by the lecturer’s use of informal reasoning about the concepts in linear algebra, in particular in the use of examples in an examples-based approach to teaching. The goal of the **acquisition of mathematical language**, in this study related to students acquiring the language of linear algebra, its body of definitions and
These two intermediate goals were designed to lead students to the (higher) goal of conceptual understanding. Conceptual understanding relates to the ability to make connections between concepts, rather than relying on a purely computational view for solving a mathematical problem, say. Achieving the goal of conceptual understanding of mathematics was designed to lead students to become mathematically competent, in particular in respect of real-life problem solving. This (last) goal related to students becoming competent in applying rules and procedures to applications that appear unfamiliar, and in those situations to be able to adapt methods and procedures if the need arose.

As the goals (based on my analysis and interpretation of the lecturer’s comments made in research meetings) were linked in a successive manner, I developed the image of a set of nested goals. This I show in Figure 5.3.

The diagram (Figure 5.3) represents my interpretation of the dependency of the goals where, according to the model, and as I said above, a higher goal was attained only after a lower one had been attained. Thus (according to the model) mathematical competence relied on conceptual understanding in working with and solving unfamiliar problems. Conceptual understanding included knowledge of (precise) mathematical language. To acquire and be able to use mathematical language was the aim of the examples-based approach taken by the lecturer. In this inductive approach to teaching,
an intuitive understanding came before a more formal understanding of mathematics, and an intuitive understanding relied on students engaging with mathematics. In the approach taken by the lecturer this meant students engaging in the lecture and with the material (the examples, the mathematical problems) presented in the lecture. Thus in respect of an informal approach to teaching I consider the first two goals and the associated actions crucial in developing student learning and understanding so that, in turn, the higher goals can be attained.

I support my interpretation of nestedness through my analysis of the comments made by the lecturer in research meetings. I give a detailed explanation of the five goals and the actions associated with them in Sections 5.4.3 to 5.4.7.

I re-visit my analysis of actions and goals in Chapter 6 (my second analysis chapter) in the context of three concepts in linear algebra: subspaces, linear independence and eigenvectors. In Chapter 6 my foci are the actions as shown in my model but, in addition, I include also the lecturer’s actions in face-to-face teaching. I consider the role of examples and the role of language in bringing about student engagement and an intuitive understanding of the concepts in linear algebra (as pre-requisites for conceptual understanding).

So far I have discussed the goals. In the next section I discuss the actions. I explain the coding of the actions in my model, and offer a categorisation.

5.4.2 The actions/strategies

Based on my analysis of the meetings data (and drawing also on the lecture observation data) I identified a number of actions (labelled strategies in my coding analysis). I linked actions to the goals as shown in my model (Figure 5.2). The relationship between goals and associated actions was my interpretation of the lecturer’s comments made in research meetings. I have categorised the actions as employing either a material tool or a psychological tool.

Material tools are all those tools that affect a change in an object physically. In my study the material tools were, for example, the course notes that have ‘gaps’. The action of providing course notes with gaps was designed to encourage students’ (physical) participation in lecture.

Psychological tools are all those tools that can be used to engage the learner mentally, either in relation to mathematics or in relation to mathematical learning, or to how mathematics is viewed. Thus psychological tools affect the mind. I sometimes refer
to them as thinking tools. In my study the action of presenting a mathematical problem (usually having the ‘outer’ form of an example), was designed to engage the learner mentally. The ‘mathematical problem’ was a psychological tool and related to the expression of questions and mathematical ideas in the problem.

In Chapter 6 I further distinguish and categorise the tools in terms of their function in respect of student engagement and learning. For example, I distinguish between examples that were designed to fulfil a technical function, a conceptual function or a cultural function.

First I assign each tool to one of the categories as follows.

1. The material tools:

- Examples: The action of presenting an example included the use of an example as a material tool. Examples were printed in the course notes and used in the lecture. The lecturer designed examples in order to engage students (in a physical sense) in the lecture and with the mathematics.

- Course notes with ‘gaps’: The lecturer provided students with a set of course notes that had gaps where students could write the solution to examples.

- ‘Breaks’ in the lecture: The lecturer made time available in lecture, that is, he provided a break from the flow of information that he presented. The lecturer wanted students to work on a piece of mathematics in the lecture and have the opportunity to interact (either with the lecturer or fellow students).

- The context of vector space $\mathbb{R}^n$: All examples used in lectures and all exercises set on problem sheets, etc., including any definitions and theorems, were presented in the context of the vector space $\mathbb{R}^n$. In my interpretation the vector space $\mathbb{R}^n$ is a material tool as using it affected the layout of the course notes, definition, theorems, etc. In referring only to the vector space $\mathbb{R}^n$ the lecturer simplified the material content of the module.

2. The psychological tools:

- Mathematical ‘problems’: The action of ‘presenting mathematical problems’ included the use of mathematical problems as a psychological tool. The lecturer posed mathematical problems or questions (having the ‘outer’ form of an example) for students to solve or to think about. The problems were stated and sometimes broken down into individual tasks that students could follow and solve.

- Mathematical content (of coursework and problem sheets): As was the case with mathematical problems above, the action of designing coursework and
problem sheets included the use of mathematical content (questions, problems, statements of theorems, for example) as a psychological tool. Coursework and problem sheets contained exercises for students to do outside of lectures. Coursework related to (formal) assessments whereas problem sheets related to non-assessed tasks.

- The language of linear algebra: The lecturer often referred to the language that students had to learn, meaning the language of formal mathematics in terms of definitions and theorems. Mathematical language provides a precise (and concise) formulation of a mathematical idea or concept. The action of using the language of linear algebra consisted of the use of the language of linear algebra as a psychological tool.

- Spoken language: The action of verbalising intentions was ‘special’ as it spanned all goals. Verbalisation comprises the use of spoken language, by the lecturer in addressing students in the lectures. Spoken language is a psychological tool.

In Sections 5.4.3 to 5.4.7 I discuss each goal and the actions (and tools) that I associated with each goal. I present evidence in the form of comments that the lecturer made in research meetings. In addition I provide evidence from the lecture observation data. This relates to comments that the lecturer made directly to his students in lectures. Some actions became ‘visible’ or were ‘more clearly visible’ in the lecturer’s face-to-face teaching of his students.

As I said above, the action of verbalising intentions was ‘special’ as it spanned all goals. I included this action in my discussion of the goals in Sections 5.4.3 to 5.4.7. However, the action of verbalising intentions had, as I will show, particular features in relation to this lecturer’s teaching. I, therefore, discuss this action again (separately) in Section 5.4.8 where I relate the lecturer’s comments to different levels of commenting and meta-commenting.

5.4.3 Goal 1: Engagement with mathematics

In my analysis I interpreted students’ engagement with mathematics as one of the lecturer’s goals. With reference to my theoretical model (Figure 5.2, on page 61) I associated this goal with the actions of presenting examples, providing course notes with ‘gaps’, making breaks in the lecture and verbalising intentions.

The lecturer expressed the goal of engagement with mathematics both in research meetings and in lectures. In research meetings the lecturer talked about wanting his
students to ‘engage’ and ‘get some hands-on experience’ (M12, 13:47), to ‘get a feel’ (L15, 05:12), by ‘playing with the concepts’ (of linear algebra) (M9, 00:26). For example, in a research meeting he said,

I’m not thinking so much about the class test, but maybe having some more ‘hands-on’ experience with what linear independence is about would have made it easier to go over the lecture on rank-nullity. (M9, 04:37)

The lecturer expressed the goal of engagement with mathematics also directly to his students in the lectures. For example, in Week 6 he said,

And also you should have had a look at your problem sheet where I put down a couple of problems asking you to do something with these terms. And the purpose of these problems and the purpose of the new problems for this week is that you get some experience in working with these objects and these concepts and you get a feel for how they fit together and how they work. (L15, 05:23)

I interpreted the use of the words “the purpose of these problems” as indicating a goal, and “that you get some experience in working with these objects” as an indication of engagement with mathematics. I linked the goal of engagement with the action of presenting examples (e.g. “where I put down a couple of problems”). I interpreted examples as a material tool and include exercises that the lecturer set in lectures or tutorials, or on problem sheets. The lecturer designed problem sheets that were to be completed outside of lectures and that formed either homework or part of written (assessed) coursework.

I associated the actions of providing course notes with ‘gaps’ and making breaks in lecture with the goal of engagement with mathematics (see Figure 5.2). In research meetings the lecturer talked about structuring the module to include printed course notes that had ‘gaps’ (where students could write the solutions to examples) and to provide the time in lecture for students to do that. He said,

But obviously in the lecture I wouldn’t want to give them any big pieces of work to do, rather things that are reasonably quick to do, just to see that they’re still with me and they pick up the important points and also, well, I have to experiment with that, but I am hoping to give students simple examples to do or ask questions of them before I myself show them the first example of how something is done. (M1, 06:55)
In a research meeting, in relation to the goal of conceptual understanding, the lecturer said that he wanted to ‘encourage students and challenge students to do that’ [to ‘engage’ more conceptually]. He said,

On the other hand, also that’s very difficult, and that’s, . . . that’s why I am trying to find ways to encourage students to get engaged with the material in that way. And that’s also one of the main reasons why I give them, to some extent, exploratory questions, examples to do in the lecture themselves, before I show them the first example of how things work. And as we have seen there are quite a few students who take that up, and who do it well, there are students who just stare at it and say they have not the slightest idea what to do, and there are students in the lecture who don’t even try. (M15, 46:42; italics my emphasis)

Thus the goal of engagement with mathematics included a sense of physical as well as mental engagement.

I have stressed that the theoretical model arose from my analysis of the meetings data. However, in this section (and subsequent sections) I include comments that the lecturer made in lectures. This is a deliberate act on my part. In including comments from lectures I indicate that the lecturer stated his intentions (that is, his goals) explicitly also to his students in lectures. I termed this action verbalising intentions. It was the one action that spanned all goals (see Figure 5.2). Because of future references that I make in relation to ‘commenting’ and ‘meta-commenting’ levels, I thought it desirable (and even necessary) to include quotations from my analysis of the lecture observation data, in this section and subsequent sections. I discuss the action of verbalising intentions separately (see Section 5.4.8), as I said previously.

In my interpretation and presentation so far, I have related actions to the goal of engagement (with mathematics). However, I have also identified instances where the lecturer related the goal of engagement to the next (or other higher level) goal. For example, the lecturer said,

. . . and the purpose of the new problems for this week is that you get some experience in working with these objects . . . and you get a feel for how they fit together and how they work. (L15, 05:23)
Here I interpreted the lecturer’s comment as linking two goals. The goal of engagement with mathematics ("working with these objects") represented a necessary step towards attaining the next (higher level) goal of an intuitive understanding (to "get a feel").

I present a second example to demonstrate the relationship between the goals.

In a research meeting (M18, 25:18) one of the researchers posed the question whether conceptual understanding could not also come through repeatedly practicing procedures. The researcher asked,

> If you do it lots and lots of times, won’t the concept come to you?

To which the lecturer replied,

> Perhaps. Probably not automatically, but one of the assumptions, or one of the hopes of the way I’ve taught this semester, is of course that it will, and that’s why I put in a lot of different examples that show phenomena in a different way, and I phrased essentially the same question in different ways at different points. I think, it won’t come automatically, but if it is to come it can only come from working with the objects. (M18, 25:18)

I interpreted the statement “if it is to come it can only come from working with the objects” as expressing the goal of conceptual understanding in terms of the goal of engagement with mathematics. By attaining the goal of engagement the lecturer envisaged that students progressed to the higher goal of (conceptual) understanding of the concepts of linear algebra. Thus I interpreted the goal of engagement with mathematics as a pre-requisite for attaining the goal of conceptual understanding. I indicated this relationship with Figure 5.3, and with my theoretical model (Figure 5.2 on page 61) by placing an arrow from the first goal to the next (higher) goal.

### 5.4.4 Goal 2: Intuitive understanding

Based on the analysis and interpretation of my data I identified the goal of intuitive understanding. With reference to my theoretical model (Figure 5.2) I associated this goal with the actions of presenting a mathematical problem (having the ‘outer’ form of an example), and formulating all problems, definitions and concepts in the context of the vector space $\mathbb{R}^n$. I also associated the action of verbalising intentions with this goal.
In my interpretation and as I discussed in Section 5.4.3, the goal of engagement with mathematics was a necessary step towards achieving the goal of an intuitive understanding.

In research meetings the lecturer said that he wanted students to ‘get a feel’ (M2, 44:06) or ‘develop a feel’ (M8, 06:56) for the concepts in linear algebra, and ‘to make summaries of that sort’ (M2, 18:29), meaning summaries of an informal or intuitive nature. In a lecture in Week 2, the lecturer presented the method of Gaussian elimination for solving linear equation systems. In a research meeting, when talking about teaching the topic, he said,

But the reason why I’m asking them actually to solve things by hand is so that they get a feel of what can happen, and so that they get a feel for how the algorithm works, and what the different cases are that can happen. Ultimately we’re going to diagnose the solvability, inconsistency, unique solvability or number of free parameters of a linear equation system from the echelon form. And we will see that those are all possibilities. (M2, 44:06; italics my emphasis)

I interpreted the lecturer’s comment “so that they get a feel” as expressing the goal of intuitive understanding. I associated the action of presenting mathematical problems with the goal of an intuitive understanding. With this action the lecturer used a mathematical problem as a psychological tool. I referred to a mathematical problem as the formulation of a question or exercise to students in mathematical terms (i.e. mathematical notation or language).

The second action that I associated with the goal of an intuitive understanding was the action of formulating in the context of vector space $\mathbb{R}^n$. The lecturer decided to formulate all linear algebra concepts, all definitions and theorems, etc. in Semester 1 in the context of the vector space $\mathbb{R}^n$. The vector space $\mathbb{R}^n$ denotes the space of all $n$-tuples, or the space of all $n$-component column vectors. Formulating linear algebra concepts in the context of the vector space $\mathbb{R}^n$ was an action designed to direct students towards an intuitive understanding (“develop a feel”) which the lecturer expressed in a research meeting as follows:

So that’s ... what I’m aiming for is to talk about these linear combinations, and linear independence, these crucial concepts, in the context of column
vectors where most people feel comfortable they can calculate with them. . . . That’s why I asked them today, and I ask them again on the problem sheet, “Write this vector as a linear combination of the other vectors. Can it be done?”, because I’m hoping that students . . . will develop a feel for what it means that one vector is a linear combination of others, and that vector is not. And also, that’s also why I’m putting the emphasis on, where I can, putting the emphasis on really what we’re talking about . . . (M8, 07:34)

By focussing on the vector space \( \mathbb{R}^n \) the lecturer wanted to direct students’ attention away from the computational and towards the more conceptual aspects of linear algebra. (See Section 6.2.3 for a detailed discussion of this action in relation to the concept of a subspace.) I interpreted the vector space \( \mathbb{R}^n \) as a material tool, rather than a psychological tool. The course notes were written often using, for example, only column vectors in \( \mathbb{R}^2 \) and \( \mathbb{R}^3 \). This also resulted in mathematical language that was ‘simpler’ than it would have been if the course notes had been formulated in the context of abstract vector space theory. The vector space \( \mathbb{R}^n \) as a tool affected the formation of the course notes and the language used (in the course notes and in lectures) and hence was a material tool.

The third action that I associated with the goal of intuitive understanding was the action of verbalising intentions. That is, the lecturer made the goal of an intuitive understanding explicit in lecture by telling his students of his goal. For example, in a lecture in Week 6 the lecturer said,

And also you should have had a look at your problem sheet where I put down a couple of problems asking you to do something with these terms. 
And the purpose of these problems and the purpose of the new problems for this week is that you get some experience in working with these objects and these concepts, and you get a feel for how they fit together and how they work. (L15, 05:23; italics my emphasis)

Thus I associated three actions with the second goal in my model, the goal of an intuitive understanding. As discussed in Section 5.4.3 and as set out in the diagram on page 62, I considered the goal of engagement with mathematics as contributing and supporting the goal of an intuitive understanding. That is, attaining the goal of engagement with mathematics was a pre-requisite for attaining the goal of an intuitive understanding.
5.4.5 Goal 3: Acquisition of mathematical language

I identified the *acquisition of mathematical language* as one of the lecturer’s goals for student learning. With reference to my theoretical model (Figure 5.2, on page 61) I associated this goal with the actions of *verbalising intentions* and *designing content for coursework/problem sheets*.

The lecturer expressed *acquiring mathematical language* as a goal for students’ learning. In research meetings he talked about students needing to ‘learn the language of linear algebra’ (M12, 02:03; M6, 12:22; M7, 01:51), to ‘get fluent in that notation’ (M5, 04:08), and to be ‘able to read a definition’ (M5, 27:15). For example, in a research meeting he said,

Yes, of course it’s [learning the language is] important because they’re [the students are] supposed to start reading mathematics on their own. And our students are very slow at that, and many probably even when they graduate, couldn’t take up a mathematics book and read it. But that’s why I’m putting a lot of emphasis on that language. (M12, 02:03)

and,

And what we are going to see over the next couple of weeks is a reformulation of linear equation systems in ever-new guises. That means there is a lot of new language coming up, which students will probably struggle to absorb. (M6, 12:22)

In lectures he emphasised the importance of mathematical language when addressing his students. For example, in a lecture in Week 5 he said,

That’s [Subspace is] an important new idea and also an important new word that you need to learn to use. And there’re a couple of more new words that I need to give you. . . .So these are important observations and important new words. And you will, as we go on through this chapter, you will find a lot of new words that I need to introduce to you, a lot of new words in which we talk about solutions to linear equation systems, and it’s very important that you learn how to use these words properly and to speak that language because this is the language in which we will be able to formulate observations, theorems that are much more general than we observe here.
And if you look at the problem sheet for this week there are a lot of things that are phrased in the new terminology that I’m introducing here and that I will introduce tomorrow. The calculations that I’m asking you on this problem sheet are all very easy. All questions on that problem sheet either ask you to do a certain matrix operation or they ask you to solve a linear equation system. So the calculations are things that you really know how to do. But the important thing is that you spend the time to find out what that language means. If you understand the language that we’re using for linear algebra you’ll almost certainly be able to do everything very well. But it’s important that you be able to understand the language we’re using and to use it properly. So please, pay attention to the new terms and the new ideas that we’re going to introduce over this chapter. (L12, 43:25)

The lecturer was referring to the terminology of linear algebra, the formulation of concepts in terms of definitions and theorems and general mathematical notation. For example, students needed to understand the notions of subspace, spanning set and linear independence. The lecturer stressed that the formal definitions were important and necessary as they provided a precise formulation of the concept or idea that was being discussed. In a research meeting the lecturer referred to the formalism of mathematics as ‘where the power of mathematics came from’ (M15, 44:28).

The lecturer made the goal of acquisition of mathematical language explicit in lectures when addressing students. For example, in lecture L12 (43:25) which I have already quoted above or in lecture L15 when he said,

I am hoping that you reviewed your lecture notes and you are comfortable now speaking in these terms and using this language. (L15, 05:12)

Making his goals explicit to his students related to the first action that I listed at the beginning of this section, the action of verbalising intentions.

The second action that I associated with the goal of students acquiring mathematical language was the action of designing content for coursework/problem sheets. With the term ‘coursework’ I referred to all assessed examinations and tests, while ‘problem sheets’ referred to formative and non-assessed exercises. In designing mathematical content for coursework and problem sheets, the lecturer’s aim was for students to practice and acquire correct and precise mathematical language. In a research meeting, he said,
So, this is on the one hand, this is, because I think, as I said, that reading text is an important skill and engaging with the language. On the other hand also this coursework is intended to send the message to the students, I keep saying ‘Using, being able to use the language is important’, and I mean that. And for that reason I’m also willing to assess you on that, if only to 1% of the module mark, . . . (M15, 1:06:16)

With reference to the theoretical model (Figure 5.2) that I created based on my analysis of the lecturer’s comments, I considered the goal of an intuitive understanding as a prerequisite for students acquiring mathematical language. Similarly, and in turn, acquiring correct mathematical language (and in particular the language of linear algebra) was a necessary step towards students’ conceptual understanding of the topic of linear algebra.

5.4.6 Goal 4: Conceptual understanding

Based on my analysis the goal of conceptual understanding was associated with the action of using the language of linear algebra, as well as the actions of verbalising intentions and designing content for coursework/problem sheets.

In research meetings the lecturer frequently talked about encouraging students to engage more conceptually with the material. He acknowledged that students found conceptual work difficult but that it was desirable and necessary that students worked conceptually during their undergraduate studies. The goal of conceptual understanding was closely linked to the goal of becoming mathematically competent, that is, being able to solve problems in unfamiliar situations. This was the next (higher level) goal that I discuss in the section that follows (Section 5.4.7).

The lecturer expressed conceptual understanding as a goal when talking about students’ difficulties with linear algebra. For example, in a meeting he said,

That’s the fundamental problem that we have to deal with here, to find that balance between . . . challenging students to get involved conceptually with the material and, on the other hand, asking too much of them. And that’s also, . . . I mean, to some extent our students come in and they probably never have been asked to do conceptual work on mathematics before. And so that’s not only a problem with that conceptual material being hard, it’s also a problem of what does a student expect work on mathematics to be like. (M15, 53:03; italics my emphasis)
I interpreted “challenging students to get involved” as expressing a goal, and “challenging students to get involved conceptually” as expressing the goal of conceptual understanding.

In lectures, in addressing students, the lecturer referred to ‘focussing on the ideas’ which I interpreted as ‘focusing on the concepts’ in linear algebra. Thus the lecturer made the goal of conceptual understanding overt also to students.

The goal of conceptual understanding was associated with the action of using the language of linear algebra. In research meetings, and in lectures directly to his students, the lecturer frequently talked about the importance of using ‘the language of linear algebra’ and ‘needing that language to get a hold of the concepts’. For example, in a research meeting the lecturer said,

I set up something like that last year and, which I thought went surprisingly well, so I’ll see to what extent we can extend that. But it certainly remains very difficult. And also, I mean, this is one of the reasons why I’m putting so much emphasis on the language of linear algebra here because that’s really what you need in order to get hold of the concepts. And . . . I’m hoping that this is going to help students get there, think about these formulae. (M8, 26:44; italics my emphasis)

I interpreted “in order to get hold of the concepts” as expressing conceptual understanding as a goal. The expression “that’s [the language of linear algebra is] really what you need” related to the action of using the language of linear algebra. This action involved the language of linear algebra as a psychological tool. This action was made overt in a lecture, that is the lecturer verbalised his intentions to his students. He said,

And as I told you yesterday I’m introducing a lot of new words, and the challenge really is not in doing the calculations, actually you’re all very familiar by now with solving linear equation systems, the challenge is in understanding what the language means, and when I give, . . . when I ask you to phrase in this new language you need to find out what the calculation does when I’m actually asking you to do, and in this case the calculation that you need to do is to solve a linear equation system. (L13, 20:00; italics my emphasis)
I interpreted “understanding what the language means” as expressing (mathematical) language as a psychological tool in gaining conceptual understanding.

I associated a third action with the goal of conceptual understanding, the action of designing content for coursework/problem sheets. With reference to Section 5.4.5 I defined coursework as referring to all assessed examinations and tests, and problem sheets to non-assessed exercises (used for homework or presented in lectures, for example). In a research meeting, while acknowledging that conceptual work was difficult for students, the lecturer said,

And for that reason, . . . it’s not only about giving the student sufficient help so that they can try and work on the abstract material on a more conceptual level, it’s also about encouraging them actually to do that. And I have given in the tutorial problems, in the problem sheets, the homework, and also in the tutorial sheets that we do in the Friday sessions, I have given them a lot of examples that show the important phenomena that we’re talking about, and quite often I have asked additional questions which I hope are encouraging students to look at the example and see what they can learn from the example in a more general sense. (M15, 54:18; italics my emphasis)

In this research meeting the lecturer stated that the content that he devised for students to work on was aimed at encouraging students to take a more conceptual view of linear algebra. Thus the content of “tutorial problems, problem sheets, homework, etc.” represented a psychological tool as part of the lecturer’s action of designing content for coursework/problem sheets.

In another meeting the lecturer referred to the kind of questions that were asked in the formal (end of year) examinations. He said,

But then the question is what you examine. And I think, I quoted my second year students, in Communicating Mathematics who said, ‘you want us to do what they always tell you and which you don’t need because it’s not on the exam’. And that means, and basically all in the department know that the central piece of exam preparation for most students is going over the old exam. That means if we want them to show some deep thinking we’ve got to put it on the exam. And that’s very difficult to do. (M8, 25:26; italics my emphasis)
I interpreted “deep thinking” as relating to conceptual understanding, and “it” in the expression “we’ve got to put it on the exam” as referring to content, that is mathematical questions and materials, that form part of formal examinations. With this comment the lecturer linked the action of designing mathematical content with the goal of conceptual understanding. The action included the use of mathematical content as a psychological tool.

5.4.7 Goal 5: Mathematical competence

The goal of mathematical competence was the last or highest level goal in my model. I associated this goal with one action only, the action of verbalising intentions. The lecturer frequently talked about skills training that focussed on computations and algorithms as insufficient in enabling students to deal with problem solving in a variety of situations and contexts. In a research meeting he said,

I think that’s one of the big steps that students need to take from A-level mathematics to degree level mathematics. There is, one thing is, to be able to do a calculation that you have been shown how to do, and the other thing is to learn a set of concepts with which you can work and which you can potentially apply to problems that are not precisely what you have been shown. And so, I took the opportunity there to explain to that small group that, ideally, at the university level they should be able to understand the ideas that go into the way the calculation works, and then be able to adapt that to whatever problems they are faced with. (M2, 48:18; italics my emphasis)

I interpreted to “apply to problems that are not precisely what you have been shown” as expressing the goal of mathematical competence, and “to understand the ideas” as expressing conceptual understanding. Gaining conceptual understanding was a previous goal. Thus students needed to achieve the goal of conceptual understanding first, in order to become mathematically competent.

The lecturer expressed the goal of mathematical competence to his students in the lecture which related to the action of verbalising intentions. In research meetings, he said that he considered this goal important as a general life skill. Thus the lecturer related the goal of mathematical competence not only to students who aimed at becoming research mathematicians but to all students graduating with a mathematics degree. He said,
Yes, I mean, most of our students probably are not planning to become research mathematicians, and even then I think it’s the idea of viewing a problem as something that you can think about and try different ways about and try to crack in some way or other, I think that could be one of the important qualifications that we give our students. They might not be, they might not be dealing with mathematical problems later in life the way that our problems here are, but they’ll certainly be dealing with problems. (M9, 24:32)

and,

I do think we should challenge our students to adopt more abstract and certainly more conceptual views of things because that’s where the power of mathematics comes from. As [a colleague] pointed out that day when we had the discussion over lunch with him. You are given a certain mathematical tool that allows you to attack a certain set of problems. And then, when you’re faced with a problem you have to solve, most of the time you have to adapt your procedure to the specific problem that you’re faced with. And to do that you need a sufficiently deep understanding of what the different pieces, what the different concepts are and how they fit together. (M15, 44:51)

Based on my data analysis and subsequent interpretations the lecturer considered conceptual understanding a key element in students becoming what he called mathematically competent. He described mathematical competence in terms of being able to apply mathematical knowledge in situations that are wholly unfamiliar. In order to be able to accomplish this students needed a sufficiently deep, that is conceptual understanding of the mathematics.

This completes my discussion of the five goals that I identified in my analysis of the meetings data and the lecture observation data. With reference to my theoretical model I related each goal to a number of actions. I provided evidence to support my analysis and interpretations in the form of comments that the lecturer made either in research meetings or in lectures.

I now discuss the action of verbalising intentions in more detail. This action related to my analysis of the lecture observation data, that is, of the audio-recordings of the lectures. I associated the action of verbalising intentions with each of the five goals
that I had identified and represented in my theoretical model. However, the action of verbalising intentions related to communicating with students (in the lecture) which included other forms of communication. This is the focus of the next section.

5.4.8 ‘Verbalising of intentions’

As I showed in Sections 5.4.3 to 5.4.7, the action of verbalising intentions spanned all goals (see also Figure 5.2). In research meetings the lecturer frequently talked about telling students of his intentions. This related to telling students (in lecture) what they should focus and what was important in respect of the linear algebra material that he presented in lectures, and giving reasons for why it was important. The lecturer talked about it in the research meetings and also told his students in lectures. For example, in a research meeting in Week 2 the lecturer talked about planning his teaching of the method of Gaussian elimination for solving systems of equations. He said,

But the reason why I’m asking them actually to solve things by hand is so that they get a feel of what can happen, and so that they get a feel for how the algorithm works, and what the different cases are that can happen. Ultimately we’re going to diagnose the solvability, inconsistency, unique solvability or number of free parameters of a linear equation system from the echelon form. And we will see that those are all possibilities. (M2, 44:06)

Three days later, he introduced and demonstrated the method of Gaussian elimination in the lecture and said to his students,

Now if that is the case, and if we can use a computer so easily to solve a linear equation system, then why am I asking you to do it by hand? That’s hard work after all. Well, the reason why I’m asking you to do that is that you get a feel for what can happen, you get a feel for how the algorithm progresses and what the different cases are. (L4, 37:19)

Thus the lecturer expressed his goal of intuitive understanding (“get a feel”, see Section 5.4.4) in a research meeting as well as directly to his students in the lecture. The lecturer did this for each of the goals. In Sections 5.4.3 to 5.4.7 I provided ‘evidence’ for the lecturer’s action of verbalising intentions in relation to each of the five goals.

My focus was the action of verbalising intentions taken by the lecturer (in lectures) and in pursuit of his goals (as exemplified in my model, Figure 5.2). In lectures and
tutorials he was *telling his students* what his goals were; he made his goals *overt*. For example, he expressed what he wanted students to concentrate on and why, what was important in terms of the mathematics that he had presented and why it was mathematically important.

My focus in this section is not the analysis of the meetings data (where the lecturer *said* to us, the researchers, that he will tell students of his goals). My focus is to report on the styles of talking in lecture, that is my focus is language as a means of communicating mathematical ideas. I distinguished the styles in terms of their *function* in relation to students’ learning.

In my analysis of the audio-recordings of the lectures I identified five different styles, or *modes of talking*. I refer to these as *commenting* and distinguish between *comments about mathematics* and *comments about the learning of mathematics* as follows:

(a) Expositions: The lecturer recited the mathematics as it was printed in the course notes without altering the formulation of the definition or theorem, that is, there was *no* level of ‘explaining’ in the delivery.

(b) Comments about mathematics: The lecturer *explained* the mathematical idea, definition or theorem. He used his own words rather than repeat or read out the definition or theorem, say.

Here I use a quotation where the lecturer changed from style (a) to style (b) and then back again to style (a). I show style (a) in normal font and style (b) in italic font.

So these are the three important properties of determinants that we need: The determinant is linear in every row, the determinant changes sign if I interchange the two rows, the determinant of the identity matrix is 1. _Three important properties of the determinant and now what we're going to do, we'll turn these three properties into a definition. We say, if we have a large square matrix we're going to associate a number with this square matrix and we call this number the determinant. And we do that in such a way that the determinant satisfies these three properties: the determinant is linear in every row, if I interchange two rows the determinant changes sign, the determinant of the identity matrix is 1._ (L24, 11:00)
(c) Meta-comments about mathematics. The lecturer made a statement that was a level above that of (b). He gave a reason why a piece of mathematics was important or where or how it was useful.

I’m going to ask you now temporarily to forget everything that you know about determinants. We’re going to start afresh, and I am going to explain determinants in a way that probably is new to you. . . . We’re going to focus on the properties that determinants have much more than on explicit formulas with which we can calculate them. And it turns out that the properties of determinants . . . are a very simple, and they provide a much more convenient manner to think about determinants than an explicit formula ever could, because formulas for determinants are quite complicated. (L24, 04:11)

(d) Comments about the learning of mathematics: The lecturer explained what students should focus on when engaging with a concept, definition or theorem.

Here I use a quotation where the lecturer changed between styles (a), (c) and (d). I show style (a) in normal font, style (c) in italic font, back to style (a) in normal font and then to style (d) in italic font.

So the determinant of an invertible matrix is always non-zero, the determinant of a non-invertible matrix is always zero. And that’s exactly what we wanted, that’s why we started to discuss the determinants in the first place, to get criteria for when a matrix is invertible. And we have now derived from the three properties that this is indeed the case. The determinant of a matrix is zero if and only if that matrix is not invertible. I have written up that argument again in your notes, so please look it up, and once again you should be aware of how the argument progresses, and that we’re really using only elementary row operations in every step and we think about what can possibly happen and the general theorem comes out . . . if we think about what can happen when we carry out the calculation. (L25, 26:44)

(e) Meta-comments about the learning of mathematics: The lecturer explained why he asked students to focus on a particular way of engaging with a mathematical concept or theorem.

Here I present a quotation that represents and captures this level of commenting well.

And also you should have had a look at your problem sheet where I put down a couple of problems asking you to do something with these
The action of verbalising intentions related to two modes of talking, (d) and (e). In comparing the modes above with my coding of the lecture observation data, I concluded that the lecturer expressed his goals to students by using the levels of commenting about the learning of mathematics and meta-commenting about the learning of mathematics.

I now relate the action of verbalising intentions, and the two levels of commenting that I associated with it, to the levels of commenting presented by Jaworski, Treffert-Thomas and Bartsch (2009). This analysis took place while the study was ongoing. The focus was on how the lecturer spoke in meetings about his didactical planning and thinking and how he communicated these didactical considerations to students in the lecture. Thus the analysis was of both meetings and lecture observation data. In analysing the meetings data Jaworski et al. (2009) distinguished between the lecturer talking in expository mode (talking about his own conception of the material) and didactic mode (talking about his conception of the teaching of the material). The first was a more factual account of the mathematics to be taught and the second included a value judgement as to what the lecturer perceived as being important for students’ learning. In analysing the lecture observation data Jaworski et al. (2009) distinguished between meta-comments A (what students need to attend to in their work, the mathematics) and meta-mathematical comments B (what students need to attend to in terms of their understanding, the learning of mathematics).

In comparing the two analyses, I interpreted statements that were coded meta-comment A in Jaworski et al. (2009) as meta-comments about mathematics, and equivalent to style (c). I interpreted statements that were coded meta-comment B in Jaworski et al. (2009) as comments about the learning of mathematics, and equivalent to style (d).

5.4.9 A brief summary of this section

In summary, in this section (Section 5.4) I discussed the actions and the goals that I had identified in my data analysis. I presented the theoretical model that I developed as a result of my analysis. I discussed each goal and the actions that I associated with each
goal, as depicted in my model (see Figure 5.2). I justified my model by presenting ‘evidence’ in the form of comments made by the lecturer in research meetings or in lectures. I related the goals to each other and developed the notion of a set of nested goals (see Figure 5.3). Although the model arose from my analysis of the meetings data it became more fully developed as a result of my analysis and interpretation of the lecture observation data. For example, I identified the action of verbalising intentions very clearly from my analysis of the audio-recordings of the lectures where it had been less apparent in the meetings data.

In Chapter 6 I re-visit the actions and goals that I identified in this section and relate them to three specific linear algebra topics: subspace, linear independence, and eigenvectors and eigenvalues.

This completes the second level of the structural analysis of my data. I now report on the third and last level of analysis. This is my interpretation of the meetings and lecture observation data in respect of the operations and conditions of Activity. At the third level of analysis I identified the conditions and four different types of operations.

5.5 Operations and conditions

At the third level of analysis I identified the condition(s) under which the actions (that I discussed at the second level of analysis) were carried out. In activity-theoretical terms actions are realised by operations. In my analysis of both meetings and lecture observation data, I identified a set of operations, that is ‘processes’ that the lecturer needed to perform in order to carry out an action. Whereas actions are determined by their goal orientation, operations are identified by their dependence on the conditions. My unit of analysis is the condition. The conditions are embedded in cultural, historical and social factors and limit the extent of an action. I, therefore, also refer to conditions as constraints.

The condition under which the lecturer taught the first year linear algebra module was the university lecture, and the university lecture format, in particular (e.g. teaching a large class of university students sitting in a tiered theatre). The constraints imposed by the lecture format were ‘physical’ in one sense (e.g. it was difficult to engage with individual students in such a large class), and ‘social and cultural’ in another (e.g. there were certain traditions and expectations by staff and students alike as to what ‘ought’ to happen in a lecture). The condition was a mathematics lecture. Thus the focus was the subject area of mathematics which brought with it traditions, assumptions
Figure 5.4: Depicting four operations under the condition of ‘a mathematics lecture’

and expectations (again by staff and students) of what it means to teach and/or learn mathematics.

The operations represent the ‘technical’ side of an action. In my analysis I found it often difficult to distinguish between an action and an operation. I interpreted a process as an operation if it appeared ‘automatic’, that is, the lecturer performed the process ‘without too much thinking’ or ‘without thinking too deeply’ about it [my words in quotation marks]. As part of the conditions of working at a university the lecturer performed certain duties. These included, but were not limited to, carrying out research in his field of mathematics, preparing and teaching a course in linear algebra and administrative duties associated with research and teaching. As part of teaching the module in linear algebra, the lecturer prepared examples that he presented in lectures, problem sheets for students to complete as ‘homework’ (outside of lectures), coursework and computer-based tests that contributed to assessment and a final examination.

I have defined an operation in a ‘narrow’ sense. I consider the actions in educational activity as consisting of well structured sequences of events, in order to bring about student learning and understanding. I, therefore, regard a lecturer’s ‘teaching acts’ as deliberate and conscious actions which have one, or more goal(s). Operations, on the other hand, are necessary in carrying out an action.

As a result of my analysis I identified four types of operations that I presented in a diagram (Figure 5.4).

The first operation related to the lecturer ‘delivering’ the lecture in person. I coded this operation maintaining a physical presence. The lecturer had to “be there” physically in the lecture since it was not, for example, an on-line lecture. The lecturer came to the lecture hall and, typically, stood in front of the class while delivering the
lecture. For some of the time during the lecture (while students were working on a
task, for example) the lecturer also walked around the lecture theatre interacting with
students.

The second operation related to the lecturer’s ‘mode of delivery’, i.e. talking to his
students. I describe the way that the lecturer most frequently talked to his students as
‘uni-directional’, or in the form of monologues. Within this category of talking I have
identified different modes, or levels of talking. These were not operations but formed
part of the lecturer’s actions. I referred to commenting and meta-commenting levels (see
Section 5.4.8 for a detailed analysis).

The third operation also related to the lecturer’s mode of delivery, i.e. writing during
the lecture. Whenever the lecturer was writing in a lecture or tutorial he was (most
usually) writing the solution to an example on the overhead projector.

The fourth, and last operation related to the mathematical content provided by
the lecturer in lectures. This included the materials for students’ use (such as the
course notes) and the presentation of examples in lectures. The way that examples were
introduced and dealt with (the ‘how’) was again related to a goal and as such formed
an action.

I based my analysis on the condition that it was a mathematics lecture and on the
definition of an operation that I gave above. I consider the four operations as ‘processes’
that the lecturer needed to perform in carrying out an action. I exemplify how the
operations related to an action with two examples. I draw on my analysis in Section 5.4
and present one action that involved the use of a material tool (presenting examples),
and one action that involved the use of a psychological tool (verbalising intentions).

The action of presenting examples

In Section 5.4.3 I discussed the goal of engagement with mathematics. I associated
this goal with several actions, including the action of presenting examples (see Figure
5.2 on page 61). In lectures and tutorials the lecturer presented a variety of examples.

In my analysis and interpretation, the operations that the lecturer needed to perform in
carrying out this action involved all four operations. He prepared examples to present
in lectures and tutorials that were related to linear algebra topics, that is, he provided
mathematical content. He explained the example and instructed his students in working
on the example which involved the operation of talking. In order to instruct students he
needed to be physically present in the lecture. In most cases when the lecturer presented
an example he also worked through the solution on the overhead projector. Hence the action of presenting examples involved the operation of writing.

The action of verbalising intentions

With reference to my analysis in Section 5.4 the action of verbalising intentions spanned all goals. The type of intention verbalised by the lecturer sometimes centred on the mathematical topic being taught, and sometimes on the approach to learning a specific topic. The operations used in this action involved talking to students in class (talking, physical presence) and, in the case where the intention was focussed on the mathematics, on providing mathematical content.

Apart from the lectures all students taking this module also attended a weekly tutorial with the lecturer. In tutorials the lecturer did not present any new material. However, tutorials were not unlike lectures: The lecturer addressed students in front of class and went through the solution of examples. Tutorials differed from lectures in that the lecturer talked less in front of class, and spent more time walking around the lecture hall and talking with students individually or in small groups. All operations identified in the lectures I identified also in the tutorials.

With the presentation of this, the third level analysis I complete the structural analysis of my data.

I now present my second analysis chapter. I discuss the lecturer’s approach and his didactical thinking in relation to teaching three linear algebra topics: subspaces, linear independence, and eigenvectors and eigenvalues. I re-visit the second level of analysis of the present chapter, namely the actions and goals of Activity.
Chapter 6

The Teaching of Linear Algebra Concepts

In the last chapter I presented the analysis of my data in respect of the research meetings. I used Leontiev’s theoretical account of activity-motive, actions-goals and operations-conditions, and related each aspect to my data. At the top level of analysis I discussed the lecturer’s motive and defined the teaching and learning of linear algebra ‘inductively’ as activity. I created a theoretical (i.e. a hypothetical) model of the lecturer’s intentions and strategies based on the lecturer’s comments in research meetings. This related to Leontiev’s action-goal level of analysis. I also discussed the operations and conditions of activity.

Based on my analysis I developed the notion of an EAG approach to teaching. I related the EAG approach to a ‘bottom-up’, or inductive style of teaching. The description that I gave in the previous chapter related to EAG as inherent in activity and connected to motive. Thus I presented my discussion of EAG at the activity-motive level of analysis. I contrasted this style with the more traditional DTP (‘definition-theorem-proof’) approach.

I related the lecturer’s approach to the design of the module overall. My research focussed on the first semester of a two-semester (one year) module in linear algebra. Semester 1 consisted of an introduction to the key ideas in linear algebra while the more formal treatment of the (same) topics was presented by a different lecturer in Semester 2. A detailed description of the overall course design and course structure was given in the previous chapter and in the introduction to the thesis, in Chapter 1.

In this chapter I report on my analysis of the lecturer’s teaching of three concepts in linear algebra, namely subspaces, linear independence and eigenvectors.
6.1 Introduction to Chapter 6

In deciding which concepts to focus on for my analysis I was guided by (a) the lecturer’s own view, as expressed in research meetings, and (b) by the research literature into the teaching and learning of linear algebra, in particular the work by Jean-Luc Dorier and his colleagues into students’ difficulties with linear algebra (Dorier, 2000; Dorier et al., 2000b; Dorier and Sierpinska, 2001).

I give a detailed analysis of each of the three concepts: subspaces, linear independence and eigenvectors. This includes a mathematical account of each concept in terms of how the concept is presented in textbooks (that is, in terms of what textbook authors have written), a mathematical account of how the lecturer in my study presented each concept (based on the written course notes to his students), my analysis and interpretation of the lecturer’s didactical thinking and planning, and finally my analysis and interpretation of the lecturer’s teaching practice (that is, the lecturer’s face-to-face teaching of his students). Thus I cover four distinct areas when analysing each concept:

i. A mathematical account of the concept

ii. The lecturer’s mathematical treatment of the concept in the course notes

iii. The lecturer’s didactical thinking in relation to the teaching of the concept

iv. The lecturer’s teaching of each concept (to students)

As a result this chapter is very long. I try and give good sign-posting for ease of reading.

In this introduction to Chapter 6, I first state my reasons for analysing the topics of subspaces, linear independence and eigenvectors. I then give some detailed information about the four areas that I cover for each concept. I explain my choice of the textbooks and set of course notes, copies of some pages from each are in Appendix C. I explain the notation that I have adopted throughout the thesis and relate it to the notation used in the various textbooks and course notes. I state two assumptions that I make in respect of my reader which concludes the introduction.

6.1.1 Analysing subspaces, linear independence and eigenvectors

I decided to analyse three concepts: subspaces, linear independence and eigenvectors/eigenvalues. In deciding which concepts to focus on I referred to my analysis of the research meetings and to the research literature into the teaching and learning of linear algebra. I was guided by the lecturer’s own view, expressed in the research meetings, as to what the ‘crucial concepts’ [his words] of linear algebra were. The lecturer said,
That chapter 3 on subspaces. ... It’s the heart of the course really, that’s why I told students on Friday that if they master that chapter everything else will fall into place easily. And I’m convinced it will. And, on the other hand, that chapter is, well, where all the abstract concepts really are. That’s the reason why it’s so difficult. (M18, 27:42)

Chapter 3 (of the course notes) was entitled “Subspaces of $\mathbb{R}^n$”. It was one of four chapters that constituted the course notes for the linear algebra module. In terms of linear algebra concepts, in Chapter 3, the lecturer introduced the concepts of vector, linear transformations/linear maps, null space, subspace, linear combination, span, spanning set, range of a matrix, linear (in)dependence, linear relation, basis, dimension, rank, nullity, rank-nullity theorem, and change of basis to students. In the exposition of these concepts in the course notes the lecturer did not give an indication of one concept being more important than another. However, in research meetings and during his teaching he often stressed the importance of certain concepts. For example, in a lecture in Week 5 of the module the lecturer told students explicitly that a subspace was an important concept. He said,

And because this is so important we give this a name. We call this a subspace of $\mathbb{R}^n$. A subspace of $\mathbb{R}^n$ is a subset that behaves the same way as the full space $\mathbb{R}^n$ in the sense that I can add two vectors and get another vector in the set. I can multiply a vector from the set with a number and get another vector in the set. And, the zero vector is in the set. (L12, 40:53)

A little later in the lecture he said,

That’s [A subspace is] an important new idea and also an important new word that you need to learn to use. And there’re a couple of more new words that I need to give you. (L12, 43:25)

Hence I decided to analyse the concept of a subspace in more detail.

A second concept that the lecturer considered ‘crucial’ [his words] was linear independence. In a research meeting at the beginning of Semester 1 the lecturer talked about the changes that he had made to the module. He said,

Well, I am making quite some changes to the material that I am covering in lectures as compared to last year. Because, I am hoping to put more emphasis
on the crucial concepts of linear algebra, that are there when you look at the module as a whole which is linear combinations, linear independence, bases. (M1, 41:50)

In a lecture in Week 6 the lecturer introduced and discussed the concept of linear independence. In defining this concept he said to his students,

..., when we’re given a set of vectors, a spanning set, we can investigate if there is a linear relation between the vectors. If there is, we say the vectors are linearly dependent, if there is none, we say the vectors are linearly independent. . . . That is quite possibly the most important idea you’re going to see in this chapter, and in all of the module, because something very similar crops up all over the place in mathematics. (L15, 24:42)

In addition I consulted the research literature into students’ difficulties with linear algebra. In particular, I considered the work by Jean-Luc Dorier and his colleagues (Dorier, 2000; Dorier, Robert, Robinet and Rogalski, 2000b). Concepts mentioned specifically by Dorier were vectors, basis and dimension, linear independence, and rank.

My review of the research literature into the teaching and learning of linear algebra (see Chapter 2) supports the view that linear independence is an important concept in linear algebra.

Hence I decided to analyse the concept of linear independence in more detail.

The third concept that I decided to analyse was eigenvectors and eigenvalues. “Eigenvalues and Eigenvectors” was the title of Chapter 4 of the course notes. In research meetings the lecturer talked extensively about structuring the material of Chapter 4 (and the introduction to eigenvalues/vectors, in particular) in order to present students with a more conceptual view of this topic. He said,

...trying to do the important concepts first and the computational recipes later on, in order to avoid that students think of an eigenvalue as a zero of the characteristic polynomial, which most know anyway because that’s how they’re calculated. But this point of view really is unhelpful if you want to move out of the calculation because it doesn’t give you an opportunity to do anything with it. (M15, 1:10:26)
Eigenvalues/vectors are a pre-requisite for dealing with change of bases/diagonalisation of matrices (also in Chapter 4 of the course notes). They also provided the focus for recent research into the teaching and learning of linear algebra (see Stewart, 2009; Stewart and Thomas, 2007, 2010).

I have chosen three topics, subspaces (and vector spaces in general), linear independence and eigenvalues/vectors, for further analysis. In a research meeting the lecturer listed these three topics among several that he considered as making up a ‘standard’ university module in linear algebra. The lecturer said,

...we certainly all agree we want students to be able to do matrix calculations and solve linear equation systems and to calculate eigenvectors, and we also want them to understand the concepts of vector spaces and linear independence, and so, I think we all agree on that. And to that extent Introductory Linear Algebra really is standard, not only here but everywhere. (M18, 06:30)

Guided by the lecturer’s comments in research meetings and by the research literature into the teaching and learning of linear algebra I decided to analyse the concepts of subspaces, linear independence and eigenvalues/vectors.

For each of these concepts I consider four aspects for analysis,

i. A mathematical account of the concept

ii. The lecturer’s mathematical treatment of the concept in the course notes

iii. The lecturer’s didactical thinking in relation to the teaching of the concept

iv. The lecturer’s teaching of each concept (to students)

I describe each of the four aspects and what they entailed for my analysis in more detail in the next section.

6.1.2 The four aspects of analysis

i. A mathematical account of each concept:

This is a description of the mathematics underpinning/surrounding each concept. I consulted three linear algebra textbooks and a set of linear algebra course notes that accompanied an Introductory Linear Algebra module that I took as an undergraduate in the academic year 1995/96. I have also drawn on my own experience
and knowledge of studying linear algebra during my under-graduate degree, and as part of my further degree in Fluid Mechanics. Some pages from these four sources have been re-produced for reference in Appendix C.

ii. The lecturer’s mathematical treatment of each concept:
Here I report on my analysis of the lecturer’s linear algebra course design in terms of the mathematical content and the structuring of the content. I refer to the lecturer’s actual words as recorded in research meetings, and to the printed course notes that accompanied the module to justify the interpretations that I have made. I refer to the student version of the course notes (these have ‘gaps’ where students can write the solution to an example or exercise) as well as the full, or complete set of course notes (which has no ‘gaps’ - all examples and solutions are printed). All pages of the course notes that are relevant to my analysis are re-produced in the Appendices E and F.

iii. The lecturer’s didactical thinking in relation to the teaching of each concept:
This is my analysis and interpretation of the lecturer’s didactical thinking in planning his teaching. I refer to the lecturer’s actual words as recorded in research meetings, and also to the printed course notes to justify interpretations. My analysis relates to the theoretical model that I created in Chapter 5. Some aspects of didactical thinking that I discuss are ‘specific’, that is, they relate to just one or two of the concepts. On the other hand, there are some aspects that are more general, or ‘generic’, and relate to all three concepts under consideration. These are discussed in the summary at the end of each section, while aspects that are ‘specific’ are discussed within the section that the concept was introduced.

iv. The lecturer’s teaching of each concept (to students):
This is my analysis and interpretation of the lecturer’s implementation of his planning. I refer to the lecturer’s actual words as recorded in lectures and to the field notes taken by the researchers while attending lectures. I also refer to the course notes that accompany the module to justify interpretations. Again, as above, aspects that were ‘specific’ to just one or two concepts are discussed within the section while the more ‘generic’ aspects are dealt with at a later stage.

In relation to point (i) I wish to elaborate the four sources (three textbooks and the set of course notes) that I consulted for comparison with the mathematical treatment that the lecturer in my study gave. In respect of (ii) I have explained the lecturer’s course notes in Section 5.1.
6.1.3 The reference textbooks and set of course notes

I consulted three linear algebra textbooks and a set of linear algebra course notes. The three textbooks were K. Hoffman and R. Kunze, *Linear Algebra*, W. H. Greub, *Linear Algebra: Third Edition*, and D. Poole, *Linear Algebra: A modern introduction*, published in 1961, 1967 and 2006, respectively. The fourth source that I consulted was a set of (unpublished) course notes that accompanied a linear algebra module that I took as an undergraduate. I refer to this set of notes as Sproston (1995). Dr P. Sproston, a mathematician employed at the university where I studied in 1995/96, had written the course notes and delivered the Introductory Linear Algebra module. A visiting lecturer to the university taught the module for one year (in 1995/96) when I was an undergraduate using the course notes by Sproston.

The textbooks were chosen from a range of textbooks available to me. I stipulated that each book should deal with all three concepts that I discuss in my thesis, and that they were published in different years. As a result I chose two textbooks that were fairly old, Hoffman and Kunze (1961) and Greub (1967), and two more recent publications, Poole (2006) and Sproston (1995).

The three textbooks and the set of course notes varied in respect of the presentation and the scope of the material. Mathematics as a discipline is often referred to as *Pure* or *Applied*. An applied mathematician may work with mathematics that is closely related to applications in Science, Commerce and Industry, for example. In contrast, a pure mathematician may focus on the very theoretical side of a mathematical topic without necessarily considering ‘concrete’ applications. The two (older) books, by Hoffman and Kunze (1961) and Greub (1967), focused on a pure treatment of linear algebra topics and contained material that I would describe as belonging to Abstract Algebra rather than Linear Algebra (such as commutative rings and ideals, for example). The course notes written by Dr Sproston, a pure mathematician, did reflect a pure treatment but without ‘straying’ into the area of Abstract Algebra, for example. David Poole, on the other hand, re-visited key concepts in linear algebra in different settings. As a consequence his book has a different ‘feel’ from the other three sources that I used for reference. In Poole, definitions of concepts and explanations (of applications, for example) recur as a student works through the book. Poole also (deliberately and explicitly, see the preface to the book, by the author) placed emphasis on explanations based on geometric insights and reasoning whereas the other three sources either did not, or to a much lesser degree.

In Appendix C I have reproduced one or two pages from each textbook and from the set of course notes. These reflect ‘typical’ pages in relation to the three concepts that I discuss in this chapter, and give the reader a ‘flavour’ of the textbooks/course notes.
that I used for comparison with the lecturer’s course notes.

With the exception of Poole (2006), the textbooks and the set of course notes followed (broadly) the same pattern of sequencing of the material. The chapter headings fell into three main categories:

- Vector spaces/linear equation systems
- Transformations/mappings/matrices
- Determinants/characteristic polynomial/diagonalisation

Poole (2006) also had this structure ‘superficially’, that is, by reading just the chapter headings. However, Poole’s approach was different from the other three authors in several ways. (1) Poole’s approach was heavily focussed on matrices as the main tool for explaining linear algebra concepts and procedures. (2) He used concrete applications and geometric insights (based in $\mathbb{R}^2$, $\mathbb{R}^3$ usually) in explanations. (3) He covered the same material more than once which resulted in a repetition of statements, theorems and definitions.\(^1\)

In taking this approach, Poole’s \textit{sequencing} was similar to the way that the lecturer in my study re-visited material. It was similar also to a matrix-oriented approach described in the research literature (see Uhlig, 2002).

Hoffman and Kunze (1961) and Greub (1967) followed a top-down approach in presenting the material, using the DTP (definition-theorem-proof) style. Sproston’s approach was

\(^1\)For example, Poole introduced subspaces twice: once in relation to the vector space $\mathbb{R}^n$, and once in relation to a general vector space $V$. He stated the definition for the first time in Section 3.5 of his book (page 190):

\textbf{Definition.} A subspace of $\mathbb{R}^n$ is any collection $S$ of vectors in $\mathbb{R}^n$ such that

1. The zero vector $\mathbf{0}$ is in $S$.
2. If $\mathbf{u}$ and $\mathbf{v}$ are in $S$, then $\mathbf{u} + \mathbf{v}$ is in $S$. ($S$ is closed under addition.)
3. If $\mathbf{u}$ is in $S$ and $c$ is a scalar, then $c\mathbf{u}$ is in $S$. ($S$ is closed under scalar multiplication.)

In Section 6.1 in his book he stated the definition again (page 438). On this occasion he referred to a general vector space $V$. The definition is identical in form except for replacing $\mathbb{R}^n$ by $V$ and $S$ by $W$, and for ‘incorporating’ the zero vector into ‘a non-empty subset’. He now refers to the definition as a ‘theorem’.

\textbf{Theorem 6.2.} Let $V$ be a vector space and let $W$ be a non-empty subset of $V$. Then $W$ is a subspace of $V$ if and only if the following conditions hold:

1. If $\mathbf{u}$ and $\mathbf{v}$ are in $W$, then $\mathbf{u} + \mathbf{v}$ is in $W$.
2. If $\mathbf{u}$ is in $W$ and $c$ is a scalar, then $c\mathbf{u}$ is in $W$.

I discuss Poole’s approach in more detail in Section 6.2.1 and at the end of Section 6.4.1.
also top-down and DTP style but included a lot of explanations that linked one area or discussion point with the next. Poole’s approach was more bottom-up (than the other three) in the sense that he introduced linear algebra topics in \( \mathbb{R}^2 / \mathbb{R}^3 \) first before translating the arguments to \( \mathbb{R}^n \). He (in contrast to the other three authors) included extensive geometric reasoning to support the written explanations.

All the authors of the textbooks/set of course notes used the DTP style for presenting linear algebra concept. In a DTP style of presenting the author states the definition of a concept first, followed by a theorem that relates to the concept, and proceeds to prove the theorem. For example, all the authors of the textbooks/set of course notes that I consulted stated the axiomatic definition of a vector space, and proceeded to verify all ten axioms for a particular example of a vector space. The example was most often drawn from a list of examples that the author had provided in the textbook.

Some of the textbook authors gave more ‘space’ to written explanations before and after presenting a theorem, definition or proof. Virtually all authors gave examples after introducing a new concept or idea.

Poole produced a textbook that attempted to link the teaching of linear algebra concepts more closely to applications and/or using geometric insights and arguments. I am not considering the impact of using geometry or not using geometry in the teaching of linear algebra. I use Poole (2006) as a resource alongside the other three textbooks/set of course notes. I wish to compare the lecturer’s presentation of linear algebra concepts with how these concepts were presented typically in textbooks.

### 6.1.4 Notation

Throughout this chapter I reproduce definitions, theorems, etc. from the textbooks or the set of course notes. For ease of comparison (between the sources) I used a uniform system to denote vectors, sets, matrices, transformations, and so on. For example, I always use the letter \( A \) for a matrix when a textbook may use a different letter, the letter \( T \), say. In all other respects a definition or theorem was reproduced as it appeared in the textbook/set of course notes. I illustrate my point with an example.

In Section 6.2.1 I state the definition of a subspace as follows:
(Equivalent) Definition of a subspace. Let $K$ be any scalar field. Let $V$ be a vector space over $K$, and $U$ a non-empty subset of $V$ with addition and scalar multiplication as defined for $V$. Then $U$ is a subspace of $V$ if for all $x, y \in U$ and for all $\lambda \in K$,

1. $x + y \in U$,
2. $\lambda x \in U$.

I based my formulation of this definition on past knowledge and experience, and on my reading of the three textbooks and the set of course notes. I state in Section 6.2.1 that the authors of the textbooks/set of course notes used ‘similar’ formulations to define a subspace. I now state the formulations as they appeared in Poole (2006), Sproston (1995), Greub (1967), and Hoffman and Kunze (1961), respectively.

**Theorem 6.2.** Let $V$ be a vector space and let $W$ be a nonempty subset of $V$. Then $W$ is a subspace of $V$ if and only if the following conditions hold:

a. If $u$ and $v$ are in $W$, then $u + v$ is in $W$.

b. If $u$ is in $W$ and $c$ is a scalar, then $cu$ is in $W$. (Poole, 2006, p. 438)

**Proposition 2.** Let $V$ be a vector space over $K$, $U$ a nonempty subset of $V$. The following conditions on $U$ are equivalent:

1. $U$ is itself a vector space over $K$, with addition and scalar multiplication ‘inherited’ from $V$ (i.e. defined for elements of $U$ as they are if those elements are considered as elements of $V$);
2. $x + y \in U$ whenever $x, y \in U$ and $\lambda x \in U$ whenever $\lambda \in K, x \in U$;
3. $\lambda x + \mu y \in U$ whenever $\lambda, \mu \in K, x, y \in U$. (Sproston, 1995, p. 4)
Subspaces. Let $E$ be a vector space over the field $\Gamma$. A non-empty subset, $E_1$ of $E$ is called a subspace if for each $x, y \in E_1$ and every scalar $\lambda \in \Gamma$,

$$x + y \in E_1$$

and

$$\lambda x \in E_1.$$  

Equivalently, a subspace is a subset of $E$ such that

$$\lambda x + \mu y \in E_1$$

whenever $x, y \in E_1$. (Greub, 1967, p. 23)

Theorem 1. A non-empty subset $W$ of $V$ is a subspace of $V$ if and only if for each pair of vectors $\alpha, \beta$ in $W$ and each scalar $c$ in $F$ the vector $c\alpha + \beta$ is again in $W$. (Hoffman & Kunze, 1961, p. 34)

The reader will notice the similarities between the formulations of the notion of a subspace. I consider the formulations by Poole, Sproston part (2) and Greub (first half) to be ‘the same’ as the formulation I gave. I discuss the topic of subspaces in detail in Section 6.2. My aim here is to explain the notation that I adopted.

In order to compare one source with another, and one source with the lecturer’s formulation of a statement or definition, I adopt the same letters to denote a vector, scalar, vector space, etc. as mentioned. My aim in quoting from the four sources is to draw out the differences in formulation and emphasis that a textbook author placed as well as any commonalities between the various authors. I then use what I have learnt from this analysis to aid my interpretation of the lecturer’s introduction of a topic. Thus, in relation to the definition of a subspace, I adopt the letters $V$ to denote a vector space, $U$ to denote a subset of $V$, and $x$ and $y$ to denote vectors in $U$. I also use $\lambda$ for a scalar value and $K$ for a field. I then quote Poole as follows:

Theorem 6.2. Let $V$ be a vector space and let $U$ be a nonempty subset of $V$. Then $U$ is a subspace of $V$ if and only if the following conditions hold:

a. If $x$ and $y$ are in $U$, then $x + y$ is in $U$.

b. If $x$ is in $U$ and $\lambda$ is a scalar, then $\lambda x$ is in $U$. (Poole, 2006, p. 438)

---

2Greater mathematical sophistication may be required to appreciate Hoffman and Kunze’s statement as equivalent to the other three.
Thus whenever I reproduce, or quote, from a source the letters that I use in the quotation may be different to the letters that the textbook author used. The formulation of the definition, theorem, etc., will be reproduced faithfully in all other respects. The aim of comparing with, and quoting from other textbooks is to gain a better understanding of the didactical thinking and planning that the lecturer employed.

6.1.5 Assumptions and points of reference

I make the following assumptions: (1) The reader is reasonably familiar with linear algebra up to first year undergraduate level. For example, I do not discuss, or explain the nature or status of a definition, theorem, proposition, lemma, or proof. (2) The reader has knowledge of many of the concepts in linear algebra such as linear transformations, span and spanning sets, range, rank and nullity. (3) The reader is familiar with the terminology of first year undergraduate mathematics in general. For example, I assume that terms such as parameter, homogeneous and inhomogeneous equation systems, and trivial solution need little explanation.

Having said that, my thesis aims to be accessible to the non-mathematician who may omit a detailed study of the mathematical explanations given, many of which are part of footnotes.

In this chapter I reproduce comments that the lecturer made in research meetings and in lectures. In addition I draw on extracts from data reduction documents and field notes. For meetings and lectures I provide references in the form, for example, (M15, 13:45) or (L14, 24:25) respectively. The first example indicates that the lecturer made a comment in meeting M15, and 13 minutes and 45 seconds after I had set the voice recorder to record. In a similar way, the second example indicates that the lecturer made a comment in the lecture L14, and 24 minutes and 25 seconds after I had set the voice recorder to record. As a reminder, and as mentioned previously, lectures and tutorials were labelled chronologically as they were given so that L5, for example, related to a tutorial.

In the case where I quote from the field notes I use the format (L15, FN1) where FN1 relates to the field notes taken by researcher 1 in the lecture L15. I use FN2 to denote the field notes taken by researcher 2. Similarly, if I quote from a data reduction document I provide the reference in the form, for example, (M8, DR2) which refers to a data reduction made by researcher 2 in respect of meeting M8.

In Chapter 7 I report on my analysis of the student data and the student focus group interviews, in particular. There were six interviews and a total of fourteen students took
part. I labelled the interviews Int1, Int2, . . . , Int6 and anonymised the identities of the students by referring to them as S1, S2, . . . , S14. Whenever I quote what a student said in the group interviews I provide a reference in the form (Int4, S2), for example. This refers to a comment made by Student 2 in the focus group interview 4.

One final comment: Whenever I refer to the lecturer who took part in my study, I usually write ‘the lecturer in my study’ or ‘the lecturer’ in short.

In summary of my introduction, I have laid out the three concepts and have given reasons for my choice of the three concepts for my analysis. I have given details of the sources that I have used for reference, three linear algebra textbooks and one set of (unpublished) course notes. I have explained my notation when reproducing from the four sources, and my convention for referencing quotations, as well as any assumptions that I make of the reader.

I now begin my analysis of the three concepts (subspaces, linear independence and eigenvalues/vectors) in earnest. I start with reporting on my analysis of the teaching of subspaces and relate it to the four areas as mentioned in Section 6.1.2.
6.2 The concept of a subspace

6.2.1 A mathematical account of vector spaces and subspaces

I consulted three textbooks and a set of course notes (Greub, 1967; Hoffman and Kunze, 1961; Poole, 2006; Sproston, 1995) to see how the concept of subspace was defined and introduced to the reader. With reference to these sources, textbook authors most frequently gave an axiomatic definition of a (general) vector space before introducing the concept of subspace. ‘Axiomatic’ means that the vector space is defined with reference to a list of properties, or ‘axioms’. The axioms represent the conditions, or properties that need to be satisfied in order for a set of objects to constitute a vector space.

The concept of a subspace is then defined as a subset of a vector space (satisfying certain properties).

I will first outline the concept of a vector space and then the concept of subspace, by referring to their definition. The definitions I give are my own formulations but based on my reading of the textbooks/set of course notes.

**Definition of a vector space.** Let $K$ denote either the field of real numbers $\mathbb{R}$ or the field of complex numbers $\mathbb{C}$. Let $V$ be a set of vectors on which two operations, addition and scalar multiplication, have been defined. Then $V$ is a vector space over $K$ if for all $x, y, z \in V$, and for all $\lambda, \mu \in K$, the following ten axioms hold:

1. $x + y \in V$
2. $x + y = y + x$
3. $x + (y + z) = (x + y) + z$
4. there exists a distinguished element $0 \in V$ such that $x + 0 = x$
5. for each $x \in V$ there exists a corresponding element $-x \in V$ such that $x + (-x) = 0$
6. $\lambda x \in V$
7. $(\lambda + \mu)x = \lambda x + \mu x$
8. $\lambda(x + y) = \lambda x + \lambda y$
9. $\lambda(\mu x) = (\lambda\mu)x$
10. $1x = x$ (see Sproston, 1995, p. 1; Poole, 2006, p. 433)
The concept of a vector space is a composite notion consisting of a field of scalars, a set of objects or elements called ‘vectors’, and two operations, vector addition and scalar multiplication, that have certain properties.

The field $K$ is a field of scalars and is (most often) either the field $\mathbb{R}$ of real numbers or the field $\mathbb{C}$ of complex numbers. However, any number system constitutes a field if the four arithmetic operations of addition, subtraction, multiplication and division (excluding division by zero) can be defined, and obey the usual laws of arithmetic (commutativity, associativity, distributive law, etc). For example, the set of integers $\mathbb{Z}_p$ where $p$ is prime is a field (under the operations of arithmetic modulo $p$).

The set of objects consists of elements which are called ‘vectors’. But the ‘vectors’ need not be vectors in the geometric sense of ‘line segments that have magnitude and direction’. Linear algebra is an algebraic system, and as such we do not need to know what the objects of a vector space are since the objects are defined by the operations that are performed on them (see Sproston, 1995). In the abstract context, “the focus is not on the nature of the ‘vectors’ as objects in isolation” as Sproston (1995, p. 1) wrote. It is the way that the objects of a vector space relate to each other that defines a vector space, and not the objects themselves.

To check whether a mathematical entity is a vector space means checking that all ten axioms are satisfied.

As an alternative to the definition stated above, a vector space can be defined as follows.
(Equivalent) **Definition of a vector space.** Let $V$ be a set of vectors with the operations of addition and scalar multiplication defined such that $x + y \in V$ and $\lambda x \in V$, for all $x, y \in V$ and $\lambda \in K$. Then $V$ is a vector space over $K$ if for all $x, y, z \in V$, and for all $\lambda, \mu \in K$, the following eight axioms hold:

1. $x + y = y + x$
2. $x + (y + z) = (x + y) + z$
3. there exists a distinguished element $0 \in V$ such that $x + 0 = x$
4. for each $x \in V$ there exists a corresponding element $-x \in V$ such that $x + (-x) = 0$
5. $(\lambda + \mu)x = \lambda x + \mu x$
6. $\lambda(x + y) = \lambda x + \lambda y$
7. $\lambda(\mu x) = (\lambda\mu)x$
8. $1x = x$

Most textbooks gave examples of vector spaces after the definition was stated. Examples that were frequently used were $\mathbb{R}$ and $\mathbb{C}$ (the set of all real and of all complex numbers), $\mathbb{R}^n$, $\mathbb{R}^3$, the set of polynomials, the set of functions, the set of all $m \times n$ matrices, the set of all square $(n \times n)$ matrices, and the set of linear maps.

A **subspace** is defined as a subset of a vector space satisfying certain properties. As a subset the subspace ‘inherits’ the structure of the vector space (Sproston, 1995, p. 4; Poole, 2006, p. 438). In particular, the definition of the vector operations (on the elements of the vector space) is the same so that the eight axioms (axioms 2. to 5. in relation to addition, and axioms 7. to 10. in relation to scalar multiplication) can be assumed to hold. No further verification of these eight axioms is necessary. Hence, to show that a subset of a vector space is a subspace it suffices to verify that the subset is not empty and that it is closed under addition and scalar multiplication. Thus it suffices to verify just three axioms. This results in the following definition of a subspace.
Definition of a subspace. Let $V$ be a vector space over $K$, and $U$ a subset of $V$ with addition and scalar multiplication as defined for $V$. If for all $x, y \in U$ and for all $\lambda \in K$,

1. $x + y \in U$,
2. $\lambda x \in U$,
3. the zero vector $\in U$,

then $U$ is a subspace of $V$.

If any one of the three properties is not satisfied then the subset is not a subspace. Hence the distinction between a subset and a subspace is an important one. Understanding the concept of a subspace is a pre-requisite for understanding other concepts in linear algebra such as null space and range, basis and basis vectors, dimension and rank, and the links between them.

An alternative, and equivalent definition can be formulated if we define $U$ as a non-empty subset of $V$. If the set $U$ is not empty then it must contain at least one vector, and for the case of a set $U$ being a subspace, this single vector is the zero vector. This subsumes axiom 3 above. Thus we have the following equivalent definition of a subspace:

(Equivalent) Definition of a subspace. Let $V$ be a vector space over $K$, and $U$ a non-empty subset of $V$ with addition and scalar multiplication as defined for $V$. Then $U$ is a subspace of $V$ if for all $x, y \in U$ and for all $\lambda \in K$,

1. $x + y \in U$,
2. $\lambda x \in U$.

This definition was given by Greub (1967, p. 23), Poole (2006, p. 438) and Sproston (1995, p. 4).

This definition states that a non-empty subset of a vector space is a subspace if it is closed under addition and scalar multiplication. To determine whether a given set is a subspace means verifying that (1) the set is not empty, (2) adding two vectors in the set results in a vector in the set, and (3) multiplying a vector by a scalar results in a vector in the set. These three steps correspond to the definition given first, and are used when performing calculations to verify that a given set of objects is a subspace. I, therefore, refer to the first definition as a ‘working definition’ [my words]. I use this expression again, for example, in Section 6.2.2 and in my discussion of the lecturer’s didactical thinking and planning in Section 6.2.3.
All the authors of the textbooks/set of course notes that I consulted used the DTP (definition-theorem-proof) style for presenting the concept of a subspace (as well as a vector space and other concepts in linear algebra). This meant that the authors stated the axiomatic definition of a subspace and proceeded to verify the three properties (or ‘axioms’) listed in the definition.

I now discuss the lecturer’s introduction of the concept of subspace.

6.2.2 The lecturer’s mathematical treatment of subspace in the course notes

I analysed the student version of the course notes in relation to the teaching of subspaces, and the lecturer’s comments made in research meetings. I describe how the lecturer structured and presented the mathematical content in the course notes to his students.

The third chapter of the course notes was entitled “Subspaces of $\mathbb{R}^n$”. In this chapter the lecturer introduced many of the ‘crucial concepts in linear algebra’ [his words] as listed in Section 6.1.1. The previous two chapters of the course notes dealt with linear equation systems, including the method of Gaussian elimination, and matrices and the rules of matrix algebra.

In contrast to the three textbooks and the set of course notes that I consulted in Section 6.2.1, the lecturer completely omitted the use of the word vector space in the course notes, and the definition of a vector space in terms of the axioms as I described in Section 6.2.1. He also did not mention the notion of a field associated with a vector space. The set $\mathbb{R}^n$, that is the set of all $n$-tuples (the lecturer wrote “the set of all $n$-component vectors” which is the set of all column vectors) was the starting point for introducing the concept of a subspace, $\mathbb{R}^n$, which is closed under addition and scalar multiplication. Scalar multiplication was described as multiplication by a number where the field of real numbers was implied. In research meetings the lecturer said that he considered the verification of the properties of a vector space as largely superfluous. He said,

...yes, I mean, on the one hand, when you study vector spaces you have the axioms. ..., talking to my small group tutees, I had the impression that to some extent they were missing the forest for the trees when focussing about the axioms, when really the idea of the vector spaces is that you can add things and that you can multiply things by numbers. Once you’ve got that, the axioms most of the time are rather obvious. So I don’t think they deserve that much emphasis even in the general context. (M1, 57:50)
Thus, in his own words, students could see this quite naturally. (See also the next section (6.2.3) for a discussion of the role of the vector space $\mathbb{R}^n$.)

In contrast to the textbook and the set of course notes, the lecturer did not use the DTP (the definition-theorem-proof) style of presenting the concept of a vector space/subspace. In connection with the set $\mathbb{R}^n$ the lecturer made an “observation” (Observation 3.1 in the course notes) listing three properties of the set $\mathbb{R}^n$. I have reproduced Observation 3.1 as it appeared in the course notes (see page 219):

**Observation 3.1.** The set $\mathbb{R}^n$ of $n$-component vectors has the following properties:

1. Two vectors can be added, the result is a vector.
2. A vector can be multiplied by a number, the result is a vector.
3. The zero vector $\mathbf{0}$ is a vector.

Thus the lecturer defined the set $\mathbb{R}^n$ as a set with three properties. He defined the set $\mathbb{R}^n$ as a vector space with addition and scalar multiplication defined over the real numbers. But this was not made *explicit*. In the course notes the three properties of $\mathbb{R}^n$ were presented as facts. No further explanations were given and no proof. Observation 3.1 corresponds to the definition of a subspace (with three axioms, see page 102). A subspace is a vector space. In Section 6.2.3 I argue that the lecturer defined the set $\mathbb{R}^n$ as a ‘parent set’ [my words] from which subsets could be, and were formed (such as the null space), that were then proven to be subspaces. I also discuss the lecturer’s didactic decision not to include the axiomatic definition of a vector space in his teaching of this module.

In general, the words “Observation” and “Remark” that the lecturer used for labeling statements/theorems/expressions replaced the more formal terminology of a “Theorem” or “Lemma”. The choice of labeling was linked with the lecturer’s didactic decision of following a less formal approach in his teaching of linear algebra. I discuss the lecturer’s didactic thinking and planning in Section 6.2.3.

In a similar way to Observation 3.1, the lecturer listed three properties of linear transformations. These corresponded to the three properties in Observation 3.1 (above), and consisted of one set of properties in relation to a linear map in terms of a function and another in terms of a matrix $A$. Again there were no further explanations. The lecturer defined *linearity* here but the significance of this concept was not made explicit. In particular, the Observations 3.1 and 3.3 (see page 219) look very similar. But they relate to different mathematical notions. Observation 3.1 relates to a set of vectors being *closed under addition and scalar multiplication*, the defining properties of a subspace,
and Observation 3.3 to the linearity of a transformation. The distinction between these two notions was not made explicit in the course notes or in the lecture to students.

The lecturer used Observation 3.1 to introduce one vector space, the vector space $\mathbb{R}^n$. (In Section 6.2.3 I again use the term ‘parent set’.) He defined a subspace as a subset of $\mathbb{R}^n$ and having the same three properties (as listed in Observation 3.1) as the set $\mathbb{R}^n$. He did not mention that as a subset of a vector space, a subspace ‘inherits’ the operations of addition and scalar multiplication (as defined on the vector space). The lecturer used a particular example of a subspace, namely the null space of a matrix (see Example 3.4) in order to introduce the concept of a subspace. I have reproduced the example here as it appeared in the course notes (see page 220):

**Example 3.4.** Assume that $A$ is an unknown $2 \times 3$ matrix. We know that

$$
\begin{pmatrix}
1 \\
3 \\
2
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
-2 \\
1 \\
3
\end{pmatrix}
$$

are solutions of a homogeneous linear equation system $Ax = 0$. Find more solutions of this equation system.

**Solution:**

Half a page was left blank here.

The lecturer presented this example as an exercise for students to complete in lecture (L12, Week 5). This consisted of generating solutions from two known solutions (via the operations of addition/subtraction and multiplication by a scalar). This was an open-ended type problem. Students were not required to calculate or find a particular solution, that was known in advance to the lecturer. The lecturer had asked students to generate their own solutions, and many different solutions were possible. In the course notes the area where the solutions ‘should be’ was left blank. In the lecture students could write in the blank spaces. At the bottom of the page, and visible to students before attempting the exercise, was Observation 3.5. This represented a summary or generalisation of the types of solutions that could be found. Observation 3.5. stated (see also page 220):
**Observation 3.5.** For any matrix $A$, the solution set $S$ of the **homogeneous** linear equation system $Ax = 0$ has the properties

1. Two vectors in $S$ can be added, the result is again in $S$.
2. A vector in $S$ can be multiplied by a number, the result is in $S$.
3. The zero vector $0$ is in $S$.

Observation 3.5 was *(virtually) identical* to Observation 3.1 but with the set $\mathbb{R}^n$ replaced by a set $S$. With Observation 3.5 the lecturer defined the set $S$ ‘closed under addition and scalar multiplication’, and a subspace of the set $\mathbb{R}^n$. In an explanation written below Observation 3.5 the lecturer (a) made the property of a subspace (as closed under addition and scalar multiplication) explicit (not reproduced here, but see page 220) and (b) defined the concept of a subspace *informally* by writing:

\[ \text{Compare this to Observation 3.1:} \]
\[ \text{The set } S \text{ has the same properties as the full set of vectors } \mathbb{R}^n. \]

A formal definition of a *subspace* (Definition 3.6, see page 221) was then given over at the top of the next page. The lecturer also stated the definition of a null space of a matrix (Definition 3.7, see page 221).

The concept of subspace was introduced via an *example*. The example centred on the null space of a matrix, and asking students to generate the null space led to the definition of a subspace. The lecturer took an inductive, or bottom-up approach to the teaching of linear algebra concepts (as expressed in research meetings) which relied on presenting examples *before* any formal (or informal) definitions were given. This was part of what I had termed the EAG approach in Chapter 5. The lecturer used informal (written) language to describe the concept of a subspace. For example, in the course notes he wrote that a subspace has “the same properties as the full set of vectors”. He also used informal (written) language when listing the axioms that are satisfied when a subset is a subspace of a known vector space. For example, in Observation 3.5 he wrote,

\[ \text{A vector in } S \text{ can be multiplied by a number, the result is in } S. \]

A more formal expression of this statement would be

\[ \text{A vector } x \in S \text{ can be multiplied by a scalar } \lambda, \text{ the result is a vector } \lambda x \in S. \ [\text{My formulation.}] \]

Or
\[ \lambda x \in U \text{ whenever } \lambda \in K, x \in U. \] (Sproston, 1995, p. 4)

A discussion of the structure of the course notes (including the language used to describe a concept) will be made in Section 6.2.3. The course notes will be discussed again with reference to how they were used in the lectures in Section 6.2.4.

In summary, the concept of subspace was introduced via an example. The example described a specific subset of the vector space \( \mathbb{R}^n \). The focus was on the property of a set being closed under addition and scalar multiplication as the defining characteristic of a vector space. Any discussion of the mathematical content and of the structuring of content inevitably leads me to consider the didactical choices that the lecturer made in planning his teaching of the linear algebra module. These I consider in the next section.

### 6.2.3 The lecturer’s didactical thinking in relation to the teaching of subspaces

I discuss here the lecturer’s didactical thinking in planning his teaching of the concept of a subspace. I draw on my previous discussion in Section 6.2.2 in relation to the lecturer’s structuring of the mathematical content. I draw again on the course notes and on the comments made by the lecturer in research meetings to aid my interpretations.

I discuss in detail the role of the vector space \( \mathbb{R}^n \) and the role of the example (Example 3.4) that the lecturer used to introduce the topic of a subspace.

**The vector space \( \mathbb{R}^n \)**

Recapping some of the arguments made in Section 6.2.2, a discussion of (abstract) vector spaces over a field and defined by two operations on the elements of the vector space such that a set of (ten) axioms were satisfied, was completely omitted in the course notes. Such a definition is called an axiomatic definition and represented the way that vector spaces (and as a consequence subspaces) were introduced in the textbooks/set of course notes that I consulted. To show that a set of vectors is a vector space means verifying ten axioms which is often tedious and time consuming. In advanced mathematics it can also be very difficult to do. That is the reason why mathematicians generally aim to show that a vector space is a subspace of a known vector space. It reduces verification to checking just three axioms. The lecturer did not state an axiomatic definition, or explain the notion of a field, etc. in the course notes. He did not use the word ‘vector space’ in the course notes. However, he did refer (briefly) to a vector space and the

---

3 The use of more informal language included the words “observation” and “remark” in place of, for example, “theorem” or “lemma” in the course notes.
axioms that define a vector space in his introduction to subspaces in the lecture (that is verbally) to his students.

In focussing on the vector space $\mathbb{R}^n$ with the definition of ‘ordinary’ addition and multiplication the lecturer reasoned that the axioms were “obvious” (see quotation below) and that checking all axioms distracted students from forming important conceptual understandings of a vector space. In a research meeting he recalled a past experience with his personal tutees. He said (as quoted previously in Section 6.2.2),

...yes, I mean, on the one hand, when you study vector spaces you have the axioms. . . . , talking to my small group tutees, I had the impression that to some extent they were missing the forest for the trees when focussing about the axioms, when really the idea of the vector spaces is that you can add things and that you can multiply things by numbers. Once you’ve got that, the axioms most of the time are rather obvious. So I don’t think they deserve that much emphasis even in the general context. (M1, 57:50)

In the course notes the lecturer introduced the set $\mathbb{R}^n$ with the sentence, “The set of all $n$-component vectors is denoted by $\mathbb{R}^n$”. He defined the set $\mathbb{R}^n$ by referring to the three properties as listed in Observation 3.1 (see page 219). Thus the concept of a vector space was “reduced” [my words] to the ‘status’ of a subspace with the need to only check three axioms, two in relation to the set being closed and the third to ensure that the set is not empty. The lecturer’s didactic choice of $\mathbb{R}^n$ as a set of vectors from which other subsets and subspaces were formed, led to an increased focus on the properties of subspaces as being closed (under a given operation) rather than on the properties of a vector space. That was an indication of the lecturer’s intentions, of what he wanted to teach and students to learn. As in other contexts the lecturer stated his intentions explicitly in lectures to his students, including on this occasion. In the first lecture on subspaces (in Week 5) he emphasised the properties of a subspace when he said,

And because this is so important we give this a name.\(^4\) We call this a subspace of $\mathbb{R}^n$. A subspace of $\mathbb{R}^n$ is a subset that behaves the same way as the full space $\mathbb{R}^n$ in the sense that I can add two vectors and get another vector in the set. I can multiply a vector from the set with a number and get another vector in the set. And, the zero vector is in the set. (L12, 40:53)

The lecturer referred to $\mathbb{R}^n$ as a “full space” (see quotation above) which implied [to me] a ‘parent space’ [my words] from which ‘sub-spaces’ can be formed. Thus the lecturer

---

\(^4\)The lecturer referred to set $S$ which represented the null space of the matrix in Example 3.4. In Observation 3.5 he defined the set $S$ a subset of $\mathbb{R}^n$ and having the same three properties as $\mathbb{R}^n$. 
focussed more on the teaching of subspaces (the title of the chapter was “Subspaces of $\mathbb{R}^n$”, “this is so important” in quotation above) than general vector spaces. I claim that the vector space $\mathbb{R}^n$ was taken for granted and a ‘baseline’ from which to build subsets and subspaces. In addition, the fields $\mathbb{R}$ of real numbers and $\mathbb{C}$ of complex numbers were also ‘assumed’, that is stated without further explanation.

By defining “ordinary” [my words] addition and multiplication by a scalar the lecturer was aiming to divert students’ attention away from focussing on calculations and more towards the ideas and concepts in linear algebra. Based on his previous experiences in teaching, and on his current experiences in teaching a small group of students (in small group tutorials), he formed an “impression” of students as focussing on calculations rather than concepts. In a research meeting, in relation to problem solving activities, he said,

...my impression from our incoming students is that there hasn’t been a lot of emphasis on that [problem solving], because rules and algorithms is what they’re good at. Anything that goes beyond that is very difficult, and my impression is many don’t have the idea even that they could try ... (M3, 08:17)

He decided to focus on the vector space $\mathbb{R}^n$ where the elements are vectors, namely column (or row) vectors in the sense known to students. In research meetings he referred to his experience in teaching the linear algebra module in the previous year and his reasons for focusing on $\mathbb{R}^n$. He said,

...and we [he and a colleague] noticed that there is a lot of this language of linear combinations and linear independence and linear maps and injective and surjective maps which I’m not planning to talk about but that was in ... that core collection of new terminology that came up in the second semester, and all that in the context of abstract vector spaces. ... So that’s why I’m, what I’m aiming for is to talk about these linear combinations, and linear independence, these crucial concepts, in the context of column vectors where most people feel comfortable they can calculate with them. (M8, 06:56)

So, the reason for focussing on $\mathbb{R}^n$ was to be able to concentrate on the “ideas” of linear algebra, those “crucial concepts” (see quotation in Section 6.1 on page 88), and to forget or not think too much about the calculations that you needed to perform. Since the vector space was a space of vectors, and often vectors in two or three dimensions, the vector space was potentially very concrete and visual, and any calculations that
were needed, were based on simple arithmetic, “where most people feel comfortable” (see quotation above). From an activity-theoretical point of view, reducing calculations to basic arithmetic meant asking students to perform calculations at the operation-condition level. Students’ focus in lecture and while working on exercises could then be more fully directed towards conceptual understanding at action-goal level. I discuss this point again in Chapter 8.

Reducing the level of abstraction to the vector space \( \mathbb{R}^n \) had a knock on effect on the language necessary to explain the mathematics which was now simpler and more concrete. In the course notes the lecturer wrote, for example, ‘multiplication by a number’ and not ‘multiplication by a scalar’, and ‘a set with properties’ and not ‘a vector space with axioms’. Combined with an examples-based (and more inductive) approach to teaching the lecturer aimed to make the concepts of linear algebra more accessible to students. What that meant, the didactical decisions that formed the lecturer’s design of “how to teach” will be the focus of the next section.

The role of (formal and informal) language in the lecturer’s design and approach to teaching permeates all his teaching and of all the topics that I considered in my analysis in this chapter. It is one aspect that I discuss in detail in relation to the topic of linear independence.

The lecturer did not introduce any dynamic or geometric aspects into his teaching of linear algebra concepts. This was part of the design of the module which was agreed by the mathematics department at the university.

**The role of Example 3.4**

In the last section (6.2.2) I addressed the lecturer’s didactical decision of “what to teach” in the module, in relation to the concept of vector spaces and subspaces. In this section I discuss the lecturer’s decision of “how to teach” vector spaces and subspaces.

Addressing the didactics of “how to teach” was twofold. In general, the lecturer presented examples and combined it with designing course notes that contained ‘gaps’. Both aspects (presenting examples and producing notes) were aimed at engaging students with the mathematics (physically and mentally), and through engagement (and intuitive reasoning) leading to an informal understanding of the concepts in linear algebra (first). This I discussed in Chapter 5.

Recapping previous discussions, the lecturer based his approach on presenting an example first, followed by a definition or theorem. I had termed this approach EAG where the
initials stand for Example - Argument - Generalisation (see Section 5.3.1), and describe the process of:

we introduce an Example,
we make an Argument on the example, and then
we Generalise to an observation (a definition or theorem).

The lecturer’s aim was to present an example, and then to make an argument on the example in order to derive a rule or observation, that could highlight a generality in what was observed in the example.

To ensure engagement (that is, to ensure that students accessed the mathematics, and in the way that the lecturer envisaged) the lecturer produced course notes for use in the lectures. These contained ‘gaps’, blank areas in the notes, where students could write the solutions to examples presented in the lecture. The course notes were designed so that students tried solutions themselves. The lecturer wanted students to engage (mentally) with the mathematics. He also wanted them to participate more (and more actively) in lectures rather than be passive listeners (as was often the case in ‘traditional lectures’, see Wu, 1999, or Pritchard, 2010).

In a research meeting the lecturer made his approach explicit, when he said,

Generally speaking, I decided that I would focus on doing the development of the argument on examples, and then trying to abstract a general fact from the example, as I have done in most cases so far. And so then, what I am doing is going through the example, and then highlight[ing] the important facts on the example, and then condens[ing] them into a general observation. And I have several times mentioned to students that this is what we’re doing, and that it’s a good idea to see an example not as an isolated example but rather as a representative of a big class. (M9, 11:58)

In this sense formulating an example is a means (a psychological tool) to an informal understanding. The informal understanding then acts as the bridge to the formal definition of a concept, and to the formal language of mathematics when settings become more abstract. In this sense the formulation of the example functioned as a conceptual tool. The lecturer also formulated the idea that examples encouraged students to think about mathematics and to view mathematical examples in a certain way (i.e. to understand the role they play in mathematics). In a research meeting he said,
And the way that the observations work is that, most of the time, we go over an example first and then we extract a general statement from the example we have gone through. And the entire thing is based on the observation that the same argument we have just used in the example will also apply in a very large class of other examples, which makes the step from the specific to the general. And I have discussed that step with students in class many times. (Slight pause.) And again, I don’t know how much of that actually sticks. I repeat it because I think it’s important, and then I’m hoping that students go away and start thinking about things the way I demonstrate in class. (M15, 56:10)

In this sense the example functioned as a \textit{cultural tool}.

In the course notes the concept of subspace was introduced with Example 3.4 (see Section 6.2.2 or page 220). The lecturer made the didactical decision to choose an example that was a very \textit{specific subspace} of $\mathbb{R}^n$, namely the \textit{null space} of a matrix $A$. This is just one of many possible subsets of $\mathbb{R}^n$. The lecturer chose this particular example (above any other) because he was designing a task for students that involved generating further solutions to the equation $Ax = 0$ when two solutions were known. The solutions $x$ to the equation $Ax = 0$ generate the \textit{set} of elements that form the null space of the matrix $A$. I interpreted the design of this example as an attempt to \textit{engage} students. \textit{Engagement} was (a) \textit{physical}, in lecture in the sense of more active participation, and (b) \textit{mental}, in the sense of engaging with the mathematical concept, and (c) designed to be accessible to all. The example represented a \textit{material tool} and the formulation of the example a \textit{psychological tool}. Example 3.4 carried out different, or more than one function.

In a research meeting the lecturer talked about anticipating students’ reaction to the exercise that he was going to set for them.

The way I’m going start on subspaces, I’m going to summarise addition and multiplication, and ... the linearity properties of linear maps, ... and then I’m going to tell students, “Assume we’ve got some matrix that I don’t tell you, but I tell you this vector and that vector are solutions of the homogenous equation system $Ax = 0$, find more of them”. Very curious to see how that’s going to go, they’ll probably be shocked ... If they do that well, they will understand what subspaces are all about. And ultimately in that context, saying that the null space of a matrix is a subspace, means you only need a few solutions and then you’ve got all of them. That’s really an amazing
statement, and I’m hoping I’ll be able to make students appreciate that.
(M6, 14:29)

The lecturer had mathematical reasons for asking students to generate solutions in this instance. For the null space further solutions can be found fairly easily. This is not necessarily the case for any subspace of $\mathbb{R}^n$. The lecturer also had a didactical reason for asking students to work with the null space in the form of a matrix equation involving column vectors. The lecturer was hoping that students “feel comfortable” (see quotation on page 109). He frequently also talked of providing students with “hands-on experience” (see quotation below on page 113). In introducing students to working inductively with examples (in this case with generating solutions) and in the more concrete setting of $\mathbb{R}^n$ (with vectors and matrix equations), the lecturer was hoping that students would be able to make the step to more formal settings when required to do so in the second semester. That was one of his goals for using examples in an inductive way in his teaching in the first semester. In a research meeting he said,

> And so my goal in this semester is to give the students the hands-on experience on handling vectors so that when they see things again in the abstract . . . . . . as much as possible they feel they know what they’re talking about. And that also means that, in that context, being able to generate an example is quite important, because to understand the abstract concept, well, certainly for me and I know for most colleagues, the first thing [we] do is ask ourselves for an example, have we seen that anywhere before. (M9, 18:13)

Asking students to generate solutions introduced the concept of a subspace in a way that was accessible to students. Choosing the null space as an example ensured that further solutions could be found relatively easily since the equation was set equal to zero. Students were able to access this exercise and formulate new - their own - solutions (see also next Section 6.2.4). In doing so they were participating in the lecture and engaging with the mathematics, and with each other. In this ‘activity’ which was mathematical and carried out in the social plane, students were encouraged (by the lecturer and his teaching approach) to engage and through engagement to learn ‘intuitively’ about the concept of a subspace.

This was the intention of the lecturer in presenting this example, and in presenting examples in general. His teaching focussed on the use of an example from which to build conceptual understanding, and with which to engage students with ‘doing’ mathematics (in a practical way). Using course notes with ‘gaps’ that needed ‘filling in’ in lecture, the lecturer encouraged participation in lectures, and interaction between students and
between himself and individual or small groups of students in the form of mathematical discussions while working on the exercise.

Thus the role of Example 3.4 in introducing the concept of a subspace was manifold. The lecturer had a mathematical reason for choosing the null space of a matrix as example of a subspace. The example was aimed at engagement: As a material tool the example was aimed at increasing students’ participation in lectures (physical engagement); as a psychological tool it was (a) aimed at students’ mental engagement with the mathematics (thinking about mathematics) and (b) designed to be accessible to all students.

From analysing the lecture observation data (that is, the audio-recordings of the lectures) I have been able to make some comments on how students worked with this example in the lecture. This is part of the focus of the next Section 6.2.4.

6.2.4 The lecturer’s teaching of subspace (to students)

In this section I consider how the lecturer translated his didactical thinking and planning into actions in the classroom. I listened to the audio-recording of the lecture (L12, the first lecture of the chapter “Subspaces of \( \mathbb{R}^n \)”) which captured all that the lecturer actually said to his students. I also read the field notes taken by one researcher who attended the lecture. (The second researcher was not present.) This gave an indication of what occurred in the lecture that an audio-recording could not capture.

From this analysis I gained insight into the extent to which engagement with Example 3.4 took place, and the didactical perspectives that could be interpreted from the lecturer’s actions. What I found striking was the extent of verbal explanations given in the lecture to students, using both informal and more formal language. The lecturer expressed his intentions, at various levels and in respect of what to focus on and how to look at it and why, in the lecture to students. In particular, he expressed explicitly in this lecture (and other lectures), that he wanted students to acquire precise mathematical language which was necessary for conceptual understanding.

The lecturer followed the course notes and presented the content in the order that it appeared in the notes. Where there were ‘gaps’ in the notes the lecturer asked students to work on the solution to the example by themselves or in small groups. What became apparent by listening to the audio-recording was the extent of verbal explanations: At the beginning of the lecture the lecturer recapped what had been covered in lectures in the previous week(s). He also introduced new content or ‘projected’ ahead to the
content to be covered in the lecture ‘today’. He gave extensive verbal explanations at the beginning of the lecture (recapping, introducing, projecting ahead) and at the end of the lecture (summarising). As a result of this analysis I noticed that the course notes did not contain many explanations at all.

The lecturer followed the course notes closely in terms of the mathematics that he presented. He projected the student version of the notes (containing ‘gaps’) during the lecture onto a large screen. He covered the first page of Chapter 3 in relation to Observations 3.1 to 3.3 (see page 219) by reading out in the main what was written down, but with further explanations and linking to previous work on linear equation systems in Chapters 1 and 2. In particular, he linked the Observation 3.1 to Observation 3.3 and made remarks to the effect that they were ‘very similar’ (the lecturer wrote that they “corresponded” in the course notes, and in the lecture he said that they were “closely related” - see quotation below). He said about Observation 3.3,

So these three properties are very closely related to the important properties of the vectors and of the set \( \mathbb{R}^n \). And they might seem quite elementary, just simple rules of matrix algebra, but they have very deep consequences. And what we’re going to do for most of the rest of the semester really is find out what the consequences of these simple properties are. And what we’re going to start with today is try to understand in a little more detail why linear equation systems work the way they work. (L12, 12:30)

At this point approximately fifteen minutes had passed since the start of the lecture (which had a duration of fifty minutes). In introducing the Example 3.4 and asking students to generate more solutions, the lecturer posed a question to students,

But it can never have two solutions and nothing else. Why is that? (L12, 13:24)

and continued,

We know that if there are two, there must be infinitely many. I want you to find some now. (L12, 14:50)

He then invited students to work on the exercise, saying

And I’ll give you depending on how it goes between five and ten minutes to do that. So, please, get together with a couple of people around you and
see if you can find out more solutions to this homogeneous equation system.

(L12, 14:58)

Altogether the lecturer gave students sixteen minutes to generate solutions. The audio-recording of the lecture ‘showed’ background noise that could indicate that students were engaging and working on the exercise. One of the researchers captured the following in the field notes:

10:15 Students work on this task - a buzz in the room. As I look around I see some students writing as they talk, some seemingly not working on the problem - are they waiting for him to fill in the answer? (L12, FN1)

Thus it appeared that some students were working while others were not. During this time the lecturer walked up and down the aisles at the side of the lecture hall, talking to students individually or in small groups from time to time. After sixteen minutes the lecturer returned to the front of the lecture hall and began writing up the first solution as it was set out in the (full) notes. Then he asked students if anyone was willing to give the solution that they had found. Several students had generated their own solutions, and readily offered them in the lecture which the lecturer ‘accepted’ and wrote on the overhead projector. There was an exchange of ‘Question-and-Answer’ type between the lecturer who was asking for solutions and ‘why what was offered was a solution’, and individual students who answered. It seemed, based on my listening of the audio-recording of the lecture that the lecturer enjoyed the verbal exchanges with the students. He appeared to get excited as more and more responses were offered. In research meetings the lecturer stated that he saw engagement with the examples and hence the mathematics as necessary for acquiring a conceptual understanding. He said,

I think, it [conceptual understanding] won’t come automatically, but if it is to come it can only come from working with the objects. (M18, 25:58)

He seemed pleased that this kind of engagement had occurred for some of his students. The lecturer often used ‘we’ when referring to himself and his students. In a research meeting he said,

I mean . . . , when I say ‘we’ in class, at least most of the time I mean ‘us’ who are there, the students and me. (M5, 57:07)

With reference to the following quotation I have interpreted engagement as being at the heart of the lecturer’s didactical thinking. In a research meeting he said,
‘We’ is the community. Now, you could say, implicit in that phrase is the assumption that they’re going to be with me and they’ll engage with that. I mean . . . , if they don’t there isn’t much point in doing things at all, . . . (M5, 50:36)

With this quotation the lecturer expressed the view (and I repeat it here):

I mean . . . , if they don’t [engage in lecture/engage with the mathematics] there isn’t much point in doing things at all, . . .

I interpreted this statement as representing the ‘core’ of his didactical thinking and decision-making which rested on being able to engage students, in the activity of learning mathematics in the lecture.

In working with Example 3.4 students were creating (new) solutions that were different from those that the lecturer had written down himself in the course notes and which eventually (at the end of the module) became available on LEARN, the university’s virtual learning environment. With reference to the field notes, students’ participation seemed to vary. This view was supported by the lecturer in a research meeting much later in the semester when he reflected on his introduction of the concept of subspace and the example that he had presented to students. He said,

At some points I realised I need to find different ways of phrasing the questions in order to make them more accessible. One example of that was the introductory example on subspaces, where I had asked students to find solutions to a homogeneous equation system with unknown coefficient matrix, given that they know a couple of solutions that I’ve given them. That was one question where I saw quite clearly that some of the students found it very easy, and some of the students didn’t have the slightest idea even if they tried. (M15, 48:18)

The reasons for this variation in participation were not the focus of my study and remained largely speculative. The lecturer suggested that this kind of task may have been too unfamiliar, something students were rarely asked to do at university. A number of students took part in focus group interviews some time later in Semester 2. Students talked about their participation in lectures and their engagement with the examples. Some students gave reasons for participating and some for why not. This confirmed that students’ response to the exercise that the lecturer had set them, varied. Some students ‘knew’ what to do and tried to generate solutions while others did not know what to
do or how, and still others indicated knowing what to do but opting out and waiting for the lecturer to present the solution ‘neatly’. A question on one of the questionnaires resulted in a similar range of responses. I discuss students’ responses in more detail in Chapter 7 where I report on my analysis of the student data.

**In summary**, in Section 6.2.3 I have discussed the lecturer’s didactical thinking and in Section 6.2.4 how he implemented his thinking into practical actions in face-to-face teaching. I discussed the teaching of subspaces in the context of the vector space $\mathbb{R}^n$ and the different functions of the Example 3.4 that the lecturer used to introduce the concept of subspace.

This concludes the first part of Chapter 6. I have analysed the concept of a subspace in relation to four areas of consideration: a mathematical account based on my reading of three textbooks and a set of course notes; a mathematical account of the lecturer’s course design in relation to the concept of subspace; an analysis of the lecturer’s didactical thinking and planning, and an analysis of his face-to-face teaching of students. I now discuss the concept of linear independence in relation to these same four areas.
6.3 The concept of linear independence

In Section 6.1.1, I gave reasons for analysing the concept of linear independence. I referred to the lecturer’s own words as recorded in research meetings and to the research literature into the teaching and learning of linear algebra.

As with ‘subspaces’ (in Section 6.2), I analysed the concept of linear independence in respect of four areas:

i. A mathematical account of linear independence

ii. The lecturer’s mathematical treatment of linear independence in the course notes

iii. The lecturer’s didactical thinking in relation to the teaching of linear independence

iv. The lecturer’s teaching of linear independence (to students)

I consulted the same four textbooks/set of course notes that I used previously, namely Hoffman and Kunze (1961), Greub (1967), Poole (2006) and Sproston (1995). Whenever I quote from the textbooks/set of course notes I may alter the letters that the authors used in their definitions and mathematical statements, etc. but not the formulation of the statements (as stated in my introduction to Chapter 6, in Section 6.1.4). In this section (as I have done in the last section) for ease of comparison, I adopt the letters \(x_1, x_2, ..., x_n\) and \(v_1, v_2, ..., v_n\) to denote vectors, and \(U\) and \(V\) to denote sets of vectors. I also use the letters \(c_1, c_2, ..., c_n\) and \(\lambda_1, \lambda_2, ..., \lambda_n\) for scalar values, and \(K\) for a field.

I again assume that the reader has some familiarity with the mathematics of a first year, introductory linear algebra module (including general mathematical notation and terminology) as detailed in Section 6.1.5. For example, in my analysis in this section (Section 6.3) I use the terms definition and proposition without further explanation as well as the clause if and only if. I also refer to several linear algebra concepts, in particular span and spanning set, linear combination and linear relation, and range.

6.3.1 A mathematical account of linear independence

As in Section 6.2.1 on the topic of subspaces I again consulted three textbooks and a set of course notes to see how the concept of linear independence was defined and introduced to the reader. With reference to these sources, textbook authors gave a (virtually) identical account of the concept of linear (in)dependence. Linear (in)dependence is defined in terms of a linear combination so that without first defining a linear combination the definition of linear (in)dependence does not make sense. There was little variation
in the approach taken by the four different authors. For three of the four authors the sequence for introducing the concept of linear (in)dependence consisted of (first) defining a linear combination, followed by span and/or spanning set and then the definition of linear (in)dependence. Greub (1967) defined span (using the to some degree outdated terminology of a “generator”) after he defined linear independence. I state here the definition of a linear combination as given by Poole.

**Definition.** A vector \( \mathbf{v} \) is a linear combination of vectors \( \mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k \) if there are scalars \( c_1, c_2, ..., c_k \) such that \( \mathbf{v} = c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + ... + c_k \mathbf{v}_k \). (Poole, 2006, p. 12)

Authors differed in defining linear independence first or linear dependence. Although mathematically the definitions are equivalent, the precise formulation of the definitions (the syntax and the wording of the statements) is different. Based on my reading I reproduce here a definition of linear dependence and a definition of linear independence. The versions I give (Definition (1) and Definition (2)) are based on my reading of the three textbooks and the set of course notes.

**Definition (1) of linear dependence.** Let \( V \) be a vector space over \( K \). A subset \( U \) of \( V \) is linearly dependent if there exist distinct vectors \( \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, ..., \mathbf{x}_n \) in \( U \) and scalars \( \lambda_1, \lambda_2, \ldots, \lambda_n \) in \( K \), not all of which are 0, such that

\[
\lambda_1 \mathbf{x}_1 + \lambda_2 \mathbf{x}_2 + \cdots + \lambda_n \mathbf{x}_n = 0.
\]

A set which is not linearly dependent is called linearly independent. (see Hoffman & Kunze, 1961, p. 40)

Alternatively, authors gave the definition for linear independence.

**Definition (2) of linear independence.** Let \( V \) be a vector space over \( K \), \( U \) a non-empty subset of \( V \) with \( n \) elements. Then \( U \) is linearly independent if and only if the following condition holds: If

\[
\lambda_1 \mathbf{x}_1 + \lambda_2 \mathbf{x}_2 + \cdots + \lambda_n \mathbf{x}_n = 0
\]

where \( \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, ..., \mathbf{x}_n \) are all the elements of \( U \) and distinct, and \( \lambda_1, \lambda_2, \ldots, \lambda_n \in K \), then

\[
\lambda_1 = \lambda_2 = \cdots = \lambda_k = 0.
\]

A set which is not linearly independent is called linearly dependent. (see Sproston, 1995, p. 6)

Both are “formal” definitions. They use mathematical notation which is deeply embedded in mathematical conventions (for expressing definitions, theorems and other
mathematical statements). The definition of linear independence includes the equation

$$\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n = 0 \quad (\lambda_i \neq 0)$$

which expresses a linear relation between the vectors $x_i$ for $i = 1, 2, \ldots, n$. By re-arranging the equation each vector $x_i$ can be expressed as a linear combination of the other vectors. For example,

$$x_1 = -\frac{\lambda_2}{\lambda_1} x_2 - \frac{\lambda_3}{\lambda_1} x_3 - \cdots - \frac{\lambda_n}{\lambda_1} x_n \quad (\lambda_i \neq 0)$$

where I chose $x_1$ to denote the left-hand side. Choosing each $x_i$ in turn by taking $i = 2, 3, \ldots n$, results in linear combinations that are similar to the one I stated above.

Definition (1) states that for a linearly dependent set of vectors it must be possible to write one or more of the vectors as a linear combination of one or more of the remaining vectors. This implies that not all values of $\lambda_i$ can be equal to 0. Definition (2) states that for a linearly independent set this is not possible, that is, none of the vectors $x_i$ can be written as a linear combination of one or more of the remaining vectors. This means also that the equation $\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n = 0$ cannot be solved, unless all the values $\lambda_i$ are 0.

All the authors that I consulted had defined the concept of a linear combination prior to introducing the definition of linear independence. Both Poole (2006) and Hoffman and Kunze (1961) stated the definition of linear (in)dependence as “definitions”, Greub (1967) attached no label, while Sproston (1995) attached the label “proposition” (which required a proof).

---

5 In the definition above, the statement

If $\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n = 0$,

\ldots, then $\lambda_1 = \lambda_2 = \cdots = \lambda_k = 0$,

means that only the trivial solution (that is, the solution $\lambda = 0$) exists for solving

$$\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n = 0.$$ 

The definition also contains the clause “if and only if” in its statement of linear independence, namely, $U$ is linearly independent if and only if the following condition holds: \ldots

Hence it is possible to define linear independence as follows.

If only the trivial solution exists for solving

$$\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n = 0,$$

then the set of vectors $U$ is linearly independent.

6 For example, the equations $v_3 = v_1 + v_2$ and $v_1 + v_2 - v_3 = 0$ can be referred to as linear relations. They express a linear relationship between the vectors $v_1, v_2$ and $v_3$. In contrast, a linear combination is the expression of the form $v_1 + v_2 + v_3$. 


Sproston (1995) and Poole (2006) gave less formal definitions alongside the formal definitions. Sproston (1995) first wrote,

| Let $V$ be a vector space over $K$, and $U$ a non-empty subset of $V$. If $U$ has two or more elements, we say $U$ is a linearly independent set if no element of $U$ is a linear combination of other elements of $U$ (if $U$ has only one element $x$, we say that $U$ is linearly independent if $x \neq 0$). |

and followed with Definition (2). He referred to Definition (2) as a ‘criterion’ for linear independence, and wrote “which is in practice what one checks”, and “in many books, indeed, this is given as the definition of linear independence” (Sproston, 1995, p. 5, italics in original).

Poole (2006), on the other hand, approached linear (in)dependence by stating Definition (1) first and followed it with a theorem that said,\(^7\)

| A set of vectors ... is linearly dependent if and only if at least one of the vectors can be expressed as a linear combination of the others. (Poole, 2006, p. 448) |

In the approach taken by Poole, the mathematical content was structured in such a way that the concept of linear independence was introduced three times. The first time the author introduced linear independence in the context of the vector space $\mathbb{R}^n$ with \textit{n-dimensional vectors} (often represented as column vectors) as elements of the set; he used geometry and ‘pictures’ as visual aids in relation to vectors being independent in $\mathbb{R}^3$; he introduced further theorems to relate linear independence to solution sets and in terms of the columns of a matrix $A$. The second time he introduced linear independence in the context of matrices and matrix algebra and referring to row and column space, and the third time in the context of an abstract vector space $V$. Thus this textbook author presented multiple ways of looking at linear independence which, as I will show, was not unlike the way the lecturer in my study presented this topic.

All authors presented the concept of linear (in)dependence in the same ‘DTP’ (definition-theorem-proof) style as they had done in relation to the concept of a subspace (see Section 6.2.1).

In all textbooks the concept of linear independence was part of a chapter on vector spaces. All the important concepts in linear algebra were introduced in that chapter: linear combination, linear independence, span and spanning sets, basis and change of basis, rank and nullity. As I have shown the concept of linear (in)dependence is defined in terms of a linear relation. In a similar way the concept of span is defined in terms of a

\(^7\)I noted that Poole (2006) should have written “if at least one of the vectors can be expressed as a linear combination of \textit{one or more of} the others”. This was possibly an error not discovered at the proof-reading stage of publication.
linear combination, and the concept of a basis in terms of a span and linear independence. Dorier (2000) pointed out the interdependence of linear algebra concepts and reported on his research into students’ difficulties with these concepts.

In the next section I discuss how the lecturer in my study designed his teaching of the concept of linear independence, and compare his approach with the one taken by the textbook authors.

### 6.3.2 The lecturer’s mathematical treatment of linear independence in the course notes

I analysed both the student version and the full (complete) version of the course notes in relation to the teaching of the concept of linear (in)dependence, and the lecturer’s comments made in research meetings.

In the course notes, Section 3.4 (of Chapter 3) had the title “Linear Independence”. I considered this section an appropriate starting point for my analysis of the lecturer’s mathematical treatment of linear independence. The relevant pages of the course notes to which I refer in my analysis are attached (see page 222 to page 227).

Vector spaces and subspaces marked the beginning of Chapter 3 of the course notes which was, as the lecturer had remarked the “heart of the course . . . where all the abstract concepts really are” (expressed in meeting M18, 27:42). After discussing the concept of a subspace the lecturer introduced several other concepts prior to linear (in)dependence. He introduced the concept of a linear combination, span and spanning set in relation to the null space of a matrix, and span and minimal spanning set in relation to the range of a matrix. The latter led to the definition of linear dependence. I now discuss the lecturer’s development of these ‘steps’ towards defining linear (in)dependence in more detail.

The concept of linear (in)dependence was introduced via two examples. In the first example, Example 3.16 (see below), the lecturer asked students to find a set of vectors that spanned the range of a given matrix \( A \). In the second example, Example 3.17 (see...
page 222), he asked students to find a smaller set that still spanned the range of the same matrix $A$. I have reproduced Example 3.16 as it appeared in the course notes.

**Example 3.16.** Find a set of vectors that spans the range of the matrix

$$A = \begin{pmatrix} 1 & 4 & 5 \\ 1 & -2 & -1 \\ 2 & 1 & 3 \end{pmatrix}$$

**Solution:**

Half a page was left blank here.

The lecturer presented the solution to the example in the lecture (see Section 6.3.4 for a fuller discussion). He wrote that a vector $b$ is in the range if

$$b = x_1 \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + x_2 \begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix} + x_3 \begin{pmatrix} 5 \\ -1 \\ 3 \end{pmatrix}.$$  

The lecturer then wrote on the overhead projector,

Thus, a vector $b$ is in the range of $A$ if and only if it can be written as a linear combination of the column vectors of $A$ [which are]

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix}, \quad \text{and} \quad v_3 = \begin{pmatrix} 5 \\ -1 \\ 3 \end{pmatrix}.$$  

He also wrote the sentence,

“The column vectors of $A$ span the range”.

This solution became part of the complete set of course notes and available to students towards the end of the module (see page 239).

In the second example, Example 3.17 (see page 222), the lecturer asked students if it were possible to find a smaller set that could still span the range. He wrote,

**Example 3.17.** Can you find a smaller set of vectors that spans the range of the matrix $A$ in Example 3.16?
In the course notes the lecturer obtained the linear relation $v_3 = v_1 + v_2$, and hence wrote the solution set in terms of $v_1$ and $v_2$ alone. The lecturer obtained the linear relation $v_3 = v_1 + v_2$ by inspection [my interpretation] (and not by calculation, say). This relation as well as the alternative form $v_1 + v_2 - v_3 = 0$ were printed in the course notes that students brought to the lecture. That is, students could see part of the solution to the example before the lecturer presented it.

In both examples (3.16 and 3.17) the lecturer had provided blank spaces after the example (as he had done throughout his teaching of this module) in anticipation of student involvement with the mathematical problems that he had posed. The first page of the course notes, in relation to the topic of linear (in)dependence (page 7), contained the outline of two examples with the remaining text area blank. The examples were presented without any written explanations apart from the initial instruction to students (which I have reproduced above).

Following Example 3.17, the lecturer stated the formal definition of linear dependence (Definition 3.18). The lecturer’s definition of linear dependence is identical to the ones stated in the textbooks/set of course notes that I consulted for my analysis in Section 6.3.1. I have reproduced Definition 3.18 here from the course notes.

**Definition 3.18.** The vectors $v_1, v_2, \ldots, v_p$ are linearly dependent if there are numbers $\lambda_1, \ldots, \lambda_p$, not all zero, such that

$$\lambda_1 v_1 + \cdots + \lambda_p v_p = 0.$$  \hspace{1cm} (3.1)

In this case, Eq. (3.1) is called a linear relation between the $v_i$. The vectors $v_1, v_2, \ldots, v_p$ are linearly independent if they are not linearly dependent.

The lecturer then presented an example (Example 3.19) where students were asked to show that the first set of vectors was linearly dependent, and that the second set was linearly independent. This example drew information about the vectors from Example 3.16. Thus the lecturer used the content of one example to develop and formulate further examples.

In terms of the sequencing of the material, the lecturer’s introduction of linear independence mirrored the way that the textbook authors introduced this topic. The concept of linear (in)dependence depends on an understanding of linear combinations. As stated in Section 6.3.1 the textbook authors introduced the concepts of a vector space and

---

10 ‘By inspection’ is part of general mathematical terminology whereby a mathematical fact or result is obtained without a calculation or deduction, that is, by ‘seeing’ a relationship.
subspace first, followed by a linear combination, and span and spanning set before introducing linear (in)dependence.\footnote{11} The lecturer in my study also followed this sequence.

Although the lecturer introduced the concept of linear independence in this place in the module it was not the first time that he mentioned the term. The lecturer introduced the term ‘independent equations’ in Chapter 1 of the course notes in connection with the solution sets of linear equation systems (see Example 1.12 on pages 234 and 235). Example 1.12 relates to a system of two equations in two unknowns where one equation is a scalar multiple of the other equation.\footnote{12} The lecturer wrote,

\begin{tabular}{|l|}
\hline
We say: The equations are not \textbf{independent} because they satisfy \\
\hline
$(\text{eq. } 2) = 2 \times (\text{eq. } 1)$ \\
or $2 \times (\text{eq. } 1) - \text{eq. } 2 = 0$ \\
\hline
\end{tabular}

This example was introduced by the lecturer in a tutorial (in Week 2 of the module). He defined independent equations by referring to them as giving ‘no additional independent information’. He said to his students,

Now some of you are asking, what precisely does it mean for equations to be independent. And for the moment, unfortunately, I can give you only an informal answer to that. . . ., if we have, say, three equations we can solve for three variables. . . . We have seen in the examples on Wednesday that sometimes it goes wrong. If one equation is a multiple of another equation, those equations aren’t independent. If I know one equation is satisfied I automatically know the other equation is also satisfied. So that other equation doesn’t give me additional independent information. That’s what I mean by that [an independent equation]. . . . (L5, 46:54)

With this explanation, the lecturer defined independence \textit{informally} [in the lecturer’s own words] in Chapter 1, by referring to a relationship between the equations. I discuss the lecturer’s \textit{informal approach} in relation to the teaching of linear independence in more detail in Section 6.3.3.

\footnote{11}There was one exception. Greub (1967) introduced span after the concept of linear independence. \footnote{12}In this case, the system of equations has many solutions. In general, a linear equation system has either no solution, one (unique) solution or many solutions. If a linear equation system is inconsistent, no solution can be found. If the system is uniquely determined, one solution can be found. If the system is underdetermined, many solutions exist, and one or more parameters are introduced in order to write down the solution set. (This related to Sections 1.2 and 1.3 in the course notes which have not been appended.)
In Chapter 3 the lecturer stated the *formal* definition of linear (in)dependence (Definition 3.18, as above). In addition, in Chapter 3, the lecturer re-visited linear equation systems which was the topic of Chapter 1. This time the equation systems were written in matrix form. I refer to the explanation following Observation 3.32 in the course notes (see page 225). This states that, if the columns of a matrix $A$ are linearly independent then the homogeneous system $Ax = 0$ has only the solution $x = 0$. Referring to the ‘number of linearly independent columns of $A$’ describes what Dorier et al. (2002) called ‘pivot counting’ in their critique of Frank Uhlig’s matrix based approach to the teaching of linear algebra (see Uhlig, 2002). Again, I discuss this in more detail in Section 6.3.3 when I consider the lecturer’s didactical thinking and decision-making.

The lecturer also referred to linear (in)dependence in Chapter 4, in relation to eigenvectors forming a linearly independent set of basis vectors. Thus the lecturer in my study introduced and discussed linear (in)dependence on four separate occasions in his teaching of this module. He ‘re-visited’ the topic of linear (in)dependence as part of his approach as did Poole (2006). However, while Poole discussed linear algebra concepts in more than one setting ($\mathbb{R}^n$ and an abstract vector space $V$), the lecturer in my study formulated statements in different contexts (linear equation systems, sets of vectors, linear equation systems in matrix form, sets of basis vectors) but *always* in the setting of $\mathbb{R}^n$. I have discussed the role of $\mathbb{R}^n$ in my didactical analysis of the teaching of subspaces in Section 6.2.3.

In summary, the concept of linear independence was introduced on several occasions and through reliance on span and spanning sets. I have presented a multiple view of this concept which reflects my analysis of the course notes and the comments made by the lecturer in research meetings.

In the next section I discuss the lecturer’s didactical thinking and decision-making in connection with the teaching of linear independence, and compare with what I have learnt from my analysis in this section.

### 6.3.3 The lecturer’s didactical thinking in relation to the teaching of linear independence

In Section 6.2.3 (in connection with subspaces) I particularly considered the role of the vector space $\mathbb{R}^n$ and the role of Example 3.4 in the inductive approach to teaching. In this section I discuss the lecturer’s didactical thinking in planning and designing the teaching of linear (in)dependence. My discussion centres again on (a) the role of
examples (Examples 3.16 and 3.17), and (b) on the role of informal language as used by
the lecturer in the writing of the course notes and in his face-to-face teaching of students.

In (a) I discuss the examples in terms of their function and, in doing so, compare with
what I have learnt from my analysis in Section 6.2.3. In (b) I discuss the role of both
formal and informal language in teaching, and refer to the written form (such as the
course notes) and to the spoken form (as used by the lecturer in lectures). The latter
involves quoting what the lecturer said to his students in face-to-face teaching. I will,
therefore, need to refer ahead to the next section from time to time.

(a) The role of Examples 3.16/3.17

In planning the teaching of linear independence the lecturer presented a specific
example, namely Example 3.16 (see page 124) for finding the spanning set of the range
of a given matrix $A$. The lecturer had introduced the concept of span in the previous
week and designed teaching materials that included students working on problems in
the lecture and in the tutorial. I assume (and both audio-recordings of lectures and
tutorials, as well as the written course notes support my assumption) that students
have had access to practicing finding the span and spanning set of a matrix. In this
sense I assume that Example 3.16 represented a ‘familiar’ mathematical problem for
students to solve. I further claim that the lecturer, in formulating and choosing to use
these two examples, also assumed that the problem was a ‘familiar’ problem for his
students. The lecturer left a blank space where students could write the solution and
hence participate in the lecture and engage with the material.

He then presented Example 3.17 as an extension to the initial example. With Example
3.17 the lecturer posed the question “Can you find a smaller set . . .?” Again, he left a
blank space for students to write the solution. The lecturer stated the formal definition
of linear dependence immediately afterwards.

Example 3.16 was a concrete example. It contained numerical values for the matrix
and vector and required students to calculate (and obtain a vector equation as a re-
sult). Students needed to know how to multiply a matrix and a vector (a mathematical
‘technique’ or an ‘algorithm’) in order to perform this calculation. The resulting vector
equation represented a unique solution, a result that students either obtained or failed
to obtain.

In presenting Example 3.17 the lecturer posed a question. The answer to the question
required students to look for a linear relationship between the vectors in order to re-
duce the number of vectors in the set. I interpreted this example as requiring students
to *guess*. I interpreted this example as a ‘concrete’ example as it contained numerical values, that required students to *calculate* alongside *guessing*. However, the required relation \( v_3 = v_1 + v_2 \) was printed in the course notes that students brought to the lecture. This meant that students could ‘fill the blank space’ after the example by performing the calculation alone. Students did not need to ‘find the relationship’ between the vectors (for themselves).

With Examples 3.16 and 3.17 the lecturer *structured* the transition from the concept of *span* to *minimal span*. It required students to have knowledge of the concepts of span and spanning sets and how to calculate with them. I am interpreting the use of these two examples as providing a ‘bridge’ between the concept of span and the concept of linear (in)dependence. In this sense the problems posed in the examples for students to solve functioned as *conceptual tools*.

In my interpretation the examples were *designed* as a *conceptual tool*. However, the relation was ‘given’. Students were required to perform a calculation where the calculation may or may not lead students to a conceptual understanding of linear independence.

In the next section (6.3.4) I show that the lecturer did not ask students to work on the solutions themselves *first* before he himself presented the solutions. The audio-recording of the lecture shows that the lecturer worked out the solution to both examples at the front of the lecture hall on the overhead projector immediately after introducing them.

Thus, based on my analysis of the lecture observation data, that is the lecturer’s face-to-face teaching, the lecturer used Example 3.16/3.17 to *demonstrate* the transition (in a very concrete sense) from one spanning set to a smaller spanning set (showing the importance of a *minimal spanning set*), and hence to the definition of linear dependence. The examples did not function as a *technical tool* for student participation in the lecture since the lecturer *showed* the solution. The examples were a *cultural tool* giving a view of mathematics as a subject with which to calculate and obtain a result.

There is one further point I wish to make in respect of the role of the examples.

In presenting the solution to Example 3.16 the lecturer made the observation that the columns vectors of \( A \) span the range, that is the vectors in the range of \( A \) can be written as a linear combination of the column vectors of the matrix \( A \) (see page 239). The lecturer made the didactic decision to present this ‘short-cut’\(^\text{13}\) to his students by

\[ \{ (1,0,0,\ldots), (0,1,0,\ldots), \ldots, (0,0,0,1) \}. \]

\(^\text{13}\)I am referring to the method of determining the spanning set of the range of a matrix without actually calculating it. The method ‘works’ because the basis is the standard basis \( \{ e_1, e_2, \ldots, e_n \} = \{ (1,0,0,\ldots), (0,1,0,\ldots), \ldots, (0,0,0,1) \} \).
actually and formally calculating the spanning set of the matrix $A$. Unlike some of the textbooks authors that I consulted, the lecturer ‘derived’ this ‘shortcut’ rather than ‘telling’ his students about it and how to use it. However, in deriving this ‘shortcut’ the lecturer’s focus was not on the *concept*, but on the *method or technique* for ‘writing down’ the spanning set of the range of given matrix (without a calculation). In the course notes, the lecturer wrote, “…a vector $b$ is in the range of $A$ if and only if it can be written as a linear combination of the column vectors of $A$”. This ‘result’ was based on a *calculation*, and *not* on ‘an argument made on the example’ as envisaged by the ‘EAG’ approach that I discussed in Chapter 5.

**(b) The role of informal language**

In the last section I discussed the lecturer’s didactical thinking in using and formulating examples in order to engage students in the lecture and with the mathematics. In this section I consider the lecturer’s *use of language*, that is, both the formal and informal language used in the course notes (the *written* form), and both the formal and informal language used in face-to-face teaching (the *spoken* form).

In my interpretation, in Chapter 3 of the course notes, the lecturer made the didactical decision to develop the concept of linear independence through working with span and spanning sets (see Section 6.3.2). I therefore analyse Example 3.8 as well as Examples 3.16/3.17 in relation to the language used in *the course notes*.

The lecturer *first* introduced the concept of *span* with Example 3.8 as follows:

**Example 3.8.** Find the null space of the matrix

$$A = \begin{pmatrix} 1 & 2 & -3 \\ -2 & -4 & 6 \\ 3 & 6 & -9 \end{pmatrix}.$$

The solution set contained two parameters, $\lambda$ and $\mu$. The lecturer wrote that all solutions could be written as a linear combination of the two vectors $v_1$ and $v_2$. I have reproduced the relevant section from the course notes.
Thus, all solutions can be written as $x = \lambda v_1 + \mu v_2$ with the two vectors

$$v_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad v_2 = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$$

We say: The vectors $v_1$ and $v_2$ span the null space of $A$.

The lecturer did not use the words “basis” or “basis vectors”. In the course notes the lecturer gave no explanations in terms of everyday (or informal) language. Based on my analysis and interpretation of the course notes, the lecturer introduced the concept of span with reference to a mathematical procedure, by ‘performing’ a calculation.

In the lecture in which Example 3.8 was presented, the lecturer talked (and read out) as he wrote on the overhead projector, as he had done in lectures generally. After writing down the solution to the example, namely the set of vectors that denotes the null space of the matrix $A$, he said,

And we see now, we have two fundamental vectors here that we know are solutions. There is this one which I’ll call $v_1$, there is this one which I’ll call $v_2$, and we know from this result that every solution of the homogeneous equation system can be written as $\lambda$ times $v_1$ plus $\mu$ times $v_2$.\textsuperscript{14} (L13, 08:37; italics correspond to emphasis audible in the recording.)

After defining span with the sentence,

We say: The vectors $v_1$ and $v_2$ span the null space of $A$.

the lecturer repeated,

Every solution can be written as $\lambda$ times $v_1$ plus $\mu$ times $v_2$.

Thus the lecturer referred to the basis vectors $v_1$ and $v_2$ as “fundamental” vectors. He gave no further explanation than to refer to the linear combination $\lambda v_1 + \mu v_2$ as the required solution to the example, and as meaning that $v_1$ and $v_2$ span the null space.

Hence, in the course notes, in the very first introduction to span the lecturer used an outline of an example with (almost) no explanation in terms of informal or everyday

\textsuperscript{14}The lecturer was referring to the system $Ax = 0$. To find the null space of the matrix $A$ involved solving $Ax = 0$ using Gaussian elimination.
language. Since he had adopted an informal approach to teaching that avoided the
traditional ‘DTP’ format, for example, the course notes also contained no explanation in
terms of formal mathematical language. (He did state formally the definition of linear
combination and span but this did not constitute an explanation.) An analysis of the
lecture resulted in a description of the informal approach based on verbal communication
of the key mathematical ideas in relation to linear algebra. This discussion is in Section
6.3.4.

The lecturer followed this introduction by posing several problems in the form of exam-

ples that explored the notion of linear combination, and span and spanning sets. In my
analysis of these examples I concluded that students were asked both to calculate and
to guess the linear relation between the vectors in a spanning set, and to decide if the
number of vectors could be reduced. I interpreted the design of an example around the
idea of reducing the number of vectors in a set as providing a “feel for” what it means
for a set of vectors to be linearly independent. It required students to look for linear
combinations between the vectors in the (spanning) set. However, a set is a minimal
spanning set if all vectors in the vector space (be it the null space or the range or any
other subspace or vector space) can be written as a linear combination of the vectors
in the (spanning) set while those very vectors cannot.

I now consider the examples that the lecturer used to introduce linear independence,
namely Examples 3.16 and 3.17.

As stated in Section 6.3.2 the lecturer introduced the formal definition of linear depen-
dence after working with Examples 3.16 and 3.17 that explored the notion of a minimal
spanning set. As was the case in relation to the introduction of span, the course notes
contained almost no explanations in respect of the introduction to linear dependence.

The lecturer used neither formal nor informal language in the course notes (apart from
stating the definition itself). After the lecturer stated the definition he made several
observations in the course notes which he labelled ‘Remarks’. The observations were
explanations of the definition and of possible implications. For example, the lecturer

---

15E.g. Examples 3.11. In this example the lecturer used the same numerical values for the matrix \( A \) as he had done in Example 3.8. Example 3.11 is a ‘continuation’ of Example 3.8.

**Example 3.11.** (a) Show that the vector \( \mathbf{x} = \begin{pmatrix} 13 \\ -2 \\ 3 \end{pmatrix} \) is in the null space of the matrix \( A \) of Example 3.8.

(b) Because \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \) span the null space of \( A \), and \( \mathbf{x} \) is in the null space of \( A \), it should be possible to write \( \mathbf{x} \) as a linear combination of \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \). Find such a linear combination.

---

16E.g. Example 3.17 as discussed previously in this section and Section 6.3.2.
wrote that linear (in)dependence applied to a set of vectors and not to individual vectors within that set.

In the lecture, in face-to-face teaching, the lecturer used informal, everyday language extensively to explain the notion of minimal span and to demonstrate the transition to linear (in)dependence. This relates also to the commenting and meta-commenting levels as discussed in Chapter 5. I discuss the use of informal language again in the next section, Section 6.3.4 (alongside the discussion in respect of Example 3.8).

I have linked the use of informal language in written form (in the course notes) and in spoken form (in face-to-face teaching) to an informal approach to the teaching of linear algebra concepts. The lecturer had formulated this approach with the aim of leading students to acquiring an intuitive understanding of linear algebra concepts first. Based on my analysis I claim that students’ intuitive understanding of linear independence took place in working with the idea of span and spanning sets. It is in connection with span that students could develop “a feel” for linear combinations which is necessary for forming an understanding of linear independence.

I wish to make one more point in relation to the word independent as used by the lecturer in the course notes and in his teaching. The word independent was first used in the course notes, in the context of linear equation systems, in Chapter 1 (as detailed in Section 6.3.2, page 126). The lecturer defined independent equations by referring to one equation being a multiple of another. He wrote,

\[
\text{We say: The equations are not independent because they satisfy}\\
(eq. 2) = 2 \times (eq. 1)\\
or \quad 2 \times (eq. 1) - eq. 2 = 0
\]

This statement related to equation ‘eq. 2’ being expressed as a linear combination (a multiple) of a second equation ‘eq. 1’. The term linear combination had not been introduced at that point. Since the lecturer had not defined a linear combination, he used an informal expression. He said that equations were not independent when one equation was a multiple of the other.

The lecturer also defined independent equations in a tutorial (in Week 2 of the module). He referred to the equation as not giving ‘any additional independent information’.
Quoting the relevant section (as in the quotation on page 126) the lecturer said to his students:

Now some of you are asking, what precisely does it mean for equations to be independent. . . . If one equation is a multiple of another equation, those equations aren’t independent. If I know one equation is satisfied I automatically know the other equation is also satisfied. So that other equation doesn’t give me additional independent information. . . . (L5, 46:54)

In this case the lecturer used ‘everyday’ language to relate ‘independence’. I interpreted the phrase “. . . that other equation doesn’t give me additional independent information” as referring to an ‘inter-equation’ relationship. There is an analogous discussion in the research literature (Dorier et al., 2000b), whereby an ‘inter-equation’ relationship is akin to a ‘between vectors’ relationship. I mention the research by Jean-Luc Dorier and his colleagues but I will not expand on it further as part of my thesis.

This concludes my didactical analysis of linear independence. Some aspects that I have discussed here are elaborated in the next section (the lecturer’s face-to-face teaching of his students), and some aspects are revisited in connection with eigenvectors and eigenvalues (in Section 6.4).

6.3.4 The lecturer’s teaching of linear independence (to students)

In this section I consider how the lecturer’s didactical thinking and planning was translated into practical actions in the classroom. In Section 6.3.2 I discussed the lecturer’s mathematical treatment of linear independence. I focussed on how the lecturer introduced the topic with Examples 3.16 and 3.17. In the course notes this relates to Chapter 3 Section 3.4. As part of my analysis of the lecture observation data I analysed and listened to the audio-recording of the lecture that corresponded to the lecturer’s introduction of linear (in)dependence. I also read the field notes taken by the researcher who attended the lecture. The audio-recording captured all that the lecturer said to his students while the field notes gave an indication of what occurred in the lecture that the audio-recording could not capture (such as students’ reactions and behaviours as perceived by the researcher).
In Section 6.2.4 I presented my analysis of the lecturer’s face-to-face teaching of subspaces. In this section I report on my analysis of the lecturer’s face-to-face teaching of linear (in)dependence. I discuss the similarities as well as the differences between the teaching of these two topics based on my analysis and interpretations. What was the same in the lectures on subspaces and linear independence was the extent of verbal explanations and instructions given by the lecturer, the presenting of examples as a primary tool for introducing and working towards a new concept, and the blank spaces after the examples for students to write solutions in lecture. What was different was the use of the blank spaces since the lecturer did not always ask students to work on the solutions themselves first, and the type of verbalisations that he used. In relation to linear independence I noted the extent of the lecturer’s repetition of mathematical statements and instructions which was not a feature in the teaching of subspaces.

As discussed in Section 6.2.4 and referred to in Section 6.3.3, I noticed the extent of verbal instructions and explanations that the lecturer gave in the lecture. In comparison, as I showed in 6.3.3, the written course notes contained few explanations.

At the beginning of lecture on linear independence the lecturer discussed a coursework assignment and summarised the material from the previous week, as he had done in the lecture on subspaces. He gave a detailed explanation of the mathematics, that is of the concepts introduced in the previous week, as well as statements that represented instructions on how students should view the problems that he had set them.

In the lecture he also stated his intentions for posing a range of problems on the problem sheets. He said to his students that he was introducing a lot of different examples in respect of the same concept, so that students could look at it ‘from different sides’ and ‘get a feel’. He said,

> And the purpose of these problems ... is that you get some experience in working with these objects ... and you get a feel for how they fit together and how they work. So that’s the importance of these problems, ... why I’m giving you a lot of different problems, and I’m asking you to look at these things from a lot of different sides, in the lecture, in our tutorial and on your problem sheets. (L15, 05:34)

As discussed previously, I highlighted engagement in terms of participation in the lecture and mental engagement with the mathematics, as one of his aims in taking an inductive approach to teaching. The lecturer wanted his students to gain experience with the new concepts by having some ‘hands-on’ experience with the concepts and not,
say, being presented with a theorem and corresponding proof, as was the case in the more traditional ‘DTP’ style. In research meetings the lecturer said that he wanted to introduce students slowly and more intuitively to the more formal reasoning (in the style of a ‘Definition-Theorem-Proof’) as was required at university.

He introduced the focus of the lecture by stating that the topic was “spanning sets for the range of a matrix” and by posing the question “How can we tell if we have a spanning set that is as small as possible?” In introducing the first example, Example 3.16 (see page 124 or page 222), the lecturer used words such as ‘wasteful’ and ‘having more vectors than were needed’. This was captured in this quotation when the lecturer addressed students just before projecting Example 3.16:

That’s what we’re going to talk about today, and then we’ll think about how many elements in that spanning set do we really need. We’re going to see that sometimes if we set up a spanning set for a subspace we can be wasteful, we can have more vectors in it than we actually need. So our big question for today and tomorrow, ‘How can we tell if we have a spanning set that is as small as possible?’, and therefore as easy to work with as possible, or how can we tell that we’re wasteful and we should simplify things. (L15, 08:22)

The lecturer projected the example and proceeded to work out the solution on the overhead projector in front of the class. He pointed out that the columns of the matrix $A$ have a connection with the solution sought. That is, the set of vectors that span the range can be written as a linear combination of the column vectors of the matrix $A$. The lecturer added this sentence in the lecture on the overhead projector, and also in the complete set of course notes that became available to students later on in the module:

Thus a vector $b$ is in the range of $A$ if and only if it can be written as a linear combination of the **column vectors** of $A$.

and

The column vectors of $A$ span the range.

In the lecture, after writing down the set of vectors that spanned the range of the matrix $A$, he explained to students,
And now look back at the matrix A, and you will see that these vectors are the
columns of A, the first column, second column, third column of A. That tells
us a vector \( \mathbf{b} \) is in the range if it can be written as a linear combination of the
columns of A . . . [The lecturer writes.] . . . That’s an important observation
because that tells us how we find a spanning set for the range of a matrix.
And it turns out it’s very easy. Once we know the matrix, the columns of
the matrix will span the range, no calculation required, nothing, just copy
down the columns and they will span the range of A. So finding a spanning
set is quite an easy thing to do for the range of a matrix. (L15, 14:10)

This explanation to students seemed to stress “not thinking” and “just copy down”. He
called it an “observation” made on this example which I interpreted as forming part of
the inductive approach to teaching.

In research meetings the lecturer talked about the creative side of mathematics, that
there were many things that students could try out for themselves and investigate. For
example in one research meeting he said,

But obviously in the lecture I wouldn’t want to give them any big pieces
of work to do, rather things that are reasonably quick to do, just to see
that they’re still with me and they pick up the important points and also,
. . . I am hoping to give students simple examples to do or ask questions of
them before I myself show them the first example of how something is done.
Hoping to get students into the habit of regarding a mathematical problem
as something that you can attack rather than something for which you look
up the solutions in your lecture notes. (M1, 06:54)

In Example 3.16 the lecturer did not ask students to work on the solution by themselves
first. The lecturer wrote down the solution, showing the correct answer that students
could copy and write into the blank spaces in the course notes.

He proceeded to introduce Example 3.17 where he asked students to find a smaller
spanning set. This example was an extension of the previous example. Again the
lecturer wrote down the solution without providing students with an opportunity to try
to find the solution themselves. He determined a smaller spanning set by writing the
vector \( \mathbf{v}_3 \) as a linear combination of the vectors, \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \). The lecturer ‘found’ this
linear combination \( by \text{ inspection} \) (and not by calculation). Furthermore, as detailed in
Section 6.3.2, the relation \( \mathbf{v}_3 = \mathbf{v}_1 + \mathbf{v}_2 \) was printed in the notes that students brought
to the lecture. Thus students could see part of the solution prior to attempting the example.

The lecturer proceeded to state the definition of linear dependence by projecting the relevant section of the course notes onto the screen, and saying,

So, what we are now led to do, when we’re given a set of vectors, a spanning set, we can investigate if there is a linear relation between the vectors. If there is, we say the vectors are linearly dependent, if there is none, we say the vectors are linearly independent. And that’s a very important concept, because if the vectors are linearly dependent, that means some of them are superfluous, we’ve been wasteful. If the vectors are linearly independent, we’re working with as few vectors as we possibly can. That is quite possibly the most important idea you’re going to see in this chapter, and in all of the module, because something very similar crops up all over the place in mathematics. And this really is a question of how simple can we make a description of the things we’re working with. And how simple can we make our calculations. That’s what linear independence is for. And here we’ve got the definition in its full beauty. (L15, 24:42)

Following this definition the lecturer presented Example 3.19, which had two parts (a) and (b), and with part (b) being a continuation of part (a). Students had to decide, using the formal definition of linear dependence, whether a given set of vectors was linearly independent or not (it was not) and then to determine the minimal spanning set. In asking students to work on this example the lecturer gave further instructions and explanations. He explained that a definition was a tool for communicating. He said to students before they attempted this example,

As I said already before a definition really is the way how mathematicians give instructions to each other. So the definition tells you what you need to do if you want to check if those vectors are linearly independent. (L15, 28:21)

In relation to Example 3.19 the lecturer gave students time to work on the solution, either by themselves or with their neighbour, whereas in relation to Examples 3.16 and 3.17 he had shown the (correct) solution, that is without providing an opportunity for students to try the solution themselves. In the field notes in relation to the Example 3.19 the researcher wrote,
Based on this entry in the field notes by one of the researchers, participation appeared to vary, with some students working on the solution while others did not.

As a result of the analysis I noticed that the type of verbalisation in relation to linear independence included extensive summarising and an increased focus on making connections with previously presented concepts. In contrast, the lecturer’s verbalisations in relation to subspace stayed closely within the context of the vector space $\mathbb{R}^n$ and subspaces.

In my analysis of the lecturer’s didactical thinking and planning I focussed on the role of Examples 3.16/3.17 and on the role of informal language. The latter included a discussion of both formal and informal language that the lecturer used in the course notes (the written form) and also in his face-to-face teaching (the spoken form).

In contrast to the teaching of subspaces the lecturer used informal language much more extensively in relation to the teaching of linear independence. He also repeated statements and intentions more frequently in lectures. The lecturer’s ‘verbalisation of intentions’ was one of the actions that I discussed in Chapter 5. Jaworski et al. (2009) characterised these ‘verbalisations’ in terms of commenting and meta-commenting levels. Further analyses led me to distinguish between commenting in relation to mathematics and commenting in relation to the learning of mathematics, with each aspect also having a category of meta-commenting. These I defined in Section 5.4.8.

In summary, I have discussed the mathematics and the didactics of linear (in)dependence. In connection with linear (in)dependence two aspects emerged during analysis: the role of examples and the role of informal language. Both these aspects also played a role in the teaching of subspaces. In both settings the examples lent structure to the course notes, and functioned as a technical tool, a conceptual tool and a cultural tool.

As a technical tool the examples (as part of the course notes that had ‘gaps’) were designed to encourage participation in the lecture. An analysis of the lecturer’s face-to-face teaching showed that the lecturer did not always use the examples in that way, and that students varied in their response to the lecturer’s request to engage and participate by working on the solution to examples in the lecture.

As a conceptual tool the examples were designed to engage students with the mathematics that the lecturer presented in the course notes as well as in the lecture. In
relation to linear independence I claim that the example designed to introduce linear independence depended on previous knowledge of span and minimal span.

As a **cultural tool** the examples were designed to engage students in *a certain way* with the mathematics, an intention made explicit by the lecturer in research meetings. It is here that the greatest differences appeared. With the example in connection with subspaces the lecturer gave a view of mathematics that was creative and open to many different answers all of which were ‘correct’. With the example in connection with linear independence the lecturer gave a view of mathematics as posing a question to which corresponded only one correct answer. He also presented a view of linear algebra as consisting of concepts that were inter-related and inter-dependent.

A second aspect that I focussed on in connection with linear independence was the *role of informal language*. In lecture the lecturer explained in detail the mathematics using informal language and introducing the formal definition only after students had worked with an example. This applied to subspaces and linear independence. As a result of this analysis I noticed that the course notes contained the outline of examples and very few explanations. Definitions were stated but little further details were given, nor were the formal aspects of the definitions highlighted and explained. The lecturer gave *detailed verbal explanations* in lecture only.
6.4 The concept of eigenvectors/values

This is the third part of Chapter 6. In this section I analyse the concept of eigenvectors and eigenvalues. In the previous two sections, Sections 6.2 and 6.3, I analysed the concepts of subspace and linear independence.

One of the reasons for analysing the topic of eigenvectors and eigenvalues (as detailed in Section 6.1.1) derived from the lecturer’s comments in research meetings. The lecturer talked extensively about structuring the module in order to shift students’ views towards the more conceptual nature of this topic, and mathematics in general.

I analyse the concept of eigenvectors/values in respect of the four areas (a mathematical account of the concept, the lecturer’s mathematical treatment of the concept, the lecturer’s didactical thinking and the lecturer’s face-to-face teaching of students) as I have done previously in relation to subspace and linear independence.

I again consulted three textbooks and a set of the course notes. The course notes and the textbooks were the same ones that I used before (Greub, 1967; Hoffman and Kunze, 1961; Poole, 2006; Sproston, 1995).

Again, whenever I quote from the textbooks/set of course notes, for ease of comparison, I may alter the letters that the authors used in their definitions or mathematical statements, but not the formulation of the statements.

In this section, in particular, I use the letter $A$ for a matrix, $T$ for a transformation, $v$ for an eigenvector and $\lambda$ for an eigenvalue. In addition I use $V$ for a vector space, $v_1, v_2$ and $w$ to denote (general) vectors and $K$ a field. I also use det, as is the convention in mathematics, for the determinant function.

I assume that the reader is familiar with the mathematics of a first year, introductory linear algebra course (including general mathematical notation and terminology) as detailed in Section 6.1.5. For example, in my analysis in this section I refer to the invertibility of matrices, the determinant and the characteristic polynomial.\footnote{I provide some explanations of these concepts during my discussion and analysis but ask the reader who is unfamiliar with linear algebra to consult a textbook.}

I start with an overview of the mathematics as covered by the textbook authors followed by a detailed account of the introduction to eigenvectors and eigenvalues in the textbooks. I include the introduction of eigenvectors by Sproston and by Poole as separate sections. I adopt the same format (an overview of the structuring of the material
followed by a detailed account of the introduction to the concept) in Section 6.4.2, in my discussion of the lecturer’s mathematical account of the concept. Thus I have the following headings:

6.4.1 A mathematical account of eigenvectors

An overview of the topics in the textbooks
The introduction of eigenvectors - key mathematical points
The introduction of eigenvectors in Sproston (1995)
The introduction of eigenvectors in Poole (2006)

6.4.2 The lecturer’s mathematical treatment of eigenvectors in the course notes

An overview of the topics (in the course notes)
The lecturer’s introduction of eigenvectors in the course notes
The lecturer’s ‘re-visiting’ of eigenvectors in the course notes

6.4.3 The lecturer’s didactical thinking in relation to the teaching of eigenvectors

(a) The role of Examples 4.1 and 4.5

(b) The sequencing of topics for conceptual understanding

6.4.4 The lecturer’s teaching of eigenvectors/values (to students)

6.4.1 A mathematical account of eigenvectors/values

The concept of eigenvectors (and eigenvalues) depends on other mathematical concepts which are for the most part not in the school curriculum. As a result most students coming to university are not familiar with the pre-requisites of linear transformations, matrix form and set notation. There are exceptions of course. A minority of students having studied certain ‘Further Mathematics’ modules enter university being familiar with how to calculate eigenvectors and eigenvalues and also being aware of some of the applications. As an indication of the proportion, from a questionnaire administered to the students in my study, 16% of students knew of eigenvectors. Hence, for the vast majority of the students taking the linear algebra module eigenvectors and eigenvalues were unfamiliar topics.

With reference to the three textbooks and the set of course notes that I consulted for my analysis, eigenvectors and eigenvalues were generally introduced later in any course
material of linear algebra. In order to analyse this concept I found it necessary to consider
the sequencing of the material prior to their introduction, and in particular the key ideas
on which each one of the textbook authors built their discussion of eigenvalues/vectors.
In contrast, the authors introduced the topics of subspaces and linear independence
(Sections 6.2.1 and 6.3.1) much earlier in the course materials/textbooks, and in doing
so, they did not assume vastly differing material prior to their introduction. Thus the
way that the authors introduced subspaces and linear independence was ‘similar’; they
built their introduction on the same set of prior concepts. This was not the case in their
introduction to eigenvectors. The authors differed because they had positioned the topic
of eigenvectors and eigenvalues at different places in relation to the other topics in the
linear algebra course. They therefore had covered a different set of material prior to the
introduction of eigenvectors.

For my analysis of eigenvectors I found it necessary to consider the structuring of the
material in each textbook/set of course notes. In the next section I give an overview of
that material.

Overview of the topics in the textbooks

In Section 6.1.3 I gave a description of the textbooks/set of course notes in terms of their
age, style and content structure. In this section I relate the structuring of the material to
the authors’ treatment of eigenvectors and eigenvalues, in particular. I re-iterate some
of the points I made previously in Section 6.1.3.

With the exception of Poole (2006) the two textbooks and the set of course notes followed
(broadly) the same pattern of sequencing of the material which was as follows:

- Vector spaces/linear equation systems
- Transformations/mappings/matrices
- Determinants/characteristic polynomial/diagonalisation

In Section 6.1.3 I refer to Poole (2006) having this structure ‘superficially’, suggesting a
linear progression through the material. However, Poole re-visited concepts and stated
theorems more than once, often in \( \mathbb{R}^2/\mathbb{R}^3 \) first and then again in \( \mathbb{R}^n \) or a more abstract
vector space. In some cases, this resulted in minor rather than major changes to the
formulation of the statement or theorem (for an example see the footnote in Section
6.1.3 in relation to the definition of a subspace). In relation to the current context Poole
introduced eigenvalues and eigenvectors in connection with an application in \( \mathbb{R}^2/\mathbb{R}^3 \)
In this case the eigenvalue problem was stated in matrix form in $\mathbb{R}^2/\mathbb{R}^3$ first, and repeated (again in matrix form) in a later chapter in the context of $\mathbb{R}^n$.

The lecturer in my study also discussed eigenvectors and eigenvalues more than once, first in introducing the concept in terms of the matrix equation $A\mathbf{v} = \lambda \mathbf{v}$, and the second time in preparing to teach the topic of diagonalisation in the second part of Chapter 4 of the course notes. The lecturer made the didactical decision to teach the theory of determinants after he had introduced eigenvectors and eigenvalues. I discuss this in Section 6.4.3. As a result of the lecturer’s didactical thinking I found it necessary to consider the sequencing of (what is the third category above) determinants/characteristic polynomial/diagonalisation.

In respect of this category, the authors of the two older textbooks and the set of course notes (Hoffman & Kunze, Greub and Sproston, respectively) introduced determinants before eigenvectors while Poole ‘sandwiched’ determinants between eigenvectors in $\mathbb{R}^2/\mathbb{R}^3$ and eigenvectors in $\mathbb{R}^n$. The sequencing of these topics had an impact on the way the arguments and proofs were constructed in relation to eigenvectors and eigenvalues. I discuss Poole’s approach in a separate section on page 151.

Sproston arranged the topics so that both determinants and diagonalisation were introduced before eigenvectors, i.e. *eigenvectors were last*. This was a unique way to proceed (when compared with the approaches by the other three authors, and with the lecturer in my study) and resulted in a different structure to the arguments presented. Sproston emphasised linear transformations throughout his approach and used the concept of diagonalisation to motivate the definition of eigenvectors and eigenvalues. I discuss Sproston’s approach in a separate section on page 149.

I now give an account of eigenvectors and eigenvalues as a mathematical concept in linear algebra. I describe the way that eigenvectors and eigenvalues were introduced in the textbooks/set of course notes that I consulted.

The *introduction* of eigenvectors - key mathematical points

Eigenvectors are defined in terms of a transformation that has the property of returning a multiple of the input vector. Eigenvectors (and eigenvalues) have numerous applications in Mathematics due to this property (which is linked to the mathematical idea of *invariance*. However, a detailed discussion of *invariance* is beyond the scope of this chapter.)
I state the definition of an eigenvector (and eigenvalue) as it appeared in Hoffman and Kunze (1961), and in a similar form in the other three sources that I consulted for this analysis. Before giving the actual definition Hoffman and Kunze wrote,

\[ T \text{ will be a linear operator on an } n\text{-dimensional vector space } V \text{ over a field } K. \]

They then stated the definition:

**Definition (1).** A characteristic value of \( T \) is a scalar \( \lambda \) in \( K \) such that there is non-zero vector \( v \) in \( V \) with \( Tv = \lambda v \). If \( \lambda \) is a characteristic value of \( T \), then any \( v \) such that \( Tv = \lambda v \) is called a characteristic vector of \( T \) associated with the characteristic value \( \lambda \). (Hoffman & Kunze, 1961, p. 164)

Here the authors defined the characteristic value first, and the characteristic vector second. Two authors (Greub and Sproston) defined the characteristic vector first. In some of textbooks that I consulted the alternative names of eigenvalue and eigenvector were used.\(^1\)

Hoffman and Kunze (1961) defined the characteristic value first, which I interpreted as stressing the algebraic nature of the equation \( Tv = \lambda v \). Some textbooks defined the characteristic vector, or eigenvector first, which I interpreted as placing emphasis on the property of the transformation (as returning a multiple of the input vector). I interpret this also as placing emphasis on the geometric nature of the equation.

A (linear) transformation can be represented by a matrix. In all four approaches that I considered, the authors established the link between a transformation and a matrix representing a transformation. Hence, adopting the version by Poole, Definition (1) may also be written as follows:

**Definition (2).** Let \( A \) be an \( n \times n \) matrix. A scalar \( \lambda \) is called an eigenvalue of \( A \) if there is a non-zero vector \( v \) such that \( Av = \lambda v \). Such a vector is called an eigenvector of \( A \) corresponding to \( \lambda \). (Poole, 2006, p. 253)

The authors Hoffman and Kunze, Greub and Sproston stated the definition in terms of a transformation while Poole stated the definition in terms of a matrix (as did the

\(^{18}\)\(T\) is a transformation, and in the context of Linear Algebra a linear transformation (or linear map) which can be represented by a matrix. Because a transformation acts on a set of vectors (of a vector space) it is sometimes also referred to as an operator. An operator can be represented by an algebraic symbol, e. g. \( \Delta \) for the Laplacian operator, or by a matrix. I have used the letter \( T \) to denote a transformation or an operator.

\(^{19}\)The older texts used the terms characteristic value and characteristic vector in their initial definition while the newer texts used the terms eigenvalue and eigenvector.

‘Eigen’ is a German word meaning “belonging to”, or “being a characteristic of”.

---

\(^*\)The older texts used the terms characteristic value and characteristic vector in their initial definition while the newer texts used the terms eigenvalue and eigenvector.

‘Eigen’ is a German word meaning “belonging to”, or “being a characteristic of”.
lecturer in my study). All authors made the link between a transformation and a matrix representing a transformation explicit in their explanations (while the lecturer in my study did not). They all formulated statements, definitions and theorems in terms of their preference, that is in terms of either a matrix or a transformation. Hoffman and Kunze (1961) and Poole (2006) stated the link on (just) one occasion in their respective textbook. Sproston (1995), on the other hand, dedicated a whole chapter to elaborating the connection between matrices and transformations. Greub (1967) stated the connection once explicitly within a chapter. In addition, Greub expressed a definition or theorem in terms of a transformation first (in Chapter 2 of his book) and then re-stated it in terms of a matrix (in Chapter 3 of his book). For example, in Chapter 2 Greub defined the rank of a linear mapping and in Chapter 3 the rank of a matrix.

Poole and Sproston used images and explanations based in Geometry while Hoffman and Kunze and Greub did not. Again, I noted here that the first two references refer to more recent textbooks (2006, 1995, respectively) and the latter two to the older ones (1961 and 1967).

After the initial definition was given the authors proceeded in one of two ways. They either (a) defined the characteristic equation \( \det(T - \lambda I) = 0 \) by an inductive argument directly from the definition, or (b) introduced a proposition or theorem involving equivalent statements (one of which was the characteristic equation) which were then proved. I discuss both approaches and reproduce the derivation used in the textbooks as well as the equivalent statements.

The choice of presentation depended on the material that the authors had covered beforehand (such as determinants and invertibility of linear maps and/or matrices). I stated in my introduction to this section that authors varied in the sequencing of linear algebra topics (see page 143). In particular, the choice of presentation, (a) or (b), depended on whether eigenvectors had been introduced before or after determinants. Hoffman & Kunze, Greub and Sproston introduced eigenvectors after determinants. Poole ‘sandwiched’ determinants between eigenvectors in \( \mathbb{R}^2/\mathbb{R}^3 \) and eigenvectors in \( \mathbb{R}^n \). Within \( \mathbb{R}^2 \), Poole used the characteristic equation without (much) explanation. However, he made his arguments more rigorous after discussing determinants, and when moving into the context of \( \mathbb{R}^n \). I refer to the latter part of the textbook by Poole (and to his discussion of eigenvectors in the context of \( \mathbb{R}^n \)) in relation to my explanations of (a) and (b) that follow.

\(^{20}\)Only in the more recent textbooks (that also used the terms eigenvector and eigenvalue) did the authors introduce the term characteristic equation which referred to the equation \( \det(T - \lambda I) = 0 \). In the two older textbooks the authors gave no name to this equation.
I begin with a discussion of (a), the derivation of the characteristic equation. Greub deduced the characteristic equation \( \det(T - \lambda I) = 0 \) through a sequence of steps from the definition which I have reproduced here:

Assume that \( \mathbf{v} \) is an eigenvector of \( T \) and that \( \lambda \) is the corresponding eigenvalue. Then

\[ T\mathbf{v} = \lambda \mathbf{v}, \quad \mathbf{v} \neq \mathbf{0}. \]

This equation can be written as

\[ (T - \lambda I)\mathbf{v} = \mathbf{0}, \]

showing that \( T - \lambda I \) is not regular. This implies that

\[ \det(T - \lambda I) = 0. \]

(\ldots) This equation is called the characteristic equation of the linear transformation \( T \). (Greub, 1967, p. 116)

This derivation contains and uses several notions which I explain below, in the footnote.\(^{21}\)

Expanding the characteristic equation \( \det(T - \lambda I) = 0 \) gives rise to a polynomial in \( \lambda \), called the characteristic polynomial, which can be solved to give the eigenvalues \( \lambda_i \) (for \( i = 1, 2, \ldots \)) of the transformation. After the eigenvalues have been determined, the eigenvectors corresponding to each eigenvalue can be found by substituting the values \( \lambda_i \) back into the definition \( T\mathbf{v} = \lambda \mathbf{v} \), or equivalently

\[ (T - \lambda I)\mathbf{v} = \mathbf{0}. \]

Using this method, that is finding the solutions \( \lambda_i \) of the equation \( \det(T - \lambda I) = 0 \), represents the ‘standard procedure’ for calculating the eigenvalues of a transformation. By ‘standard procedure’ I refer to an explanation given by the lecturer in my study in a research meeting when he said,

\(^{21}\) A linear map refers to a linear transformation and can be represented by an algebraic symbol or a matrix. \( I \) is the identity map. Writing \( T\mathbf{v} = \lambda \mathbf{v} \), followed by \( T\mathbf{v} - \lambda \mathbf{v} = \mathbf{0} \) does not make (mathematical) sense as a re-arrangement of the first equation. Inserting \( I \) into \( T\mathbf{v} = \lambda \mathbf{v} \) (that is, imagining the equation as \( T\mathbf{v} = \lambda I\mathbf{v} \)) followed by \( (T - \lambda I)\mathbf{v} = \mathbf{0} \) makes sense (mathematically). The expression ‘not regular’ or ‘singular’ (see Theorem 5 on page 149) is equivalent to ‘not invertible’.

The notions of invertibility and determinant relates to material that Greub introduced in a previous chapter of his book, in particular in the study of linear equation systems. The reader can consult a textbook for further information if necessary.
Again [I am] inspired by trying to do the important concepts first and the computational recipes later on, in order to avoid that students think of an eigenvalue as a zero of the characteristic polynomial, which most know anyway because that’s how they’re calculated. But this point of view really is unhelpful if you want to move out of the calculation because it doesn’t give you an opportunity to do anything with it. (M15, 1:11:02; *italics my emphasis*)

The lecturer stated that ‘finding the zeros of the characteristic polynomial’ is the method for calculating eigenvalues. The characteristic equation represents a ‘working definition’ ("what one has to do", a criterion or condition) for calculating the eigenvalues (and from there the eigenvectors) of a transformation.

While I deduced the characteristic equation in terms of a transformation above, it is possible also to write the sequence of steps in terms of a matrix as follows.

<table>
<thead>
<tr>
<th>Consider the definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Av = \lambda v$, $v \neq 0$,</td>
</tr>
<tr>
<td>for some $n \times n$ matrix $A$. This can be written as</td>
</tr>
<tr>
<td>$(A - \lambda I)v = 0$.</td>
</tr>
<tr>
<td>Since $A - \lambda I$ is not an invertible matrix,</td>
</tr>
<tr>
<td>$\det(A - \lambda I) = 0$.</td>
</tr>
</tbody>
</table>

In Poole (2006) the author stated the definition in terms of a matrix, and in the approaches to the topic of eigenvectors presented in Hoffman and Kunze (1961) and Sproston (1995), the authors stated the definition in terms of a transformation. However, only Greub deduced the characteristic equation.

To make the progression from the definition of an eigenvector to the characteristic equation depended on the material covered prior to the introduction of the definition, in particular, knowledge of determinants and invertibility criteria (of transformations/matrices).

This represents part (a), the derivation through an inductive argument, of my discussion of how eigenvectors/values were introduced in the textbooks. I now discuss presentation (b).
As an alternative to deducing the characteristic equation, the authors Hoffman & Kunze and Sproston listed a set of equivalent statements. These were formulated as a theorem and as a proposition, respectively, and for each the authors gave a proof. I have reproduced the theorem by Hoffman and Kunze here:

**Theorem 5.** The following are equivalent:

- \( \lambda \) is a characteristic value of \( T \).
- The operator \((T - \lambda I)\) is singular (not invertible).
- \( \det(T - \lambda I) = 0 \).

(Hoffman & Kunze, 1961, p. 164)

The three statements contained in this theorem reproduce the steps used in deducing the characteristic equation in presentation (a). However, here the authors provided proofs.

For all authors the characteristic equation represented the ‘standard way’ of calculating eigenvalues and eigenvectors. For some authors the discussions were rooted in transformations (Hoffman & Kunze, Sproston), for some in matrices (Poole, the lecturer in my study) and for some in both (Greub).

The importance of eigenvectors lies in their usefulness in the process of transforming bases. *Diagonalisation* is the term given to the process of ‘manipulating’ transformations, or matrices representing transformations, in relation to a change of basis. I discuss this process in more detail in relation to the approach taken by Sproston (1995) below.

These are the key mathematical points that I wish to make in connection with the concept of eigenvector and eigenvalue. I now refer to two texts in more detail: Sproston (1995), because his approach (having similarities with the other textbooks) was unique in the introduction of eigenvectors, and Poole (2006), because his approach had similarities with the approach taken by the lecturer in my study.

**The introduction of eigenvectors in Sproston (1995)**

In Sproston (1995) eigenvectors were *motivated* by a discussion of diagonalisation. In particular, the author posed the question,

**How simple can we make the matrix of \( T \) [where \( T \) is a transformation] by suitable choice of basis?**

I particularly noted the use of the word ‘simple’ here, a word that the lecturer in my study also used in relation to eigenvectors and change of bases. To answer the question
posed above the author referred to a diagram that he had produced as part of the course notes to students (see Figure 6.1).

The diagram showed a mapping $T$ from a domain space to an image space where both had the same basis $e_1, e_2, e_3$, etc. This mapping was represented by the matrix $A$. The same mapping $T$ also mapped a second domain space to an image space, where this time the bases were a set of vectors $e'_1, e'_2, e'_3$ etc. This mapping was represented by the matrix $B$. The diagram also showed the matrices $P$ and $P^{-1}$ denoting the mappings from one basis to the other. In answering Sproston’s original question: The transformation $T$ is the simplest in the case of an eigenvalue problem. Say matrix $B$ represented a diagonal matrix consisting of the eigenvalues of $T$. Say a second matrix, matrix $A$, represented the mapping $T$ in respect of another basis. Then matrix $A$ is said to be diagonalisable if there is an invertible matrix $P$ such that $P^{-1}AP$ is a diagonal matrix consisting of the eigenvalues of $T$, namely said matrix $B$.

The author had spent some considerable space in the course notes on making the link between a transformation and the matrix denoting a particular transformation. Hence, the statement above (in the quotation) may also be expressed in the following way (as Sproston (1995) did),

Equivalently, given an $n \times n$ matrix $A$, how simple can we make $P^{-1}AP$ by suitable choice of $P$?

The answer to this question is, $P^{-1}AP$ can be made diagonal by finding a matrix $P$ with columns consisting of a basis of eigenvectors. Only then did Sproston state the definition of eigenvectors and eigenvalues. Hence the concept of an eigenvector was motivated by a discussion of diagonalisation, and the definition of an eigenvector was deduced from the process of diagonalisation.
The author then presented two examples showing (a) being able to find a basis of eigenvectors and hence being able to diagonalise, and (b) not being able to find enough eigenvectors for a linearly independent set of basis vectors and hence not being able to diagonalise. These two examples constituted the end of the course notes. Hence, in this approach the author finished his teaching of the module with the calculation of eigenvectors and eigenvalues (using what I have referred to as the ‘standard procedure’ above). He had given most of the course notes to forming the link between matrices and transformations and developing the arguments surrounding invertibility and determinants. All these aspects were brought together in the discussion of diagonalisation.

The introduction of eigenvectors in Poole (2006)

Poole (2006), as I said previously, presented an approach that had similarities with the approach taken by the lecturer in my study.

Poole introduced linear algebra concepts in a concrete setting first, \( \mathbb{R}^2 \), \( \mathbb{R}^3 \) and up to \( \mathbb{R}^n \), and in a more abstract setting later on. The author expressed his aims explicitly to his readers in the preface of his book. He stated that he wanted to address the difficulties that students were reported to have with the abstract nature of linear algebra (as documented in the research literature). The author also stressed that he wanted to use geometric images and insights in connection with vectors in \( \mathbb{R}^2 \) and \( \mathbb{R}^3 \). Poole introduced matrices first, and then linked these to transformations, where all other authors had done the opposite. He introduced the eigenvalue problem with matrices and based on an example involving Markov chains and one involving the (unfamiliar to me) Leslie model for population growth (towards the end of Chapter 3 in the textbook). I interpreted this introduction as informal since the author did not mention the terms eigenvalue or eigenvector. Only after having worked with the examples did he formally define the eigenvalue and eigenvector. All eigenvalue problems and exercises in Chapter 4 of the textbook were set in \( \mathbb{R}^2 \) and \( \mathbb{R}^3 \), with geometric explanations to aid understanding. The author re-visited the concept of eigenvectors in the general context (of \( \mathbb{R}^n \) and more abstract vector spaces) after a discussion of determinants (in a separate (sub)section). In re-revisiting the concept the author referred to the characteristic polynomial and the characteristic equation. He defined the characteristic polynomial as the polynomial in \( \lambda \), obtained from expanding the characteristic equation

\[
\det(A - \lambda I) = 0.
\]
He focussed on the *procedure* for finding eigenvalues and eigenvectors, and the use of similar matrices to create the link (via triangular and diagonal matrices) to diagonalisation.

In general in Poole (2006), the author focussed heavily on matrices as the main tool for explaining linear algebra concepts and procedures. Transformations were largely downplayed, but shown to be linked to matrices and geometric representations in a subsection of Chapter 3. With reference to Section 6.1.3 the two older textbooks and the set of course notes that I consulted for my analysis focussed on matrices rather than on matrices. In contrast in this (more recent) textbook Poole focussed on matrices and mentioned transformations only secondly, or hardly at all, as was the case also with the lecturer in my study (see Section 6.4.2).

In the next section I discuss the lecturer’s treatment and design of the topic of eigenvectors/values in the course notes.

### 6.4.2 The lecturer’s mathematical treatment of eigenvectors/values in the course notes

I first give an overview of the *sequence of topics* that the lecturer in my study designed for his teaching of linear algebra. I do this because the lecturer, in research meetings, had discussed his structuring of the course in order to direct students’ attention towards the more conceptual nature of eigenvectors (for example, M15, 1:11:02; more details in Section 6.4.3). I compare the lecturer’s sequencing of topics with that by the authors of the textbooks.

After presenting the overview I discuss how the lecturer *introduced* eigenvectors and eigenvalues.

#### Overview of the sequence of topics

The lecturer started the course with a discussion of linear equation systems in Chapter 1. He used the Gaussian elimination procedure *on a linear equation system*. He then introduced a matrix to represent the equation system and repeated the Gaussian elimination steps for the case of the matrix. However, he did not formally define a matrix until Chapter 2. The focus of Chapter 1 was on the types of solutions that could arise from a system of equations.

In Chapter 2 the lecturer introduced matrices and stated the *definition* of a matrix as a rectangular array of numbers. In this chapter the lecturer covered matrix operations and
(general) matrix algebra, such as adding and multiplying matrices, finding transposes and inverses, determining the identity matrix and obtaining the rules of matrix algebra. Chapter 2 built on Chapter 1 in the sense that the lecturer linked invertibility of a matrix to the unique solution of an equation system.

In the lecturer’s own words, the content of Chapter 3 was “at the heart of the course” (see quotation on page 88), where all the important concepts of linear algebra were introduced.

At the start of Chapter 3 the lecturer related vectors in $\mathbb{R}^n$, linear transformations and matrices to the notion of linearity. He introduced subspace, span, spanning sets, range, linear independence, bases, dimension, rank, nullity, etc., and the rank-nullity theorem. The lecturer first defined the term linear combination in connection with spanning sets (but without giving a detailed explanation of the term). The rank-nullity theorem was an important observation that the lecturer made in the course notes. The lecturer related the rank-nullity theorem to the complementary function and the particular integral of the solution of a differential equation.

Towards the end of Chapter 3 the lecturer discussed bases and change of bases. He introduced a basis for $\mathbb{R}^3$ that was not the standard basis in order to show how one coordinate system was related to another by a transformation. Transformations had not been discussed very much up to this point. Here the focus was on the concept or idea of a transformation (represented by a matrix), and not on matrices as a tool, for example, towards finding a solution.

The focus of Chapter 4 was eigenvectors, determinants and diagonalisation. The lecturer started with the definition $Av = \lambda v$ of an eigenvector and initially used only this definition in relation to the examples presented. He re-visited eigenvectors (introducing the characteristic equation and the characteristic polynomial for calculating eigenvalues) after a discussion of diagonalisation and the theory of determinants. As a summary of Chapter 4, the sequencing of the content was as follows:

4 Eigenvalues and Eigenvectors

4.1 Eigenvectors

4.2 Diagonalisation

4.3 Determinants

4 Eigenvalues and Eigenvectors (continued)

4.4 Row and column expansion of determinants
4.5 The characteristic polynomial

4.6 Functions of matrices

This concludes my overview. I now discuss the lecturer introduction of eigenvectors in the course notes.

The lecturer’s introduction of eigenvectors in the course notes

As in previous sections I analysed the student version and the full (complete) version of the course notes, and the lecturer’s comments made in research meetings, this time with a focus on the teaching of eigenvectors and eigenvalues. As mentioned immediately above, the lecturer re-visited the concept of an eigenvector (and eigenvalue) after introducing the theory of determinants and the process of diagonalisation. In my analysis in this chapter I focus on the lecturer’s introduction of the concept of an eigenvector and eigenvalue. I considered Chapter 4 with the title “Eigenvalues and Eigenvectors” an appropriate starting point and give a detailed analysis of the examples used.

The concept of an eigenvector was introduced via an example, Example 4.1 in the course notes (see below). This example had two parts, (a) and (b).

Example 4.1. Consider the matrix

\[
A = \begin{pmatrix}
5 & 5 & -2 \\
-4 & -4 & 2 \\
-3 & -3 & 2
\end{pmatrix}
\]

and the vectors

\[
v_1 = \begin{pmatrix}
2 \\
0 \\
1
\end{pmatrix}
\quad \text{and} \quad
v_2 = \begin{pmatrix}
-1 \\
1 \\
1
\end{pmatrix}
\]

(a) Calculate the vectors \( A v_1 \) and \( A v_2 \). Comment on your results.

\textbf{Solution:}

Blank space here.

(b) Calculate \( A^{2000} v_2 \).

\textbf{Solution:}

Blank space here.
In this example the lecturer presented a $3 \times 3$ matrix $A$ and two column vectors, $v_1$ and $v_2$, giving numerical values for both the matrix and the vectors. For part (a) he “asked” students to calculate $Av_1$ and $Av_2$ and comment on the results. One vector ($v_2$) was an eigenvector of the matrix while the other ($v_1$) was not. At this point the lecturer had not defined or used the term *eigenvalue* or *eigenvector*. In the solution he wrote that $Av_2$ was equal to $2v_2$. In the lecture, in front of students, the lecturer read this line and did not mention the term eigenvector or eigenvalue until he stated the formal definition (Definition 4.2 below), (see also Sections 6.4.3 and 6.4.4).

For part (b) the lecturer again “asked” students to *calculate*, this time $A^{2000}v_2$. He then introduced the formal definition of an eigenvector, reproduced here as it appeared in the course notes.

**Definition 4.2.** For an $n \times n$ matrix $A$, a vector $v$ (not equal to the zero vector) and a number $\lambda$ that satisfy

$$Av = \lambda v$$

are called an eigenvector and the corresponding eigenvalue of $A$.

In the student version of the course notes part of the definition, namely the line $Av = \lambda v$, was not printed. There was a blank space in the course notes. The lecturer wrote the sentence $Av = \lambda v$ during the lecture, and students could add this sentence to their notes. In the course notes, the lecturer ‘usually’ provided definitions and theorems in full, (a) so that definitions and theorems were accurately written down, and (b) in order to relieve students of the chore of ‘copying’ these definitions and theorems in the lecture.

The lecturer introduced eigenvectors/values with Example 4.1 and Definition 4.2. In the course notes he added the following explanation (which he labelled ‘Remark 4.3’ ) which I have reproduced here:

**Remark 4.3.** The zero vector satisfies $A0 = \lambda 0$ for any square matrix $A$ and any number $\lambda$. For this reason, the zero vector does not count as an eigenvector. However, the number zero is permitted as an eigenvalue.
With this remark the lecturer gave an informal reason why the zero vector \( \mathbf{v} = \mathbf{0} \) was not an eigenvector. In Poole (2006) and in Hoffman and Kunze (1961), for example, the authors stated this fact as an assumption or condition in the formal definition of an eigenvector/value (see Section 6.4.1). In contrast, the lecturer here used informal language in the course notes, namely, ‘the zero vector does not count’ (his words, as written down in Remark 4.3), while in the lecture he said, again informally but verbally to students, “That’s nothing special”. That is, he used informal language in both written and spoken form. With Remark 4.3 the lecturer gave an informal reason (my interpretation) why the zero vector is excluded as a potential solution (and hence as an eigenvector) of the equation \( \mathbf{A}\mathbf{v} = \lambda \mathbf{v} \).

The lecturer formulated a second statement after the definition of an eigenvector, namely Observation 4.4, which I have reproduced here.

**Observation 4.4.** If \( \mathbf{v} \) is an eigenvector of the matrix \( \mathbf{A} \) with eigenvalue \( \lambda \), it is also an eigenvector of \( \mathbf{A}^n \) for every \( n \), and the eigenvalue is \( \lambda^n \).

With this observation the lecturer generalised the result from Example 4.1 part (b) to calculations involving higher powers of the matrix \( \mathbf{A} \). In providing a proof of this result, the lecturer explained the steps involved in justifying the result. He wrote:

**Proof.** The argument is the same as in the example: Take the eigenvector-eigenvalue equation \( \mathbf{A}\mathbf{v} = \lambda \mathbf{v} \) and multiply it by \( \mathbf{A} \) from the left, which gives \( \mathbf{A}^2\mathbf{v} = \lambda \mathbf{A}\mathbf{v} \). Now use the eigenvector equation on the right hand side and get \( \mathbf{A}^2\mathbf{v} = \lambda^2 \mathbf{v} \). Repeating the process \( n \) times, we get

\[
\mathbf{A}^n \mathbf{v} = \lambda^n \mathbf{v}.
\]

Here the lecturer did not, for example, present a ‘proof by induction’ which is one way of proving this statement formally.

Both Remark 4.3 and Observation 4.4 formed part of the lecturer’s didactical planning and decision making. I have discussed the use and role of (in)formal language in Section 6.3.3.

After this introduction the lecturer presented a second example, Example 4.5 (see page 161 or page 229), with a (different) \( 3 \times 3 \) matrix \( \mathbf{A} \). Again, he used numerical values for both the matrix and the vector. This example had three parts. In part (a) the lecturer asked students to show that a given vector \( \mathbf{v} \) was an eigenvector of the matrix \( \mathbf{A} \) and to find the corresponding eigenvalue. In part (b) he asked students to show that \( \lambda = 3 \) was an eigenvalue of the matrix \( \mathbf{A} \) and to find the corresponding eigenvector. And in part (c) the lecturer asked students to decide if \( \lambda = -5 \) was an eigenvalue or not. After each short introduction, that is, after each statement of the mathematical problem in parts (a), (b) and (c), there was a blank space in the course notes where students could
add the solution. In Example 4.1 the lecturer had only used the definition $A\mathbf{v} = \lambda \mathbf{v}$ to obtain the solutions. This was the starting point also in Example 4.5 part (a). For part (b) the lecturer set up a system of linear equations to formulate the problem, and then Gaussian elimination on the augmented matrix in order to obtain the solution. As part of the solution to part (b), the lecturer introduced an unknown vector $\mathbf{w} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ to formulate the linear equation system. After presenting the solution he wrote the following sentence.

| Observation: | The linear equation system $A\mathbf{w} = 3\mathbf{w}$ can be rewritten as $(A - 3I_3)\mathbf{w} = \mathbf{0}$. |

That is, the lecturer deduced $(A - 3I_3)\mathbf{w} = \mathbf{0}$ from $A\mathbf{w} = 3\mathbf{w}$, or equivalently, $(A - 3I_3)\mathbf{v} = \mathbf{0}$ from $A\mathbf{v} = 3\mathbf{v}$. He used this new equation with $\lambda = -5$ to solve part (c). For the value $\lambda = -5$, only the zero vector was a solution which implied that $-5$ was not an eigenvalue.

Following this example the lecturer made several observations which were printed in the course notes. He emphasised that eigenvectors were found by solving a linear equation system $(A - \lambda I_n)\mathbf{v} = \mathbf{0}$. He referred to this equation as the eigenvector equation (where none of the textbook authors that I consulted for my analysis in Section 6.4.1 gave this equation a name).

In relation to $(A - \lambda I_n)\mathbf{v} = \mathbf{0}$, the lecturer made the following observation: Either the matrix $A - \lambda I_n$ is invertible and hence the vector $\mathbf{v}$ is the zero vector (and $\lambda$ is not an eigenvalue), or the matrix $A - \lambda I_n$ is not invertible, and in that case the value of $\lambda$ is an eigenvalue of the matrix $A$.\(^{23}\)

In connection with the example presented, the characteristic equation is $\det(A - \lambda I_n) = 0$ which the lecturer in my study did not, and could not introduce at this point because he had not yet covered determinants.

The observations discussed above represented the end of the lecture and the end of the introduction of eigenvectors and eigenvalues. With these observations the lecturer made links to the invertibility criteria of matrices in Chapter 2 and to the concept of a null space in Chapter 3.

\(^{23}\)Further explanations of this observation: If the matrix $A - \lambda I_n$ is invertible, then the determinant $\det(A - \lambda I_n)$ must be non-zero. As a consequence the vector $\mathbf{v}$ must be the zero vector. It follows that $\lambda$ is not an eigenvalue. On the other hand, if the matrix $A - \lambda I_n$ is not invertible, then the determinant $\det(A - \lambda I_n)$ is zero. In that case the values of $\lambda$ are the eigenvalues of the matrix $A$. 
The lecturer’s presentation was aimed at introducing eigenvectors and eigenvalues. All the observations that the textbook authors made, and that I highlighted in my analysis in Section 6.4.1, the lecturer made, too. However, while the authors of the textbooks/set of course notes stated observations in terms of theorems and definitions (and included proofs), the lecturer in my study used the terms ‘Remark’ and ‘Observation’ and provided, in most cases, no formal proofs. In contrast to the textbook authors, the lecturer did not introduce the characteristic equation in his introduction to this topic.

The lecturer re-visited eigenvectors in the second half of Chapter 4 of the course notes. This is the focus of the next section.

**The lecturer’s ‘re-visiting’ of eigenvectors in the course notes**

Chapter 4 of the course notes consisted of two parts and were made available to students as two separate hand-outs. The lecturer had introduced eigenvectors in the first half of Chapter 4 via the definition $A\mathbf{v} = \lambda \mathbf{v}$, and had used only the definition to find the solution to the examples that he presented.

Over the next weeks of the module the lecturer discussed the diagonalisation process and the theory of determinants. He presented the row and column expansion method for calculating the determinant of a matrix of order up to $3 \times 3$. He then re-visited the concept of eigenvectors and eigenvalues. This included linking the solution of an eigenvector equation with criteria for invertibility, and hence with determinants.

In Chapter 4.5 in the course notes, under the heading “The characteristic polynomial”, the lecturer introduced Example 4.33 (which I do not discuss in detail). In presenting this example, the lecturer focussed on exploring the solutions in relation to the concepts of invertibility and determinants. The solutions to the questions he posed (to students) resulted in Observation 4.34. This observation (a) stated that the eigenvalues of the matrix $A$ are the zeros of the characteristic polynomial, and (b) gave an exposition of the ‘standard procedure’ for determining the eigenvalues and corresponding eigenvectors of a matrix. I have re-produced here what I had termed the ‘standard procedure’:
To find all eigenvalues and eigenvectors of a (small) matrix $A$, we can proceed as follows:

1. Calculate the characteristic polynomial $P(\lambda) = \det(A - \lambda I_n)$.

2. Find all zeros of the characteristic polynomial. These are the eigenvalues.

3. For each eigenvalue $\lambda_i$, solve the equation $Av = \lambda_i v$ or $(A - \lambda_i I_n)v = 0$. The solutions are the eigenvectors corresponding to $\lambda_i$.

I interpreted the use of Example 4.33, from the ‘expository’ questioning to the formulation of the observation, as representing the EAG approach as detailed in Chapter 5.

6.4.3 The lecturer’s didactical thinking in relation to the teaching of eigenvectors

In Sections 6.2.3 and 6.3.3 I discussed the lecturer’s didactical thinking in relation to the concept of subspace and of linear independence. I focused on the role of the examples used, the role of the vector space $\mathbb{R}^n$ and the role of (in)formal language.

My focus in this section is (a) the role of the Examples 4.1 and 4.5, and (b) the lecturer’s sequencing of topics for conceptual understanding.

I again analysed the comments made by the lecturer in research meetings, and the student version and the complete version of the course notes (as I have done in Sections 6.2.3 and 6.3.3). I begin with an analysis of the role of the Examples 4.1 and 4.5. I focus on the introduction of eigenvectors/values in the first part of Chapter 4 of the course notes.

(a) The role of Examples 4.1 and 4.5

As I have done in Sections 6.2.3 and 6.3.3 in relation to the concepts of subspace and of linear independence, I will show in this section in relation to eigenvectors/values that the examples presented to students were designed to fulfill more than one function. The examples functioned as technical tools when considering students’ participation in the lecture. In formulating mathematical problems the examples functioned as conceptual tools when considering thinking about and engaging with the mathematics, and as cultural tools when considering the view of mathematics and of mathematical learning that the lecturer tried to convey.
As discussed in Section 6.4.2 the lecturer introduced the concept of eigenvector via Example 4.1 (see below or page 228). He stated a $3 \times 3$ matrix and two vectors (giving numerical values for both the matrix and the vectors). He ‘asked’ students to calculate the matrix multiplication $Av$, and to notice that the resulting vector represented a multiple of the input vector. As a result of these calculations he defined the eigenvector, and noted that one of the vectors was an eigenvector, while the other was not. This introduction to eigenvectors relied on a calculation based on a purely algebraic view of linear algebra. The lecturer did not evoke any Geometry (a deliberate act, see Chapter 1), or any aspects aligned to transformations. The lecturer made the didactical decision to focus solely on the definition of an eigenvector in order to draw students’ attention to the conceptual nature of this concept. In a research meeting he said,

Again [I am] inspired by trying to do the important concepts first and the computational recipes later on, in order to avoid that students think of an eigenvalue as a zero of the characteristic polynomial, which most know anyway because that’s how they’re calculated. But this point of view really is unhelpful if you want to move out of the calculation because it doesn’t give you an opportunity to do anything with it. That’s a way of looking at things that doesn’t really go to the heart of what eigenvectors and eigenvalues are. (M15, 1:11:02)

In stressing this point to students, the lecturer made the decision to also email small group tutors asking them to focus on the definition when working with students on eigenvectors and eigenvalues in small group tutorials. The lecturer said in a research meeting,

---

**Example 4.1.** Consider the matrix

$$A = \begin{pmatrix} 5 & 5 & -2 \\ -4 & -4 & 2 \\ -3 & -3 & 2 \end{pmatrix}$$

and the vectors

$$v_1 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad v_2 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

(a) Calculate the vectors $Av_1$ and $Av_2$. Comment on your results.

**Solution:**

Blank space here.

(b) Calculate $A^{2000}v_2$.

**Solution:**

Blank space here.
...if I think there is something important about a specific tutorial sheet I send an email, as you have seen, for example, telling tutors that we are discussing eigenvectors and eigenvalues, that students don’t know determinants and don’t know the characteristic polynomial, so please focus on the linear equation side of things. (M15, 1:10:26)

Part (b) of Example 4.1 involved a repeated *use* of the definition in solving the problem. Designing this example was aimed at students *gaining* a conceptual understanding of eigenvectors, in terms of the definition alone.

The lecturer presented the second example, Example 4.5, after he had introduced the definition of an eigenvector. I have reproduced the example here as it appeared in the course notes:

**Example 4.5.** Consider the matrix

\[
A = \begin{pmatrix}
0 & -6 & 4 \\
1 & 5 & 2 \\
0 & 0 & -2
\end{pmatrix}
\]

(a) Show that the vector

\[
v = \begin{pmatrix}
3 \\
-1 \\
0
\end{pmatrix}
\]

is an eigenvector of \( A \). What is the corresponding eigenvalue?

**Solution:**

Blank space here.

(b) Show that \( \lambda = 3 \) is an eigenvalue of \( A \). Find all corresponding eigenvectors.

**Solution:**

Blank space here.

(c) Is -5 an eigenvalue of \( A \)? If so, find the corresponding eigenvectors.

**Solution:**

Blank space here.

In my interpretation, and in comparing with the different kinds of examples that I discussed previously, Example 4.5 represented a ‘routine’-type example. It was an example of ‘practicing’ the application of the definition. Here, in relation to eigenvectors/values,
the lecturer presented three parts to the example, which he labelled (a), (b) and (c). In parts (a) and (b) he presented the two ‘possibilities’ that can occur when calculating eigenvectors/values: Given a vector, show that it is an eigenvector and find the eigenvalue, or vice versa, given a value for \( \lambda \), show that it is an eigenvalue and determine the eigenvector. The lecturer wrote down the solution to each possibility in the lecture (see next section, 6.4.4), and thus ‘showed’ students the methods, the ‘how to do it’.

At the same time, in designing parts (a) and (b) of this example the lecturer presented a certain view of mathematics and of mathematical work, namely one that uses a definition as starting point for further work.

The third part, part (c) of Example 4.5 was different from parts (a) and (b). Part (c) related to noticing that the zero vector is not an eigenvector. That is, if \( v = 0 \) is the only solution to \((A - \lambda I)v = 0\), then \( \lambda \) is not an eigenvalue.

In designing part (c) the lecturer posed the question “Is \(-5\) an eigenvalue of \(A\)?” To answer the question students needed to perform a calculation that went back to the definition of an eigenvalue. Hence, students needed to first decide, or know which equation to use. When they had performed the calculation, they needed to go back to the formulation of the definition and decide on the meaning of their solution. Hence I interpreted Example 4.5 part (c) as designed for conceptual understanding.

In a research meeting, the lecturer explained his thinking when designing the examples in relation to eigenvectors/values. He said,

"But then, of course, the way I would like them to think about it is, ‘I do this calculation because I’ve got the eigenvalue equation, and this is what it means for a number to be an eigenvalue, that I check if there is a non-zero vector that satisfies that’. (M15, 36:52)"

Hence, the reason for offering this example was for conceptual understanding, for students to realise the significance of obtaining the zero vector as the only solution. This was a didactical decision that encouraged ‘thinking’ about the solution found, and how it connected with the definition of an eigenvector (\( v \) must be non-zero).

My discussion above relates to the introduction of eigenvectors/values in the course notes and corresponding lecture. This represented the first lecture on eigenvectors in the module. The lecturer had not introduced, or mentioned the characteristic equation.
or the characteristic polynomial. I now refer back to part (b) of Example 4.5.

For the solution to part (b), the lecturer derived the equation \((A - \lambda I_3)w = 0\) from the definition, that is from \(Aw = \lambda w\). (He used of the letter \(w\) instead of \(v\), see my previous discussion in Section 6.4.2). However, as I have interpreted this, he did not deduce the equation \((A - \lambda I_3)w = 0\) for his students. In setting up the example, the lecturer had focused on the definition. In solving the example, he had written the problem as a linear equation system, re-arranged the equations to collect ‘like’-terms and then translated the equations into an augmented matrix form (see below or pages 229/230). Thus he avoided any mention of the characteristic equation while at the same time using the procedure that is the direct result of solving the characteristic equation.

With reference to the structuring of the linear algebra topics, the lecturer had not introduced determinants at this point so could not evoke the characteristic equation. The lecturer thus performed the ‘standard procedure’, he was doing it but not saying. He went step-by-step through the algebraic manipulations and arrived at the characteristic equation. It was an explicit way of proceeding but the fact that this is always so, and can be encapsulated in what I have called the ‘standard procedure’, was not made explicit to students. This part (b) of the example exemplified the lecturer’s informal

\[\text{Solution:}\quad \text{We need to find a non-zero vector } w \text{ that satisfies } Aw = 3w. \text{ If we write}
\begin{pmatrix} x \\ y \\ z \end{pmatrix}
\text{in components, this equation reads}
\begin{align*}
-6y + 4z &= 3x \\
x + 5y + 2z &= 3y \\
-2z &= 3z
\end{align*}
\text{This is a linear equation system! We rewrite it as}
\begin{align*}
-3x - 6y + 4z &= 0 \\
x + 2y + 2z &= 0 \\
-5z &= 0
\end{align*}
\text{and solve it:}
\begin{pmatrix}
-3 & -6 & 4 & 0 \\
1 & 2 & 2 & 0 \\
0 & 0 & -5 & 0
\end{pmatrix}
\xrightarrow{\text{Gaussian elimination}}
\begin{pmatrix}
1 & 2 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\text{The general solution is}
\begin{pmatrix}
w \\
x \\
y \\
z
\end{pmatrix} = \alpha
\begin{pmatrix}
-2 \\
1 \\
0 \\
0
\end{pmatrix}
\text{with a free parameter } \alpha. \text{ All vectors of this form, with } \alpha \neq 0, \text{ are therefore eigenvectors.}
\text{Observation:}\quad \text{The linear equation system } Aw = 3w \text{ can be rewritten as } (A - 3I_3)w = 0.\]
approach of teaching with the aim of providing students with an intuitive understanding of the concept. He evoked only the definition of an eigenvector, and encouraged students to use the definition to inform them on how to proceed with the solution. The lecturer presented a view of mathematical working based on definitions, deductive arguments and calculations. In this sense the examples functioned as cultural tools, giving a view of mathematics that differed from my previous interpretations of examples.

In designing Example 4.1 the lecturer’s focus was on the use of the definition of an eigenvector. In formulating the mathematical content of the example he asked students to calculate. Example 4.1 was designed to move students from computational to conceptual understanding, from a calculation to a definition and hence a concept. Example 4.1 was designed for conceptual understanding.

In my interpretation and analysis of Example 4.5, parts (a) and (b) were of routine-or practice type, and part (c) was an example designed for conceptual understanding. Thus within this introduction to eigenvectors the lecturer presented a multiple view of mathematical working.

In keeping with my interpretation in Sections 6.2.3 and 6.3.3 Examples 4.1 and 4.5 were a technical tool in encouraging participation in the lecture and with the mathematics. In formulating content they functioned as conceptual tools in encouraging thinking about and working with eigenvectors, and as cultural tools in portraying a view of mathematics as a subject that is both computational, deductive and conceptual.

This concludes my didactical analysis of the role and use of examples in relation to the introduction of eigenvectors and eigenvalues. I now discuss the sequencing of the material of Chapter 4 of the course notes.

(b) The sequencing of topics for conceptual understanding

Based on my analysis, the sequencing, or ‘structuring’ of the course material was the ‘big’ didactical decision that the lecturer made in connection with the concept of eigenvectors/values. The sequencing of the topics eigenvectors/determinants/diagonalisation included a re-visiting of eigenvectors and eigenvalues, as detailed in the course notes (in written form) and put into practice in face-to-face teaching. In doing so, that is, in re-visiting eigenvectors the lecturer shifted his didactical focus from the definition to the working procedure, from conceptual to procedural ways of working with eigenvectors.
The lecturer based his decision on his experience of teaching the module in the previous year. In research meetings he said that he perceived students’ focus to be on the computational aspect of eigenvectors/values, and that students were unable to relate the concept to its definition. He also re-organised the material around the topic of determinants which he said focused students’ attention on computations and procedures. In a research meeting he said,

... [Determinants] they’re easy to use, they have a formula attached to them. If you ask one of our students when a matrix is invertible, they’ll say the determinant is not zero. They won’t say, ‘I can do something like division with it’, or ‘I can multiply it by its inverse’ and, if you ask our students at the end of the semester what an eigenvalue is, they’ll say it’s a zero of the characteristic polynomial, because that’s how you calculate it. And it’s true, of course, but it doesn’t take you to where you can actually think about eigenvalues. (M10, 38:30)

In dividing the teaching of eigenvectors/values into two distinct parts, the lecturer wanted to direct students’ attention away from the computational and algorithmic aspects, and more towards the conceptual nature of this concept. In Section 6.2.3 (in relation to the concept of subspace) I discussed the lecturer’s decision to embed the teaching of linear algebra in the setting of $\mathbb{R}^n$. This, too, was an attempt to shift students’ attention towards the conceptual nature of linear algebra concepts.

This division was made explicit through the design of the course notes and in the lecturer’s subsequent face-to-face teaching which featured eigenvectors twice. Based on my analysis of the course notes the structure and development of the content in Chapter 4 was as follows:

1. Defining an eigenvector and corresponding eigenvalue.
2. Deriving $(A - \lambda I_n)v = 0$ from the definition - but not defining the characteristic equation.
3. Introduction to diagonalisation: The diagonal matrix and the diagonalising matrix.
4. Defining determinants axiomatically (i.e. ‘by their properties’).
5. Row and column expansion of determinants.
6. Deriving the characteristic polynomial.
7. Re-visiting diagonalisation.
First the lecturer presented an introduction to both eigenvectors and diagonalisation, followed by determinants, and then a re-visiting of eigenvectors and diagonalisation. With points (1) to (4) the lecturer focussed on the conceptual nature of the topics of eigenvectors, diagonalisation and determinants, and with points (5) to (7) on the more computational aspects. Hence, at the level of topic design the lecturer replicated the approach that he had taken at the level of example design.

The lecturer decided to teach an introduction (based on the definition) of eigenvectors and eigenvalues first. He re-visited eigenvectors (and eigenvalues) and only then presented the calculations required to determine eigenvectors (and eigenvalues) from the characteristic polynomial. In a research meeting, the lecturer said that he was ...

... trying to do the important concepts first and the computational recipes later on, in order to avoid that students think of an eigenvalue as a zero of the characteristic polynomial.. (M15, 1:11:02)

The lecturer took this didactical decision in order to bring about a change in students’ view of eigenvectors as a conceptual, rather than a computational or algorithmic aspect of linear algebra.

(b)(i) Introducing eigenvectors

At the start of Chapter 4 of the course notes the lecturer referred only to the definition of an eigenvector, and focussed on an algebraic (and not a geometric, say) view. I discussed the examples that the lecturer presented to students, Examples 4.1 and 4.5, earlier in this section. The lecturer’s intention (based on the comments he made in research meetings) was to encourage students to think conceptually about eigenvectors. In my analysis of the course notes the first example, Example 4.1, was designed to move students from computational to conceptual understanding, from a calculation to a definition and hence to the concept. In my interpretation and analysis of Example 4.5, parts (a) and (b), were routine- or practice types of examples, and were designed to re-enforce the calculation based on an appreciation of the concept. Example 4.5 part (c) was designed for conceptual understanding, that is, students were required to calculate and then go back to the definition in order to evaluate and complete the solution.

(b)(ii) Re-visiting eigenvectors

On re-visiting the concept of eigenvectors and eigenvalues in the second half of Chapter 4 the lecturer focussed more on the computational aspects in connection with eigenvectors.
He made the computational aspects *explicit* in his teaching. In the course notes he gave a detailed description (that students could follow or use) of the ‘standard procedure’ for calculating eigenvalues and eigenvectors. He wrote:

<table>
<thead>
<tr>
<th>To find all eigenvalues and eigenvectors of a (small) matrix $A$, we can proceed as follows:</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Calculate the characteristic polynomial $P(\lambda) = \det(A - \lambda I_n)$.</td>
</tr>
<tr>
<td>2. Find all zeros of the characteristic polynomial. These are the eigenvalues.</td>
</tr>
<tr>
<td>3. For each eigenvalue $\lambda_i$, solve the equation $Av = \lambda_i v$ or $(A - \lambda_i I_n)v = 0$. The solutions are the eigenvectors corresponding to $\lambda_i$.</td>
</tr>
</tbody>
</table>

A fellow mathematician noticed that the last sentence should read, “The non-zero solutions are the eigenvectors corresponding to $\lambda_i$.” In providing this description the lecturer was focussing on the procedure and not the definition of an eigenvector. I interpreted the omission of ‘non-zero’ as part of the informal approach to teaching.

On re-visiting eigenvectors the lecturer shifted the focus towards the use of eigenvectors as tools for calculating a change of basis. In a research meeting the lecturer said,

Diagonalisation basically *is* the link between expressing things in a basis and eigenvectors and eigenvalues. (M15, 16:13; *italics my emphasis*)

and also,

...next week [Semester 1 Week 8], and then [we] do determinants which we need as a technical tool for doing eigenvectors, and when we’ve done that I’ll talk about the characteristic polynomial and come back to how we actually calculate eigenvectors. (M10, 37:23)

Hence, in the lecturer’s own words, determinants are a tool for calculating eigenvectors, and eigenvectors are a tool for diagonalising matrices. The lecturer’s use (and my use above) of the word tool is not to be confused with the term as used in Activity Theory. The lecturer had no knowledge of Activity Theory when making these comments in research meetings. In using the word tool the lecturer described the inter-connectedness of linear algebra concepts, a point that is made in the research literature and often cited as one of the reasons why students find linear algebra difficult (see Chapter 2, Sections 2.3.1 and 2.3.2).
The structuring of the content of Chapter 4 was, as I said above, the ‘big’ didactical decision that the lecturer had made in connection with his teaching of eigenvectors. In asking students to work with the definition he was aiming for students to gain a conceptual understanding (first). In re-visiting eigenvectors in the second part of Chapter 4, the lecturer introduced the ‘standard procedure’ for calculating eigenvectors which engaged students more computationally. The overall design was aimed at students’ conceptual understandings where determining eigenvectors became a tool for further conceptual work (this time on diagonalisation and change of bases).

This concludes my didactical analysis of the lecturer’s thinking and decision-making in connection with the topic of eigenvectors and eigenvalues.

6.4.4 The lecturer’s teaching of eigenvectors/values (to students)

In this section I consider how my interpretation of the lecturer’s didactical thinking has been translated into practice. I discuss in detail the introduction of eigenvectors corresponding to the first part of Chapter 4 of the course notes.

The lecturer introduced students to the concept of an eigenvector/value by presenting an example, Example 4.1 (see page 154 or page 228). This example had two parts, parts (a) and (b), and led to the formal definition of an eigenvector (and corresponding eigenvalue). My analysis of the presentation of this example in the lecture supports my interpretation of the lecturer’s didactical thinking and planning. I re-iterate some of the points raised in Section 6.4.2 and include comments that the lecturer made in lecture to his students.

In introducing the example the lecturer presented a $3 \times 3$ matrix $A$ and two column vectors, $v_1$ and $v_2$. For part (a) the lecturer had written, “Calculate the vectors $Av_1$ and $Av_2$” [italics my emphasis] in the course notes. In the lecture (in Week 8) the lecturer introduced this example by saying,

We have seen a matrix can be applied to a vector, gives a new vector, so what are the simplest things that we can think of that can possibly happen? And this is something that I would like you to explore in the first example that I have brought in. In the example on your sheet I've given you a matrix with two vectors, and I would like you to multiply the matrix with the two vectors, see what happens, if you can notice anything special. (L21, 04:22).

Within the introduction the lecturer posed the question,
What are the simplest things that we can think of that can possibly happen? (L21, 04:28).

The lecturer used the words 'simple' and 'special', drawing attention to a property of eigenvectors, but without making this property explicit. This forms part of the use of informal language (as discussed in Section 6.4.3) and is representative of the informal approach taken by the lecturer in this module.

The lecturer gave students some time to work on this problem before writing the solution on the overhead projector. After presenting the solution for part (a), he said,

[Multiplying $A$ by $v_1$], this is the vector that we get. Nothing special to observe in this case. But in the other case, if we multiply $A$ by $v_2$, what we’re going to get is this vector. And if we compare this vector to the vector $v_2$, that we put in, we’ll find this is just double the vector $v_2$. So in some sense the vector $v_2$ is very special for the matrix, if we multiply the matrix by the vector we get only two times the original vector back. That’s particularly simple, something very simple that the matrix does to the vector. And using that observation, you’ll be able to figure out what ‘$A$ to the power 2000 times $v_2$’, what that’s going to be . . . (L21, 13:22)

The lecturer told students to use the result of part (a) to calculate $A^{2000}v_2$ in part (b). He introduced part (b) by saying that calculating powers of matrices was an application related to Markov processes, and that these applications occurred frequently, and were important in mathematics. Thus the lecturer used a concrete example (Markov processes) to motivate the concept of an eigenvector. He again gave students time to try the solution themselves first, before he wrote the solution on the overhead projector. After presenting the solution (see Section 6.4.2 on page 154 for a copy of the solution), the lecturer again pointed to $v_2$ having a special property. He said,

Notice how we did that. We observed some special property that the vector $v_2$ has. We observed multiplying the vector $v_2$ by the matrix $A$ is the same as multiplying the vector $v_2$ by the number 2. And that allowed us to do this calculation. And if you now try to calculate ‘$A$ to the power 2000 times the vector $v_1$’, you will find that the calculation we have just done is some sort of a miracle. It works only because we’ve got this special property of the vector $v_2$. For the vector $v_1$ we have no chance of achieving anything of that sort. (L21, 24:23)
The lecturer presented the definition (Definition 4.2) which I reproduce here (again):

**Definition 4.2.** For an \( n \times n \) matrix \( A \), a vector \( v \) (not equal to the zero vector) and a number \( \lambda \) that satisfy

\[
Av = \lambda v
\]

are called an eigenvector and the corresponding eigenvalue of \( A \).

The equation \( Av = \lambda v \) was not printed in the course notes so that students did not see this equation prior to working on the problem.

After presenting the Definition 4.2 the lecturer said,

> And observe the key to everything we have done here is this special property of the vector \( v_2 \). (L21, 26:57)

The task set for students of calculating powers (Example 4.1 part (b)) relied on a simple substitution, of \( Av \) by \( \lambda v \), or in this case \( 2v \). Thus the lecturer’s first example in relation to eigenvectors consisted of a *calculation* that led to the definition. The example was based on a process, a calculation, that was algebraic in nature (a substitution).

The lecturer then projected Remark 4.3 which referred to the fact that the zero vector always satisfied the equation \( Av = \lambda v \). He said,

> And notice also this says the vector \( v \) must be non-zero. Why is that? Well, if you take the zero vector, that *always* satisfies ‘\( A \times \text{zero} \)’, is ‘\( \lambda \times \text{zero} \)’, is ‘zero’, for any number \( \lambda \), so that’s nothing special. We’re interested in vectors that have special properties with respect to the matrix \( A \). So we always require that the eigenvector \( v \) must be *not* the zero vector. (L21, 28:15; *italics my emphasis*).

In Section 6.4.2 I discussed this remark in detail in relation to the lecturer’s use of *informal language*. The lecturer introduced the next example, Example 4.5, by saying,

> “...to see if we actually understand correctly what it means for a vector to be an eigenvector...”

This example had three parts to it, parts (a), (b) and (c), and each part was of a different nature, fulfilled a different function.

In part (a) the lecturer asked students to verify that the given vector was an eigenvector and to find the corresponding eigenvalue, and vice versa for part (b). This example
was presented for students to verify their understanding, and for practice. The lecturer referred to the definition and what students had to “check” to see if a vector was an eigenvector. Part (c) was of a conceptual nature and involved a calculation first, followed by a study of the definition in order to give meaning to the solution. I have discussed Example 4.5 in detail in Section 6.4.2. In the lecture, in presenting this example the lecturer gave students some time to try each part for themselves first before he presented the solution. Field notes indicate that more students seemed to be working on this example than on previous examples.

On re-revisiting eigenvectors and eigenvalues two weeks after first introducing the concepts, the lecturer introduced the characteristic polynomial and worked exclusively with the characteristic polynomial in the examples that he presented to students. With Observation 4.34 (see page 232) he summarised the steps involved in carrying out the procedure for calculating the eigenvalues and corresponding eigenvectors. Thus in connection with eigenvalues and eigenvectors the lecturer made the didactical decision to present work first exclusively conceptually, and then exclusively computationally.
6.5 Didactical summary

In this chapter I reported on my analysis of the lecturer’s teaching of three concepts in linear algebra. I presented my interpretation of the lecturer’s didactical thinking and planning and my interpretation of his teaching of each concept in the context of the university mathematics lecture. I considered four texts, that is three textbooks and a set of course notes, to see how other authors approached the topics of subspaces, linear independence and eigenvectors/values in their writing in order to aid my interpretations.

The lecturer had adopted an inductive approach to teaching which I had termed EAG and for which I gave details in Chapter 5. Central to the lecturer’s inductive style of teaching was the presenting of examples from which to make observations and arrive at a more general, intuitive appreciation of a mathematical concept. The lecturer discussed his approach and his intentions for students’ learning in research meetings. In addition, he verbalised his intentions to students in face-to-face teaching.

The lecturer (in collaboration with the lecturer who taught the second semester) had made the didactical decision to re-structure the material of the linear algebra module so that all the concepts of this introductory linear algebra module were introduced in the first semester in the context of the vector space $\mathbb{R}^n$. The lecturer who taught the second semester then re-visited the same concepts in the more general context of abstract vector space theory.

In Chapter 5 I reported on my analysis of the lecturer’s actions which included the use of tools, that is material tools and psychological tools. In this chapter I reported on my examining the function of the material and psychological tools. I focussed on (a) examples as material tools, (b) the mathematical problems (presented within the examples) as psychological tools, and (c) the role of language, that is formal and informal language, and in written and spoken form, as a psychological tool. I examined the function of each of the tools mentioned. As a result of my analysis I categorised the material/psychological tools as functioning as technical, conceptual and cultural tools. I did this in relation to the teaching of three particular topics within linear algebra.

The examples

As a result of my analysis I reported on the lecturer’s didactical thinking in designing examples in an inductive approach to teaching. Examples were designed to encourage and increase student participation in lectures, to engage students with the mathematics by thinking about, and attempting to solve the mathematical problems...
presented in the example, and to encourage students to form a certain view of mathematics and of mathematical learning. Thus the examples functioned as technical tools in respect of increasing student participation. In formulating the content of examples, the mathematical problems functioned as conceptual tools to encourage students to think (more) conceptually about mathematics, and as cultural tools to challenge students’ view of mathematics (as the lecturer saw it) based solely on calculations and algorithms.

(a) My analysis of the lecturer’s didactical thinking and planning showed that the lecturer designed all examples with the aim of participation. Functioning as technical tools the examples lent structure to the course notes (the course notes had blank areas after the example for students to write in) and affected the format of the lecture (the lecturer provided ‘breaks’ during face-to-face teaching in order to give students time to work on the mathematical problems). My analysis of the lecturer’s face-to-face teaching of the three concepts showed that, when the lecturer did provide time for students to work on the problems presented, some students participated in lecture and worked on the problems and the tasks set while others seemingly did not. In Chapter 7 I discuss the reasons for students’ engagement or disengagement. My analysis also showed that there were many occasions when the lecturer presented an example and proceeded to solve the problem, without giving students time to work on the problems themselves first.

(b) As psychological tools the examples were designed to present mathematical problems for students to engage with in the lecture. Here I distinguished their functioning in terms of conceptual tools and in terms of cultural tools. (b)(i) Functioning as a conceptual tool the examples were designed to lead students to an intuitive understanding of the linear algebra concepts being studied and to an appreciation of their conceptual nature. (b)(ii) Functioning as a cultural tool the examples were designed to engage students in a certain way with the mathematical problem, that is, so that students could form a view of mathematics as creative and adaptable to many different situations, an intention made explicit by the lecturer in research meetings and in lectures to students.

(b)(i) In formulating the content of examples, the lecturer’s design made the mathematical problems (presented as examples) accessible. In my interpretation of the design of Example 3.4 (in relation to subspaces) students did not need any particular prior knowledge in order to attempt the problem set. Hence the example was accessible to all students. Example 4.1 (in relation to the introduction of eigenvectors) was accessible because the lecturer “asked” students to perform a calculation which students ‘knew’ how to perform, namely matrix multiplication. This was the case also with Example
3.16/3.17 (in relation to linear independence), that is the lecturer posed a problem that required the students to perform a calculation. However, unlike Example 4.1, the calculation that students needed to perform related to the concepts of span and spanning sets which were new to students and which had been introduced only in the previous lecture.

Example 3.4 demonstrated the inductive style of generalising from an example. Students were required to generate solutions and formulate more general statements of the solutions based on thinking about and spotting patterns. The examples related to linear independence and eigenvectors functioned differently. Example 4.1 (related to the introduction of eigenvectors) was based on a calculation and, through thinking about the algebra of the calculation, to the definition. Examples 3.16/3.17, related to linear independence relied, as I said above, on a previous concept that was new to students and that students may not have internalised. In my analysis in Section 6.3.3 I claim that Examples 3.16/3.17 structured the transition from the concept of span to linear independence based on mathematical definitions. This structuring relied on deducing one concept from a previous concept.

In addition to the examples mentioned thus far, the lecturer used further and follow up examples that were of a routine or practice type. That is the formulation of the mathematical problems was such that all possible cases that can occur were covered (for example, in relation to eigenvectors the lecturer’s formulation of Example 4.5 (a) and (b)).

(b)(ii) In formulating mathematical problems the lecturer presented a cultural view of mathematics and mathematical learning. His intentions as stated in meetings were to change students’ view of mathematics and to design examples that could demonstrate the creative side of mathematics and the way that mathematicians worked with problems. This had led the lecturer to design an inductive approach to teaching. With reference to (b)(i) some examples were designed to be accessible, open to investigation and could produce multiple answers (e.g. Example 3.4) while others were closed and producing just one correct answer (Examples 3.16/3.17, Example 4.1/4.5), or relying on previous knowledge (Examples 3.16/3.17) which made them less accessible to all students.

Formal and informal language

In Chapter 5 I discussed the action of ‘verbalising intentions’. This related to the lecturer’s didactical thinking as expressed in research meetings. My focus was to identify the actions and the goals based on the lecturer’s didactical thinking and not as realised in lectures, in face-to-face teaching. However, the action of verbalising intentions led me
to consider the role of language in introducing the three topics in the course notes (at the design and planning stage) and as used by the lecturer in face-to-face teaching of students.

In designing the course notes the lecturer used precise mathematical language to formulate and write down definitions and theorems etc., and informal everyday English in explanations. Explanations in the course notes generally followed a definition or theorem and the example, and took the form of a “Remark” or an “Observation”. Hence apart from stating the definition or theorem itself, the lecturer, on the whole, did not use precise mathematical language in explaining linear algebra concept in the course notes. In the remarks and observation he used informal language such as “multiplication by a number” and not “by a scalar”. At times this also led to imprecisions and omissions. For example, in writing down the procedure for finding the eigenvectors using the characteristic polynomial, the lecturer omitted that the eigenvectors must all be non-zero (see page 166).

In writing down explanations he used everyday language and comparisons to provide a “feel” for the concepts, as expressed in meetings and also in lectures to students. Presenting all the linear concepts in the first semester in the context of $\mathbb{R}^n$ made using informal language both easier and appropriate. For example, the lecturer could talk about column vectors rather than the notion of a vector as an element of an abstract vector space.

In his face-to-face teaching of students the lecturer used informal language extensively to explain linear algebra concepts. He used the formal language of mathematics to read and write the definitions and theorems that he was introducing. However, in the course of the lecture he provided all explanations verbally and using everyday English. In Section 5.4.8 I reported on my analysis of the lecturer’s modes of talking in face-to-face teaching. I referred to commenting and meta-commenting levels which related to comments made by the lecturer in relation to mathematics and mathematical learning.

The use of examples and the use of informal language were features of the lecturer’s inductive approach to teaching where examples were used to motivate definitions and theorems. In lectures, students were asked to try solutions themselves first in order to get a “feel” for the concepts that were introduced. The lecturer provided extensive verbal explanations to guide students in developing a more conceptual understanding of linear algebra concepts. These were features of the inductive approach.

In addition, the lecturer had made two ‘big’ didactical decisions in order to focus students’ attention on the conceptual nature of linear algebra. He had re-structured the
module so that all concepts were introduced in the context of the vector space $\mathbb{R}^n$. Second, he designed a sequence of topics such that students were working conceptually with eigenvectors (based on the definition) before introducing the characteristic polynomial at a later stage in the module.

The didactical decisions that the lecturer had made, had (possible) implications for students’ learning of linear algebra. I discuss some of these implications in relation to mathematics and to mathematical learning in Chapter 8. I focus on possible implications for learning due to an absence of the concept of a vector space, the use of $\mathbb{R}^n$ in place of a general vector space, the use of informal language, and the fact that nearly all explanations were provided verbally in lectures (and mostly using informal language) while few explanations were provided in written form in the course notes.

In the next chapter I report on my analysis of the student data which provided information on students’ views and responses to the lecturer’s inductive approach to teaching linear algebra.
Chapter 7

Students’ views

In this short chapter I present my analysis of the student data which consisted of two surveys (in the form of questionnaires) and six focus group interviews. As discussed in Chapter 3 the lecturer who took part in the research taught the linear algebra module in Semester 1 only. A different lecturer took over the teaching of the module in Semester 2 but little research took place with this lecturer.

I sought students’ views in the first semester with two questionnaires and in the second semester by conducting group interviews. I first give a summary of my analysis of the questionnaires and then of the interviews.

Analysis of the questionnaires

The questionnaires were designed to obtain mainly factual information regarding students’ mathematical background before coming to the university, and how they were coping with the linear algebra module up to that point. My analysis which was both quantitative (using the computer software SPSS) and qualitative (using a coding and categorisation procedure) raised the following points:

Most students indicated that linear algebra was an entirely new topic for them. However, some students had met matrices before (at school or college), in particular matrix multiplication. Students said that they were coping with the demands of the new module; they ‘liked’ the notes with gaps as they felt it kept them interested in lectures, and enabled them to listen, and hence concentrate more/better in lectures.

One of my foci in analysing the second questionnaire was in respect of students’ engagement with the in-class exercises that the lecturer set (routinely) in most lectures. This involved students working on their own or with a neighbour on a problem before the
lecturer himself presented the solution. From lecture observations (captured in the field notes) I had come to conclude that some students took up this opportunity of working in-class while others waited for the lecturer to present the solution. On the questionnaire I asked students to estimate what percentage of the time they worked on the exercises that the lecturer had set, choosing from several options, for example 25-50% etc. Students indicated that they made a serious attempt working on the exercises 75% to 90% of the time. However, lecture observations and the subsequent student interviews did not support this claim. Not all students engaged in lectures with the exercises as the lecturer had envisaged, so probing students’ views in this area became a focus in the interviews.

Analysis of the focus group interviews

I conducted the focus group interviews after I had completed the analysis of the questionnaires. I interviewed fourteen students, in groups of two to four students, in Semester 2. My focus was on learning how students had responded to the inductive approach to the teaching that the lecturer had adopted for the linear algebra module. In particular, as I wrote in Chapter 3, I wanted to know (a) how students responded in lectures to working on the exercises that the lecturer set, (b) whether the inductive approach had led to an increased focus on the concepts of linear algebra (the lecturer’s goal) and (c) if, as a result of the teaching design students were coming to view mathematics as a creative, stimulating subject that could be used to develop ways of thinking (which related to the lecturer’s motive).

At the start of the interviews I presented students with a list of written questions. (A copy of the questions can be found in Appendix A.) These related to the course notes with ‘gaps’, students’ participation in lectures and what students had found ‘difficult’ or ‘easy’. I told students that these questions were prompts and that they should feel free to add anything they wished in connection with their learning experience in the linear algebra module. Students, in general, answered these questions and made further contributions of their own. These contributions offered insights into students’ views of examinations, the pressure of exams, their study habits and their preferred ways of learning, and ‘their dependence’ on the lecturer in presenting the perfect solution (in order to gain maximum marks in the exams).

Analysis of the focus group interviews raised three points in particular. Students (a) found linear algebra difficult, (b) liked the notes-with-gaps, and (c) frequently focussed on computational aspects and algorithms rather than engaging with the conceptual
nature of the topics as desired by the lecturer. I discuss each point below and present ‘evidence’ from the student interviews.

(a) All the students who took part in the interviews acknowledged that they found linear algebra difficult and particularly challenging at the start, that it was a subject that was entirely new and that had little connection with what they had studied before at school or college.

I think this is the only module that hasn’t actually been touched in my A-level, because I know that even in Calculus we’ve covered quite a bit of it, in Statistics we’ve covered quite a bit of it at normal A-level, but this is completely new, . . . and learning so much of it, I think throughout the whole year, it’s completely thrown me. (Int3, S7)

All students found the chapter on subspaces “quite difficult” whereas the first two chapters (on linear equation systems and matrices) and the last chapter (on eigenvectors) “were okay”.

S7: I think everything was fine until we hit the subspaces, (S5: Hmm.) and I think that’s where (S5: Yeah. S6: Yeah.) I realised how hard (laughter) it was. Because until then I was absolutely fine, I understood it perfectly. (Int 3, S5, S6, S7)

One student continued, saying,

It’s quite wordy, . . . you got bases, and kernel, and image, and range, . . . I still am at times, even now, I still don’t really know, . . . I think everyone found that difficult. (Int 3, S5)

Thus on the whole students indicated that they were unprepared for the conceptual nature of the topic.

(b) The students in the focus group interviews indicated that they liked the way that the lecturer had designed the module with the use of notes-with-gaps. They said that they felt it engaged them more in lectures and helped them to concentrate and ‘listen’ more. One student compared the lecture notes to “a workbook”, and the design of the module as providing a “stepping stone” from A-level study to university level study. Despite the positive attitude towards the notes-with-gaps students did not always work actively on the solution to the examples in lectures. As these students pointed out,
S3: Sometimes [I would work on the solution]. It depended like whether or not I could do it.

S4: Yeah. I think that. It depends whether you understand it. If you obviously don’t understand it, then you don’t wanna write down the wrong answer because obviously it would just muck up your notes and everything, so I just waited . . . (omissions) . . . In all honesty, I would discuss it with my friends next door or who sat next to me, and we would discuss it. But in all fairness, if I didn’t know then I wouldn’t write it in the gaps . . . (inaudible) . . . If I did understand it I would try it, if I was confident enough to do it.

(Int 2, S3, S4)

Students in the focus groups generally acknowledged that many students waited for the solution to be presented by the lecturer. The student above cited some of the reasons why students waited rather than working on the solution themselves. Other students cited the examination and the pressure of producing answers that gained “maximum marks”. For example this student said,

You hesitated to write anything down cos you didn’t know for certain if it’s gonna be in that sort of order cos you wanted to know the perfect solution, ways to set out an answer . . . (Int4, S8)

When pressed by the researcher to explain more about the ‘perfect solution’, the student said,

Obviously, you wanna learn the way [that the lecturer does it] in the exam . . . you wanna write your answers in the way that they gonna get the most marks and the best way to set them out and stuff, . . . (Int4, S8)

Hence, learning how to write things down in the way that the lecturer preferred was important for doing well in the examinations. In fact, students said that exams were on their mind “all the time”. This may have had a greater impact on students’ ways of working and general work habits than we, the researchers, or the lecturer in my study might have anticipated.

c) Students frequently referred to computational aspects of linear algebra. The Gaussian elimination procedure was taught in the beginning of the module, in Chapter 2. One student commented that you always had to use Gaussian elimination somewhere at some time, so if she didn’t know what to do, she would always do a Gaussian elimination on
the matrix, “automatically”. She expressed the view that this was likely to gain at least some marks (in an exam, say).

When I asked, “What do you mean you just do it automatically?”, she said:

When you see a matrix I automatically think Gaussian elimination. (I asked “Why?”) Because it’s just been programmed into me that every time you see a matrix, to work out like different . . . , to evaluate what that matrix is all about you have to do Gaussian elimination. That’s what’s programmed in my mind. (Int6, S12)

And a little later on she said,

Going through Semester 1, no matter what we were working out . . . when we got onto matrices and working out like the range and . . . everything just seemed to be ‘you do it by Gaussian elimination’. From that you could tell what the range is, you could tell what the null space is, you could tell what the . . . (omissions) . . . I just knew that everything in that chapter had to be worked out by Gaussian elimination into reduced echelon form. . . . I wouldn’t have got any marks in the class test if I didn’t realise that. (Int6, S12)

Thus students’ focus frequently seemed to be on the computations that they encountered in the module and not on the concepts of linear algebra. One student said, she did not realise that definitions were important, she was revising from the exercises and examples instead, and realised [too late] that understanding definitions was a requirement for the in-class test.

One thing I kind of did wrong in my first semester was that I didn’t really understand, know . . . I did know there were a lot of definitions but I didn’t really know that they were relevant. Like in the questions you would, basically, in maths, they would ask you to write down something like a sentence. I thought it was one of the exercises so instead of me learning all the definitions, I learnt all the exercises, how to answer them and stuff like that, and didn’t really understand the concepts behind it, and this is the problem I’m having now, like, in my tutorial, he’ll ask me what a span is, where I can answer a question on what about spanning sets, things like that, but actually explain it to him I find it very difficult. I think that was a key flaw, one of the things that I didn’t do for linear algebra this semester. (Int2, S4)
Although students struggled with linear algebra most students in the focus group interviews indicated that they felt they were getting on top of it in Semester 2.

S9: I think my understanding of the subject got a bit better and I understand what a lot of the words mean a lot better now [in Semester 2], so many things like range, basis, then rank, rank-nullity, span, and there so many of them and try and cram them all in ... The way we’ve used them again and again this term and [in] my small group tutorial, we only ever do linear algebra and calculus ... We’ve gone over it so many times that I’d be pretty stupid if I didn’t get it by now ... and we went through the class test afterwards in my [small group] tutorial and I kinda thought that’s really silly, I should have done better. S8: Yeah, it did seem very easy afterwards and once we looked at the solutions for it. (Int4, S9, S8)

In addition, despite students focussing on computational aspects there were students who in the interviews expressed an ‘appreciation’ of the conceptual nature of linear algebra. Two students in particular commented on their own understanding in this way.

I don’t know, for me learning is like when you understand the concepts, like it’s not necessarily for me about answering the questions, it’s when someone says to you, you know, this is the range of this, the span of this, and like you know what it means, you can apply it to different situations and different like, for me that’s what learning is when it all clicks, and it’s all there for you to use and do whatever you like with really. (Int2, S3)

And this student expressed her understanding in this way:

I work through maths very much with the language. I think of it as a language, kind of thing, that’s how I work with it, so when he [the lecturer] said that to me I kind of clicked into that right away and thought ‘okay’, and before then, I suppose I didn’t really have a way of working with maths ... (omissions) ... and so from then on I kind of realised that that’s how I work with maths, cos before then I was just trying all different methods of working with maths and kind of bumming along and ... (Laughter) ... (Int3, S6)

When the researcher asked if it had changed the way that she saw maths, the student answered,
Not the way I saw it. (Pause) Hmm . . . Just the way I approach it, I suppose. I approach it differently to how I did before I came here, I think. (Int3, S6)

The last comment appeared to express a change of point of view which resonated with comments that the lecturer made in meetings in relation to changing how students viewed mathematics. However, this perspective was expressed by just one student.

A brief summary

The students in the focus group interviews represented a small sample of the total number of students who attended this module (less than 7%). They all expressed having had difficulties when first encountering linear algebra but felt that as they progressed through the second semester they were coping with the demands of the module.

They all cited their friends and their personal tutor (in the small group tutorials) as having been vital in helping them master the content of the linear algebra module.

The data and data analysis that I reported on in this chapter are limited as only a small number of students took part in the interviews. However, as a result of the analysis I was able to give an indication of the breadth of students’ experience of this module.
Chapter 8

Synthesis and Conclusions

In my thesis I have described an inductive approach to the teaching of linear algebra. I used activity theory as an analytical tool in analysis and as a theoretical tool to further characterise and conceptualise the teaching practice of mathematics at university. As a result I presented a theoretical model of the teaching process (in Chapter 5). In Chapter 6, I applied aspects of my model at the action and goal level of analysis to further analyse the teaching of three topics in linear algebra. And finally, I reported on my analysis of students’ responses (in Chapter 7).

In this, my final chapter, I draw together the various aspects of analyses from previous chapters. I compare and synthesise what I have learnt as a result of my analyses and reflections and report on possible implications of the inductive approach to the teaching of linear algebra that I have presented in my thesis. This includes a discussion of (possible) implications in respect of student learning and in respect of teaching.

I reflect on my theoretical and methodological choices in the research process. I discuss the use of theory alongside a traditional approach to grounded theory analysis as the basis of my methodology. In addition, I discuss the use of activity theory in conceptualising the teaching practice of mathematics at the university level of education.

8.1 On the teaching of linear algebra

In my study I presented research into a lecturer’s attempt to overcome some of the difficulties that students experience with linear algebra. The issues in teaching linear algebra are twofold. Linear algebra is highly conceptual and based on proofs. Proofs play a role in all of mathematics and, as the lecturer pointed out in a research meeting, in the calculus module that students also take in their first year. However, in linear
### Table 8.1: Aspects of Semester 1 and Semester 2 Teaching

<table>
<thead>
<tr>
<th></th>
<th>Semester 1</th>
<th>Semester 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>vector space</td>
<td>$\mathbb{R}^n$</td>
<td>general vector space $V$</td>
</tr>
<tr>
<td>style</td>
<td>informal reasoning</td>
<td>formal proving</td>
</tr>
<tr>
<td>approach</td>
<td>inductive</td>
<td>deductive</td>
</tr>
</tbody>
</table>

algebra there are so many definitions and theorems that proofs are used more often and are crucial in dealing with the conceptual basis of linear algebra. Students thus need to master the writing, reading and understanding of proofs as well as the many concepts in linear algebra and how they interlink. In addition, as Dorier et al. (2000b) stated, students also, or above all, need to understand linear algebra as a unifying theory for problems stemming from linearity. The question for the teacher of linear algebra is how to approach and plan teaching given the nature of linear algebra and students' difficulties with it. With this thesis I have presented one such approach, and the lecturer's thinking that has surrounded it.

#### 8.1.1 The lecturer's model for teaching linear algebra

The highly conceptual nature is a particular feature of linear algebra. This has been addressed in the literature leading some to say no matter how linear algebra is taught students will always struggle with it (Dorier and Sierpńska, 2001). In recognising students’ difficulties and students’ inexperience with proof, in particular, the lecturer devised a ‘gentle’ introduction to proofs through ‘reasoning about an example’ (the EAG approach). In addition, he addressed the highly conceptual nature of linear algebra by focussing solely on the vector space $\mathbb{R}^n$. Thus at this university, students met linear algebra in Semester 1 as a mathematical reasoning course using argumentation based on an inductive style in working with linear algebra concepts, and as a proof-based course in Semester 2 (with a different lecturer) using a formal approach based on a deductive style. That is, students met linear algebra concepts in Semester 1 in the vector space $\mathbb{R}^n$ and the same concepts again in Semester 2 in abstract vector spaces. I have summarised the different aspects of Semester 1 and Semester 2 teaching in Table 8.1.

Going beyond a vector space $\mathbb{R}^3$ ensured that students would not get stuck in spatial imagery and unable to abstract beyond (see for example, Sierpńska, 2005). However,
many of the examples that the lecturer presented were in fact confined to the vector spaces \( \mathbb{R}^2 \) and \( \mathbb{R}^3 \).

In devising this split the lecturer’s approach is different from both Uhlig’s and Dorier’s. Both Uhlig and Dorier were presenting linear algebra as a linearly designed module, with a linear progression through the concepts, Uhlig working inductively and Dorier working deductively. They both advocated going slowly and changing the context so that students could become familiar with the concepts when presented in different settings. The lecturer (working with his colleague), on the other hand, was repeating the linear progression and changing both the context and the way of working, from \( \mathbb{R}^n \) to abstract vector spaces as well as from informal reasoning and argumentation to formal ways of proving. This approach was the lecturer’s own invention and is novel as a model for teaching linear algebra. My research has shown that students found linear algebra difficult which corresponds to research findings in the literature. Based on my analysis of the student data, all students interviewed said that they felt confident about linear algebra at the end of the module. However, these students represented only a small proportion of the students taking this module so that any conclusions drawn must be regarded as tentative.

### 8.1.2 Some implications of the model

If we are to conclude that this approach was helpful to students, what were the features of the lecturer’s teaching? What did the lecturer do in his work as a teacher that may have made a difference? Since research only took place with the lecturer in the first semester, I highlight features of Semester 1 teaching only and not the teaching in Semester 2 (with a different lecturer). These features include:

- **Clear goals.**
  
  My research showed that the lecturer had clear views of what he wanted to achieve. Thus identifying his goals was crucial in understanding the actions that he performed. I related these terms to my activity theoretical perspectives which had implications for how I interpreted the approach.

- **Choice of examples.**
  
  Crucial in planning was the choice of the examples to use in a lecture so that argumentation was possible and would enable students to ‘see’ the concept or underlying structure.
- Planning for student engagement.
Students’ engagement was one aspect of the lecturer’s didactical planning that related to the use of examples in lectures. The lecturer regarded student engagement in lectures as central to the success of his goals.

- Meta-level explanations.
In face-to-face teaching the lecturer used explanations that I categorised as types of ‘verbalisations’ in Section 5.4.8. Here I take up Dorier’s notion of a meta-lever that I discussed in Chapter 2. The lecturer’s argumentation on examples is equivalent to (a) Uhlig’s sequence of WWHWT questions and (b) to Dorier’s meta-lever.

The first three features relate to planning and the last to implementation of teaching.

In the EAG approach choosing the example was important as was the argumentation on the example that followed. The latter relates to Dorier’s meta lever. The meta lever is hard to pinpoint but always oral and a type of explanation or action given when students are in reflective abstraction on a task (Dorier et al., 2000a). This implies that students must be engaged in a task for the meta-lever to be used effectively. The lecturer’s aim was to engage students in the lecture by presenting examples. He used verbal explanations and instructions before, during and after students were working on the problems presented with the example. My analysis of the observation data included an analysis of the meta level explanations, in which I explained five types of verbalisations (see Section 5.4.8). These were:

(a) Expositions: The lecturer recited the mathematics as it was printed in the course notes without altering the formulation of the definition or theorem, that is, there was no level of ‘explaining’ in the delivery.

(b) Comments about mathematics: The lecturer explained the mathematical idea, definition or theorem. He used his own words rather than repeat or read out the definition or theorem, say.

(c) Meta-comments about mathematics. The lecturer made a statement that was a level above that of (b). He gave a reason why a piece of mathematics was important or where or how it was useful.

(d) Comments about the learning of mathematics: The lecturer explained what students should focus on when engaging with a concept, definition or theorem.

(e) Meta-comments about the learning of mathematics: The lecturer explained why he asked students to focus on a particular way of engaging with a mathematical concept or theorem.
According to the description given by Dorier all but the expository mode could function as a meta lever. Thus for a lecturer of linear algebra, or mathematics in general, the modes of talking in a lecture are crucial for developing students’ mathematical understandings. As a result the designing of modes of talking may be a consideration for didactical planning. This represents one finding from my research. It arose from a study of linear algebra as a particularly difficult topic for students but can be seen to be applicable to the teaching of any area of mathematics at this level.

8.1.3 Cultural shifts and the devolution of learning

Students were entering university with a set of expectations that the lecturer said were not well aligned with the expectations of staff. In designing an inductive approach to linear algebra teaching, the lecturer introduced students to a new way of working with mathematics, (a) more creatively, and (b) as a subject to think about. According to the lecturer this represented a cultural shift in students’ expectation of what the learning of mathematics at university entailed. My analysis of the student data (though limited due to its small scale) showed that students were not used to working this way and were often reluctant to take it up.

At the same time the teaching approach itself also represents a cultural shift in teaching. The lecturer devised an approach in response to what he perceived as students’ inability to cope with the demands of a formal proof-based course. I have documented this approach with my thesis. Part of this approach included a devolution of learning to the student. Dorier et al. (2000a) refer to a ‘change of contract’. By creating opportunities for students to engage in the lecture with a mathematical task, the lecturer was making it overt that he expected students to work on problems themselves, that is, to be active in learning and in the acquisition of mathematical knowledge. Part of his didactical thinking included the role of communication with others in the acquisition of mathematical knowledge. The lecturer made it overt that talking with a neighbour about the mathematical problem they had to solve, was helpful in furthering one’s own knowledge. In the lecture, the lecturer himself interacted with some of the students while they were working on the task. His attempts were hampered by having a large group of students to teach (more than 200) and by the layout of a tiered lecture theatre which made interaction with all students impossible. However, his attempt represents a cultural shift in university teaching, from the traditional uni-directional talk at the front of the lecture hall to a more interactive and personal approach. My study was based on the teaching of linear algebra but the general approach is transferable to other areas of university mathematics.
8.2 Implications for the learning of linear algebra

In this section I draw on my analyses in Chapters 6 and 7. I try and balance aspects that were the result of my analysis of the lecturer’s didactical thinking and decision-making in respect of three specific topics in linear algebra with aspects that arose out of my analysis of the student data (in Chapter 7).

In various places throughout this thesis I have described the lecturer’s inductive approach based on informal reasoning around an example. Examples were crucial in trying to engage students in a lecture and with the mathematical problems presented. The lecturer’s intention was to engage students more conceptually with mathematics by presenting definitions and theorems only after students had worked on the example, and to put across a view of mathematics that could show how mathematicians themselves worked when presented with a problem to solve. As part of this approach I have discussed, in particular, the role of examples and the role of informal language. Explanations were given largely verbally in lectures. There were relatively few explanations in the course notes that the lecturer provided for students to bring to lectures, to write in and for revision. In the lectures, the lecturer projected the course notes and presented the material verbatim as well as including verbal explanations and working through examples.

In general, the lecturer provided explanations verbally only (in lectures). The course notes contained few explanations. This raises the question as to whether the lecturer’s style of teaching based on providing all explanations verbally only (without also writing them down in the course notes, say) privileged some students who like student S6 were ‘naturally’ inclined to work with mathematics the way that the lecturer did.

8.2.1 Accounting for students’ views

The analysis of the student data that I presented in Chapter 7 was based on a small sample of students and hence is limited to draw conclusions from. However, it gives an indication of the breadth of experience among students. All students that I interviewed found linear algebra difficult to start with which resonated with findings published in the research literature (Dorier and Sierpinska, 2001, for example). All said that they liked the notes with gaps. Some indicated that having to fill the gaps in a lecture kept them interested, and having the definitions and theorems written down in the course notes meant that they could concentrate and ‘listen’ more. However, students also reported that they did not always work on the solutions of the examples in the lecture. They gave reasons for participating, and reasons for why not. My analysis of the functioning of examples showed that some examples were more accessible than others. Thus in
considering students’ responses and balancing with my analysis, the question arises as to whether some examples could be formulated to make them more accessible and increase student participation.

All students reported that by the end of the second semester they felt they were getting on top of things. This included the student who said that she “did not understand anything” in Semester 1 and relied on Gaussian elimination “to gain at least some marks” in the in-class test. When taking part in the focus group interview in Week 11 of Semester 2 she talked fairly conceptually about linear algebra and recognised Gaussian elimination as a tool to gain information about the matrix (student S12, see Chapter 7). While this student appeared ‘slow’ in recognising what was at the heart of the linear algebra module, another student said that she ‘clicked straightaway’ into the lecturer’s instruction to focus on the language that she had to learn (student S6, see Chapter 7).

8.2.2 Mathematical considerations

The lecturer had re-structured the first year linear algebra module so that all concepts were introduced in Semester 1 in the context of the vector space $\mathbb{R}^n$. Students then met all concepts again in Semester 2 in the context of abstract vector space theory.

In Chapter 6, I discussed some instances in the course notes (and read out in the lectures) where the lecturer used formulations of mathematical statements that were expressed in language that was closer to everyday language than, as is usual in mathematical texts, more abstract mathematical language. Being able to use more concrete expressions was a direct result of the lecturer working in the vector space $\mathbb{R}^n$.

In addition, apart from a brief mention of the term vector space verbally in a lecture, the lecturer never stated the axiomatic definition of a vector space. He provided the definition of a subspace and formulated problems for students that involved checking whether a given set of vectors formed a subspace of $\mathbb{R}^n$ or not.

At this point in time I do not have any information that could shed light on how students viewed the vector space $\mathbb{R}^n$ or a vector space in general. Nor do I know what conceptual understanding students (may) have of a subspace. A subspace is a subset of a vector space satisfying certain properties. Since a subspace is a subset of a vector space of which students have no conception, it raises the question of how they viewed the concept of a subspace.

Expressing definitions and theorems in the context of $\mathbb{R}^n$ and providing explanations using everyday language led to minor omissions and/or inaccuracies. For example, at one point in the course notes the fact that the eigenvector must always be non-zero
was omitted. Again, I have no information regarding students’ conception, and whether minor omissions could make a difference to their understandings since the aim of introducing the linear algebra concepts in Semester 1 was to build students’ intuitive and informal understandings of the abstract ideas that were to come in Semester 2.

However, what I have raised in this section are possible implications of the inductive approach for students’ mathematical understanding. The lecturer’s aim was for students to form an intuitive understanding, knowing that students would meet all concepts again in the second semester. In research meetings he stated that he saw the first semester as ‘paving the way’ for the more formal treatment that was to follow. From the responses given by students in the focus group interviews it appeared that all were able to form conceptual understandings towards the end of Semester 2. However, my analysis of the student data is limited due to only a small number of students being represented. In addition, students seemed to progress at varying rates throughout Semesters 1 and 2.

In having a one year long module of linear algebra and presenting students with two different approaches in two contexts, one concrete and one abstract, the question arises to what extent this helped students to learn linear algebra and overcome some of the difficulties that the lecturer had aimed at tackling. I have no information of students’ progress through years 2 and 3 of their study. A follow up study that could capture the student experience more fully would be helpful. I have the agreement of the lecturer in my study to re-enter his lectures and conduct research with the students. The lecturer is teaching this module again in the next academic year as he has done every year since I conducted the research with him (making some changes in content and focus of the module but retaining the inductive approach).

In any follow up study one focus would be to explore the underlying reasons for students’ participation and non-participation in lectures, and to understand how they engaged at the interface of the lecturer’s presentation of mathematics and the mathematics of linear algebra itself.
8.3 The theoretical and methodological contribution of my research to the field of mathematics education

8.3.1 Conceptualising a university mathematics teaching practice from an activity theory perspective

Part of research is the development or application of theory. I have used activity theory in a new area, namely in research in higher education, in the teaching of university mathematics. In particular, I have used Leontiev’s development of activity theory, that is Leontiev’s structural elements of activity theory, to conceptualise and theorise the teaching practice of linear algebra.

As reported in Chapter 5 I related the activity-motive, the action-goal and the operation-condition levels of ‘activity’ to the educational setting of university linear algebra teaching. As a result, in my thesis, I offer a theoretical model of the teaching process in relation to linear algebra (Figure 5.1, page 61). The model arose out of the data and was at the same time embedded within Leontiev’s activity theory framework. The model relates the lecturer’s intentions and strategies in his teaching of linear algebra to the actions and goals of activity theory. The model arose out of data that was specific to the teaching of linear algebra. At the same time, the model incorporates aspects that are general in teaching, for example clear goals and the use of meta-level explanations (as detailed in Section 8.1.2), aspects that are generalisable to contexts outside of linear algebra, that is, to other areas of university mathematics teaching.

In Chapter 6 I presented a detailed analysis of the functioning of the tools used in the teaching of subspaces, linear independence and eigenvectors/values. In doing so I took my analysis further and applied my model to the data. That is, I used my model to overlay the data and categorise the tools (used in actions) at the level of designing and planning and at the level of implementation and practical use. As a result I concluded that the tools functioned differently at the design and planning stage and at the point of delivery, in the lectures.

In Chapter 5 the model arose out of the data and was based mainly on an analysis of the research meetings with the lecturer. In Chapter 6 I re-applied my model to the data that now included lecture observations as well as linear algebra textbooks and the lecturer’s course notes. Hence an individual perspective (statements by the lecturer in research meetings) gave rise to a more general model of the teaching process while my analysis of the social aspects of teaching (face-to-face teaching and representations in textbooks) gave rise to the categorisation of the tools that was specific to the teaching of three linear algebra concepts.
Using Leontiev’s activity theory allowed me to model a complex human behaviour, namely teaching with a relatively ‘simple’ framework. However, despite or maybe because of its simplicity, I was able to capture a wide spectrum of the teaching process.

I have used a ‘simple’ but powerful theory, and in doing so I have gone back to the foundations of activity theory. I have not used the more detailed developments by third generation activity theorists such as Engeström. The cultural, historical and social factors that contribute to and shape human activity enter, for example, at the activity-motive level of analysis and give expression to the lecturer’s motive, in this case enculturation into mathematical practice. Furthermore I analysed the actions and goals in detail. Here, in my analysis of the tools that the lecturer used in his teaching, I made reference to the functioning of tools. The lecturer created tools (course notes, examples, for example) and a learning situation in the lecture hall (verbalising his intentions, as an example of the use of a psychological tool) with the aim for his students to engage in activity. The lecturer’s motive, his actions and the creation of tools are a representation of the lecturer’s personal cultural heritage and of the culture of university mathematics teaching. In and through engagement in activity in the lecture hall students interpret and negotiate the use of tools and the lecturer’s actions. I regard the interactions in the lecture hall as an area that would benefit from further research. In developing my analysis and theoretical position in my thesis, I came to view teaching as a process that is as not static but liable to change over (relatively short or longer periods of) time. Activity theory can be used to categorise a teaching practice as well as explain changes in that practice.

8.3.2 The development of methodology in an interpretive paradigm

As Speer et al. (2010) noted we urgently need more studies into mathematical teaching practices at the university level of education. “Such research will require researchers to move into the classrooms and offices of collegiate teachers in order to collect data that can support analyses of practice”. (Speer et al., 2010, p. 13). Speer et al. defined teaching practice as all that a teacher does and thinks about, both inside and outside the classroom. With this thesis I have made a contribution in this area.

I have collected data in respect of the lecturer’s didactical thinking and planning and his face-to-face teaching of students. That is I have collected data that made possible an analysis at both the individual level (didactical thinking and planning) and the social level (face-to-face teaching of students) of teaching, essential in order to investigate a teaching practice as defined by Speer et al.
As detailed in Chapter 3, I based my research into a mathematics teaching practice within an activity theoretical framework and grounded theory methodology. In this framework theory and methodology were not separate entities. A grounded theory approach to data collection and analysis was initially supported by activity theory as an analytical tool and then extended by it. That is, initially in Chapter 5, an analysis using a combined approach of grounded theory and activity theory led to a conceptualisation of the teaching practice. Extending the analysis and using activity theory as a theoretical tool in Chapter 6 led to a characterisation of the tools. Hence my methodology enabled me to research both the theory and the praxis (practical side) of teaching. It is in this context and sense that I offer the wider research community a methodology for researching a teaching practice and teaching, in general.

Within my methodological framework of a grounded theory approach, the lecturer in my study gained access to the data collected and to ongoing analysis. This informed the lecturer of the research process within the field of mathematics education and within a qualitative study in particular. He gained knowledge of our research aims and the methods and methodologies used in the field of mathematics education. As mentioned he became more involved as the study progressed. Building a relationship that enabled a free exchange of ideas was crucial for the success of the study, and it is in this sense that I offer a methodology for conducting research collaborations between the disciplines of mathematics and mathematics education.

\section{Concluding remarks}

With this thesis I have contributed to the didactics of mathematics in four distinct areas: I have provided a characterisation of the mathematics teaching practice at the university level of education; I have applied activity theory in the higher educational setting in respect of teaching; I have shown that the development of my methodological basis is suitable for researching and analysing a complex human behaviour such as the teaching of university mathematics; and I have contributed to the teaching of university mathematics by documenting an inductive approach to the teaching of linear algebra, in particular to the teaching of three concepts: subspaces, linear independence and eigenvectors/values.

To the mathematics education community I offer a theorising of teaching and learning from a second generation activity theory perspective. In particular, I relate the lecturer’s intentions and strategies to Leontiev’s structural elements of actions and goals. I further present a detailed analysis and categorisation of the use of tools in mediating activity.
To the mathematical community I offer a detailed description and analysis of the teaching of subspaces, linear independence and eigenvectors/values. The inductive approach adopted by the lecturer presents an alternative to the ‘DTP’ style for introducing linear algebra concepts. This defines a novel way to teach linear algebra, not documented in the research literature. To make this analysis more broadly accessible to those who teach linear algebra I aim to publish in relevant mathematical journals.

For the mathematics department at the university where the research took place I offer insights into an alternative approach devised by a member of staff, a mathematician in their department. This approach took account of students’ mathematical competence as viewed by the lecturer. Interim results of the study were communicated to members of the department in seminars but also informally in discussions that the lecturer had with colleagues. This raised questions and comments that the lecturer at times communicated in research meetings to us, the researchers. As part of my validation exercise I contacted three mathematicians to check my thesis for mathematical accuracy. Again the involvement of other members of staff raised the profile of the study and encouraged discussion in relation to the design and outcome of the study and the teaching approach that I documented.

As mentioned in Section 8.3.2, through engagement in the study the lecturer gained knowledge of qualitative research methodologies and learning theories in mathematics education. He read all of my ongoing analysis, participated in presentations in seminars and co-authored papers. He reported on his participation in the study as a positive experience. He said that he benefitted from student feedback that we, the researchers provided, often informally after lectures, but also through the analysis of the questionnaires. As a result of participating in this research the lecturer (a) became aware of the different commenting levels that he used in lectures, and (b) made several, gradual changes to the material content of the course notes.

In research meetings he often explained mathematical ideas in general and linear algebra concepts in particular to us, the researchers. I benefitted mathematically from these explanations which added to my knowledge of linear algebra. Close interactions with the lecturer allowed me to view the teaching and learning of a particular area of advanced mathematics, linear algebra, from a mathematician’s viewpoint. Because of the frankness in communication I gained a close knowledge of the lecturer’s beliefs in relation to the teaching and learning of mathematics as well as his beliefs in relation to the nature of mathematics itself. This I found fascinating and, although only touched on briefly in this thesis will be the content of a forthcoming paper.

This thesis is the culmination of a long process that included my developing expertise as a researcher in mathematics education. During the data collection period (in research
meetings with the lecturer and attending his lectures with students) I deepened my own understanding and knowledge of linear algebra, in some cases acquiring knowledge that was entirely new to me. In research meetings with the lecturer, under the leadership and guidance of my supervisor, I developed an understanding of qualitative research processes. This provided me with a firm grounding in the demands and rigour of working in an interpretive paradigm.

In the long term I wish to continue studying the teaching and learning of linear algebra from an activity theory perspective. I aim to collect evidence that can give insight into students’ responses to the teaching of the three concepts that I analysed in Chapter 6: subspace, linear independence and eigenvectors/values. I have the lecturer’s agreement it re-enter his ‘classroom’ and observe and interview students. As mentioned previously the lecturer has taught the module several times since the study took place, and although he has made changes to the material content of the module, he has retained the inductive approach. Thus in a further study of students’ learning, insights into how students approach, think about and work with the three linear algebra concepts could be set against my current analysis into the teaching of these three concepts. This, I think, is a unique opportunity for a longer term study of the teaching and learning of mathematics at the university level of education.
Appendix A

The Focus Group Interviews
Participant Information Sheet

Focus Group Interview, Monday, 23rd March 2009

Please read the following section and then sign below.

1) My research is into the teaching of mathematics at university, with a focus on first year first semester linear algebra.

2) The purpose of this interview is to gather data on student perception and experience of semester 1 of linear algebra. I gave out 2 questionnaires and my questions today are very much a follow-up of these questionnaires.

3) For practical reasons I would like to audio-tape our conversation. During the interview we will use real names if we wish to address one another. Later I will transcribe the full interview and use pseudonyms throughout. This will ensure anonymity. The tape and the transcript will be kept until the completion of my PhD (and no longer than 6 years).

4) Your identity will not be revealed to anyone outside the research, that means only Barbara Jaworski and I know who you are. In dealing with the data that you provide today I will follow the ethical guidelines as set out in the documentation on how to conduct research at Loughborough University.

5) The research that I am engaged in is about increasing understanding of the teaching and learning of mathematics at university. Ultimately the aim is to improve students' mathematical experience whilst at university.

Interviewer: Stephanie Thomas

Figure A.1: Participant Information Sheet
INFORMED CONSENT FORM
(to be completed after Participant Information Sheet has been read)

The purpose and details of this study have been explained to me. I understand that this study is designed to further scientific knowledge and that all procedures have been approved by the Loughborough University Ethical Advisory Committee.

I have read and understood the information sheet and this consent form.

I have had an opportunity to ask questions about my participation.

I understand that I am under no obligation to take part in the study.

I understand that I have the right to withdraw from this study at any stage for any reason, and that I will not be required to explain my reasons for withdrawing.

I understand that all the information I provide will be treated in strict confidence.

I agree to participate in this study.

Your name

Your signature

Signature of investigator

Date

Figure A.2: Consent Form
Some questions you may like to reflect on:

There were four chapters in semester 1:
Linear equation systems, matrices, subspaces of $\mathbb{R}^n$, eigenvalues and eigenvectors (2)

1) What did you find easy/difficult at the time? What do you think now was easy/difficult in semester 1? Why do you think that?
What if I changed the words ‘easy’ and ‘difficult’ in the questions above to ‘interesting’, ‘confusing’, ‘like’, ‘dislike’, …… ?

2) Students were asked to print off the lecture notes (which had gaps) from the Learn server and bring these to the lecture. Did you do that, and what is your view on these ‘gappy’ notes?

3) If you recall, lectures in semester 1 often had breaks when students were asked to work on an exercise by themselves or with a neighbour. What did you do in these time intervals? What does working on an exercise look like? What does ‘engaging with an exercise’ or ‘engaging with the material’ look like?

4) What, or how much, do you think you have ‘absorbed’ from semester 1? Is it helping you with the work in semester 2? How do you learn (best)? What do you do?
Appendix B

Student Questionnaires
### Initial Responses To The First Year Linear Algebra Module At LU

This questionnaire is designed to get a sense of how you are responding to the Linear Algebra module. This is related both to your previous experience before arriving at LU, AND to your current experience in the module here. We are genuinely interested in your response to the module and want to know how you are experiencing it. Your responses here are anonymous and can in no way affect assessment of your achievement in the module. We appreciate your honest and frank replies. Thank you.  

**Barbara Jaworski, Stephanie Thomas and Thomas Bartsch**

<table>
<thead>
<tr>
<th>Question</th>
<th>Answer Options</th>
<th>Yes</th>
<th>No</th>
</tr>
</thead>
<tbody>
<tr>
<td>1a.) Had you <strong>seen</strong> Gaussian elimination before you came to Loughborough University? (circle one)</td>
<td>Yes</td>
<td>No</td>
<td></td>
</tr>
<tr>
<td>1b.) Where did you see this (e.g. school sixth form, college, other university)?</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1c.) Were you able to carry out Gaussian elimination before coming to LU? (circle one)</td>
<td>Yes</td>
<td>No</td>
<td></td>
</tr>
<tr>
<td>1d.) Please rate your performance/success in carrying out Gaussian elimination (before coming to LU) (e.g. ‘I could do it well’, ‘I could get it right most of the time’, ‘I always made mistakes’, ‘I was okay with it’). (Please use your own words)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2a.) Had you <strong>seen</strong> matrix multiplication before you came to Loughborough University? (circle one)</td>
<td>Yes</td>
<td>No</td>
<td></td>
</tr>
<tr>
<td>2b.) Where did you see this (e.g. school sixth form, college, other university)?</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2c.) Were you able to carry out matrix multiplication before coming to LU? (circle one)</td>
<td>Yes</td>
<td>No</td>
<td></td>
</tr>
<tr>
<td>2d.) Please rate your performance/success in carrying out Gaussian elimination (before coming to LU) (e.g. ‘I could do it well’, ‘I could get it right most of the time’, ‘I always made mistakes’, ‘I was okay with it’). (Please use your own words)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3a.) Had you <strong>seen</strong> linear equation systems with infinitely many solutions before you came to LU? (circle one)</td>
<td>Yes</td>
<td>No</td>
<td></td>
</tr>
<tr>
<td>3b.) Where did you see this (e.g. school sixth form, college, other university)?</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3c.) Were you able to solve such systems before coming to LU? (circle one)</td>
<td>Yes</td>
<td>No</td>
<td></td>
</tr>
<tr>
<td>3d.) Please rate your performance/success in solving such systems (before coming to LU) (e.g. ‘I could do it well’, ‘I got it right most of the time’, ‘I always made mistakes’, ‘I was okay with it’). (Please use your own words)</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

---

**Figure B.1: Student Questionnaire 1, page 1**
You have now covered the first four weeks / the first two chapters in this module. Please answer the following questions with regard to module content.

Please indicate how strongly you agree / disagree with a statement by circling
SA (strongly agree)     A (agree)     N (neutral/no opinion)     D (disagree)     SD (strongly disagree)

<p>| | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>4a.) Chapter 1 was easy.</td>
<td>SA</td>
<td>A</td>
<td>N</td>
<td>D</td>
</tr>
<tr>
<td>4b.) Chapter 2 was easy.</td>
<td>SA</td>
<td>A</td>
<td>N</td>
<td>D</td>
</tr>
<tr>
<td>5a.) I found Gaussian elimination difficult to grasp at first.</td>
<td>SA</td>
<td>A</td>
<td>N</td>
<td>D</td>
</tr>
<tr>
<td>5b.) I am confident that I have mastered Gaussian elimination now.</td>
<td>SA</td>
<td>A</td>
<td>N</td>
<td>D</td>
</tr>
<tr>
<td>6a.) I found matrix multiplication difficult to grasp at first.</td>
<td>SA</td>
<td>A</td>
<td>N</td>
<td>D</td>
</tr>
<tr>
<td>6b.) I am confident that I have mastered matrix multiplication now.</td>
<td>SA</td>
<td>A</td>
<td>N</td>
<td>D</td>
</tr>
</tbody>
</table>

Please answer the following questions with regard to module delivery.

<p>| | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>7.) I can understand the lecturer’s voice clearly.</td>
<td>SA</td>
<td>A</td>
<td>N</td>
<td>D</td>
</tr>
<tr>
<td>8.) I can understand the lecturer’s explanations clearly.</td>
<td>SA</td>
<td>A</td>
<td>N</td>
<td>D</td>
</tr>
<tr>
<td>9.) The pace of the lecture is (please circle)</td>
<td>too fast</td>
<td>a little fast</td>
<td>just right</td>
<td>quite slow</td>
</tr>
</tbody>
</table>

In relation to the printed lecture notes which are available on LEARN, please answer:

<p>| | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>10a.) It is helpful to have the lecture notes in advance of lectures.</td>
<td>SA</td>
<td>A</td>
<td>N</td>
<td>D</td>
</tr>
<tr>
<td>10b.) Please explain your answer to 10a). Are there any advantages/disadvantages you would like to highlight?</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Advantages:</td>
<td>Disadvantages:</td>
<td>Other:</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<p>| | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>11a.) I like the way the lecture notes are written up with gaps.</td>
<td>SA</td>
<td>A</td>
<td>N</td>
<td>D</td>
</tr>
<tr>
<td>11b.) Please explain your answer to 11a). Are there any advantages/disadvantages you would like to highlight?</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Advantages:</td>
<td>Disadvantages:</td>
<td>Other:</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

12.) Please write any other comments you wish to make in relation to the first four weeks of Linear Algebra lectures and tutorials:

---

**Figure B.2: Student Questionnaire 1, page 2**
Responses (II) To The First Year Linear Algebra Module At LU

We should like to follow up on your earlier responses to our questionnaire about your experiences in the Linear Algebra module. Again there are questions relating to your previous experience as well as to your current experience with this module. Your responses here are anonymous and can in no way affect your assessment of the module.

Thank you. 

Barbara Jaworski, Stephanie Thomas and Thomas Bartsch

Please answer the following question about your mathematical knowledge before you came here to Loughborough University:

1) Had you seen the following topics before you came to Loughborough University?

<table>
<thead>
<tr>
<th>Topic</th>
<th>Yes</th>
<th>No</th>
</tr>
</thead>
<tbody>
<tr>
<td>subspaces and vector spaces</td>
<td></td>
<td></td>
</tr>
<tr>
<td>linear independence</td>
<td></td>
<td></td>
</tr>
<tr>
<td>spanning sets and bases</td>
<td></td>
<td></td>
</tr>
<tr>
<td>nullity and rank</td>
<td></td>
<td></td>
</tr>
<tr>
<td>eigenvalues and eigenvectors</td>
<td></td>
<td></td>
</tr>
<tr>
<td>determinants</td>
<td></td>
<td></td>
</tr>
<tr>
<td>diagonalization</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

You have now completed the first semester/ first half of this module. Please answer the following questions:

Please indicate how strongly you agree / disagree with a statement by circling

SA (strongly agree) A (agree) N (neutral/no opinion) D (disagree) SD (strongly disagree)

2) Chapter 3 was easy.

3) Chapter 4 was easy.

4a.) I found the work on spanning sets and bases difficult at first.

4b.) I am confident that I have mastered this work now.

5a.) I found the work on eigenvalues and eigenvectors difficult at first.

5b.) I am confident that I have mastered this work now.

6.) The pace of the lecture in the second half of the semester was

(circle one) too fast a little fast just right quite slow too slow

7a.) My attendance at lectures (Wednesdays and Thursdays) was approximately

(circle one) 0 - 20% 21 - 40% 41 - 60% 61 - 80% 81 - 100%

7b.) My attendance at tutorials (Fridays) was approximately

(circle one) 0 - 20% 21 - 40% 41 - 60% 61 - 80% 81 - 100%

---

**Figure B.3:** Student Questionnaire 2, page 1
The majority of students indicated in the last survey that they liked having lecture notes in advance. Please answer the following question:

8a.) I have printed out the lecture notes before coming to lectures. (circle one)

<table>
<thead>
<tr>
<th>Frequency</th>
<th>Percentage</th>
</tr>
</thead>
<tbody>
<tr>
<td>never (0%)</td>
<td></td>
</tr>
<tr>
<td>hardly ever (10%)</td>
<td></td>
</tr>
<tr>
<td>occasionally (25%)</td>
<td></td>
</tr>
<tr>
<td>sometimes (40%)</td>
<td></td>
</tr>
<tr>
<td>quite often (60%)</td>
<td></td>
</tr>
<tr>
<td>most of the time (75%)</td>
<td></td>
</tr>
<tr>
<td>nearly every time (90%)</td>
<td></td>
</tr>
<tr>
<td>every time (100%)</td>
<td></td>
</tr>
</tbody>
</table>

8b.) I have read through the lecture notes before coming to lectures. (circle one)

<table>
<thead>
<tr>
<th>Frequency</th>
<th>Percentage</th>
</tr>
</thead>
<tbody>
<tr>
<td>never (0%)</td>
<td></td>
</tr>
<tr>
<td>hardly ever (10%)</td>
<td></td>
</tr>
<tr>
<td>occasionally (25%)</td>
<td></td>
</tr>
<tr>
<td>sometimes (40%)</td>
<td></td>
</tr>
<tr>
<td>quite often (60%)</td>
<td></td>
</tr>
<tr>
<td>most of the time (75%)</td>
<td></td>
</tr>
<tr>
<td>nearly every time (90%)</td>
<td></td>
</tr>
<tr>
<td>every time (100%)</td>
<td></td>
</tr>
</tbody>
</table>

8c.) I have worked through the lecture notes quite thoroughly before coming to lectures. (circle one)

<table>
<thead>
<tr>
<th>Frequency</th>
<th>Percentage</th>
</tr>
</thead>
<tbody>
<tr>
<td>never (0%)</td>
<td></td>
</tr>
<tr>
<td>hardly ever (10%)</td>
<td></td>
</tr>
<tr>
<td>occasionally (25%)</td>
<td></td>
</tr>
<tr>
<td>sometimes (40%)</td>
<td></td>
</tr>
<tr>
<td>quite often (60%)</td>
<td></td>
</tr>
<tr>
<td>most of the time (75%)</td>
<td></td>
</tr>
<tr>
<td>nearly every time (90%)</td>
<td></td>
</tr>
<tr>
<td>every time (100%)</td>
<td></td>
</tr>
</tbody>
</table>

In the last survey a number of students indicated that having lecture notes with gaps made them listen more and participate more in lecture.

9a.) Having lecture notes with gaps made me listen/participate more in lecture.  SA   A   N   D   SD

9b.) Having lecture notes with gaps has helped me learn linear algebra.  SA   A   N   D   SD

Dr. Bartsch gives lecture and tutorial time for students to work on their own (or with a neighbour) on an example or exercise. Please answer the following:

10a.) I make a good attempt at the exercises/examples that Dr. Bartsch writes up/projects during lectures. (circle one)

<table>
<thead>
<tr>
<th>Frequency</th>
<th>Percentage</th>
</tr>
</thead>
<tbody>
<tr>
<td>never (0%)</td>
<td></td>
</tr>
<tr>
<td>hardly ever (10%)</td>
<td></td>
</tr>
<tr>
<td>occasionally (25%)</td>
<td></td>
</tr>
<tr>
<td>sometimes (40%)</td>
<td></td>
</tr>
<tr>
<td>quite often (60%)</td>
<td></td>
</tr>
<tr>
<td>most of the time (75%)</td>
<td></td>
</tr>
<tr>
<td>nearly always (90%)</td>
<td></td>
</tr>
<tr>
<td>always (100%)</td>
<td></td>
</tr>
</tbody>
</table>

10b.) Please give a reason for your answer in 10a):

11.) Please write any other comments you wish to make in relation to the Linear Algebra lectures and tutorials:

Figure B.4: Student Questionnaire 2, page 2
Appendix C

Extracts from Textbooks
EXERCISES

1. If \( F \) is a field, verify that \( F^n \) (as defined in Example 1) is a vector space over the field \( F \).

2. If \( V \) is a vector space over the field \( F \), verify that
   \[
   (\alpha_1 + \alpha_2) + (\alpha_3 + \alpha_4) = [\alpha_1 + (\alpha_3 + \alpha_1)] + \alpha_4
   \]
   for all vectors \( \alpha_1, \alpha_2, \alpha_3, \alpha_4 \) in \( V \).

3. If \( C \) is the field of complex numbers, which vectors in \( C^n \) are linear combinations of \( (1, 0, -1), (0, 1, 1), \) and \( (1, 1, 1) \)?

4. Let \( V \) be the set of all pairs \( (x, y) \) of real numbers, and let \( F \) be the field of real numbers. Define
   \[
   (x, y) + (x_1, y_1) = (x + x_1, y + y_1)
   \]
   \[
   c(x, y) = (cx, y).
   \]
   Is \( V \), with these operations, a vector space over the field of real numbers?

5. Let \( V \) be the set of all pairs \( (x, y) \) of real numbers and let \( F \) be the field of real numbers. Define
   \[
   (x, y) + (x_1, y_1) = (3y + 3y_1, -x - x_1)
   \]
   \[
   c(x, y) = (3cy, -cx).
   \]
   Verify that \( V \), with these operations, is not a vector space over the field of real numbers.

2.2. Subspaces

In this section we shall introduce some of the basic concepts in the study of vector spaces.

**Definition.** Let \( V \) be a vector space over the field \( F \). A **subspace** of \( V \) is a subset \( W \) of \( V \) which is itself a vector space over \( F \) with the operations of vector addition and scalar multiplication on \( V \).

A direct check of the axioms for a vector space shows that the subset \( W \) of \( V \) is a subspace if for each \( \alpha \) and \( \beta \) in \( W \) the vector \( \alpha + \beta \) is again in \( W \); the 0 vector is in \( W \); for each \( \alpha \) in \( W \) the vector \( -\alpha \) is in \( W \); for each \( \alpha \) in \( W \) and each scalar \( c \) the vector \( c\alpha \) is in \( W \). The commutativity and associativity of vector addition, and the properties (4)(a), (b), (c), and (d) of scalar multiplication do not need to be checked, since these are properties of the operations on \( V \). One can simplify things still further.

**Theorem 1.** A non-empty subset \( W \) of \( V \) is a subspace of \( V \) if and only if for each pair of vectors \( \alpha, \beta \) in \( W \) and each scalar \( c \) in \( F \) the vector \( c\alpha + \beta \) is again in \( W \).
Sec. 2.2  VECTOR SPACES

Proof. Suppose that $W$ is a non-empty subset of $V$ such that $c\alpha + \beta$ belongs to $W$ for all vectors $\alpha, \beta$ in $W$ and all scalars $c$ in $F$. Since $W$ is non-empty, there is a vector $\rho$ in $W$, and hence $(-1)\rho + \rho = 0$ is in $W$. Then if $\alpha$ is any vector in $W$ and $c$ any scalar, the vector $c\alpha = \alpha + 0$ is in $W$. In particular, $(-1)\alpha = -\alpha$ is in $W$. Finally, if $\alpha$ and $\beta$ are in $W$, then $\alpha + \beta = 1\alpha + \beta$ is in $W$. Thus $W$ is a subspace of $V$.

Conversely, if $W$ is a subspace of $V$, $\alpha$ and $\beta$ are in $W$, and $c$ is a scalar, certainly $c\alpha + \beta$ is in $W$.

Example 6

(a) If $V$ is any vector space, $V$ is a subspace of $V$; the subset consisting of the zero vector alone is a subspace of $V$, called the zero subspace of $V$.

(b) In $F^n$, the set of $n$-tuples $(x_1, \ldots, x_n)$ with $x_1 = 0$ is a subspace; however, the set of $n$-tuples with $x_1 = 1 + x_2$ is not a subspace ($n \geq 2$).

(c) The space of polynomial functions over the field $F$ is a subspace of the space of all functions from $F$ into $F$.

(d) An $n \times n$ (square) matrix $A$ over the field $F$ is symmetric if $A_{ij} = A_{ji}$ for each $i$ and $j$. The symmetric matrices form a subspace of the space of all $n \times n$ matrices over $F$.

(e) An $n \times n$ (square) matrix $A$ over the field $C$ of complex numbers is Hermitian (or self-adjoint) if

$$A_{jk} = \overline{A_{kj}}$$

for each $j, k$, the bar denoting complex conjugation. A $2 \times 2$ matrix is Hermitian if and only if it has the form

$$\begin{bmatrix}
  z & x + iy \\
  x - iy & w
\end{bmatrix}$$

where $x, y, z,$ and $w$ are real numbers. The set of all Hermitian matrices is not a subspace of the space of all $n \times n$ matrices over $C$. For if $A$ is Hermitian, its diagonal entries $A_{11}, A_{22}, \ldots$, are all real numbers, but the diagonal entries of $iA$ are in general not real. On the other hand, it is easily verified that the set of all complex Hermitian matrices is a vector space over the field $R$ of real numbers (with the usual operations).

Example 7. The solution space of a system of homogeneous linear equations. Let $A$ be an $m \times n$ matrix over $F$. Then the set of all $n \times 1$ (column) matrices $X$ over $F$ such that $AX = 0$ is a subspace of the space of all $n \times 1$ matrices over $F$. To prove this we must show that $A(cX + Y) = 0$ when $AX = 0, AY = 0$, and $c$ is an arbitrary scalar in $F$. This follows immediately from the following general fact.
1.12. **Subspaces.** Let $E$ be a vector space over the field $\Gamma$. A non-empty subset, $E_1$, of $E$ is called a *subspace* if for each $x, y \in E_1$ and every scalar $\lambda \in \Gamma$

$$x + y \in E_1 \quad (1.14)$$

and

$$\lambda x \in E_1. \quad (1.15)$$

Equivalently, a subspace is a subset of $E$ such that

$$\lambda x + \mu y \in E_1$$

whenever $x, y \in E_1$. In particular, the whole space $E$ and the subset $(0)$ consisting of the zero vector only are subspaces. Every subspace $E_1 \subset E$ contains the zero vector. In fact, if $x_1 \in E_1$ is an arbitrary vector we have that $0 = x_1 - x_1 \in E_1$. A subspace $E_1$ of $E$ inherits the structure of a vector space from $E$.

Now consider the injective map $i: E_1 \to E$ defined by

$$ix = x, \quad x \in E_1.$$

In view of the definition of the linear operations in $E$, $i$ is a linear mapping, called the *canonical injection* of $E_1$ into $E$. Since $i$ is injective it follows from (sec. 1.11) that a family of vectors in $E_1$ is linearly independent (dependent) if and only if it is linearly independent (dependent) in $E$.

Next let $S$ be any non-empty subset of $E$ and denote by $E_s$ the set of linear combinations of vectors in $S$. Then any linear combination of vectors in $E_s$ is a linear combination of vectors in $S$ (cf. sec. 1.3) and hence it belongs to $E_s$. Thus $E_s$ is a subspace of $E$, called the *subspace generated by $S$*, or the *linear closure of $S$*.

Clearly, $S$ is a system of generators for $E_s$. In particular, if the set $S$ is linearly independent, then $S$ is a basis of $E_s$. We notice that $E_s = S$ if and only if $S$ is a subspace itself.

1.13. **Intersections and sums.** Let $E_1$ and $E_2$ be subspaces of $E$ and consider the intersection $E_1 \cap E_2$ of the sets $E_1$ and $E_2$. Then $E_1 \cap E_2$ is again a subspace of $E$. In fact, since $0 \in E_1$ and $0 \in E_2$ we have $0 \in E_1 \cap E_2$ and so $E_1 \cap E_2$ is not empty. Moreover, it is clear that the set $E_1 \cap E_2$ satisfies again conditions (1.14) and (1.15) and so it is a subspace of $E$. $E_1 \cap E_2$ is called the *intersection* of the subspaces $E_1$ and $E_2$. Clearly, $E_1 \cap E_2$ is a subspace of $E_1$ and a subspace of $E_2$.

The *sum* of two subspaces $E_1$ and $E_2$ is defined as the set of all vectors of the form

$$x = x_1 + x_2, \quad x_1 \in E_1, x_2 \in E_2 \quad (1.16)$$
Figure C.4: Introducing the concept of a subspace in Poole (2006, p. 438)
Subspaces

Proposition 2  Let \( V \) be a vector space over \( K \), \( U \) a nonempty subset of \( V \). The following conditions on \( U \) are equivalent:

1. \( U \) is itself a vector space over \( K \), with addition and scalar multiplication 'inherited' from \( V \) (i.e., defined for elements of \( U \) as they are if those elements are considered as elements of \( V \));
2. \( x + y \in U \) whenever \( x, y \in U \) and \( \lambda x \in U \) whenever \( \lambda \in K \), \( x \in U \);
3. \( \lambda x + \mu y \in U \) whenever \( \lambda, \mu \in K \), \( x, y \in U \).

Proof

(1)\(\Rightarrow\)(2) is clear, in view of axioms (i) and (vi), applied now to \( U \).

To show that (2)\(\Rightarrow\)(1), note that if (2) holds then certainly axioms (i) and (vi) hold in \( U \). Moreover, whenever \( x \in U \) we have \((-1)x \in U \), so that (Proposition 1(ii)) \(-x \in U \). This shows that axiom (ii) is satisfied in \( U \). Further, it then follows that \( x + (-x) \in U \), i.e. that \( 0 \in U \); so axiom (iv) is satisfied in \( U \). The other six axioms are 'inherited' from \( V \), that is, they hold in \( U \) because they hold throughout \( V \) and therefore hold, in particular, in \( U \).

(2)\(\Rightarrow\)(3): let \( \lambda, \mu \in K \), \( x, y \in U \). Given that (2) holds, we have \( \lambda x \in U \), \( \mu y \in U \), and then \( \lambda x + \mu y \in U \), which gives (3).

(3)\(\Rightarrow\)(2): if (3) holds, put \( \lambda = \mu = 1 \) to get the first statement in (2), and put \( \lambda = 1 \), \( \mu = 0 \) to get the second.

Definition 2  Let \( V \) be a vector space over \( K \), \( U \) a nonempty subset of \( V \). \( U \) is called a vector subspace of \( V \) (or just a subspace of \( V \)) if it satisfies the equivalent conditions of Proposition 2.

Thus a nonempty subset \( U \) of vector space \( V \) is a subspace of \( V \) if and only if it is closed under addition and scalar multiplication (Proposition 2, condition (2)). Condition (3) of Proposition 2 combines the two statements 'closed under addition' and 'closed under scalar multiplication' of condition (2) into a single statement which is sometimes more convenient to use.

Note. As the proof of (2)\(\Rightarrow\)(1) in Proposition 2 emphasizes, if \( U \) is a subspace of \( V \), then \( 0 \in U \). It follows that a subset of \( V \) which does not contain \( 0 \) cannot be a subspace of \( V \).

Examples of Subspaces

1. In any vector space \( V \), \( \{0\} \) and \( V \) are subspaces. A proper subspace is one that is not the whole of \( V \).
2. In \( \mathbb{R}^2 \), \( \mathbb{R}^3 \), sets of points corresponding to lines and planes through the origin are subspaces (and there are no others apart from \( \{0\} \) and the whole space). We shall deal algebraically with this later.
3. In \( P_n \), if \( n \geq 1 \) the subset \( P_n \) of all polynomials with degree \( \leq n \) is a subspace, because if \( p \), \( q \) are polynomials with degree \( \leq n \) then so are \( p + q \) and \( \lambda p \) for any \( \lambda \in \mathbb{R} \): \( \lambda p = \lambda a_0 + a_1 x + \cdots + a_n x^n \), where \( a_n \neq 0 \), has degree \( n \).
4. Let \( I \) denote an interval in \( \mathbb{R} \) (possibly the whole of \( \mathbb{R} \)). Then in \( F(I) \) the subset \( C(I) \) consisting of all continuous real-valued functions on \( I \) is a subspace, because if \( f \), \( g \) are continuous on \( I \) then so are \( f + g \) and \( \lambda f \) for any \( \lambda \in \mathbb{R} \). Also, the subset \( D(I) \) consisting of all differentiable real-valued functions on \( I \) is a subspace, because if \( f \), \( g \) are differentiable on \( I \) then so are \( f + g \) and \( \lambda f \) for any \( \lambda \in \mathbb{R} \). In fact \( D(I) \subseteq C(I) \) so that \( D(I) \) is also a subspace of \( C(I) \).

Proposition 3  Let \( V \) be a vector space over \( K \), and let \( U, W \) be subspaces of \( V \). Then \( U \cap W \) is also a subspace of \( V \).

Proof

Let \( x, y \in U \cap W \) and let \( \lambda, \mu \in K \). Since \( U \) is a subspace of \( V \), \( \lambda x + \mu y \in U \); and since \( W \) is a subspace of \( V \), \( \lambda x + \mu y \in W \) (Proposition 2). Thus \( \lambda x + \mu y \in U \cap W \). This proves that \( U \cap W \) is a subspace of \( V \).

Figure C.5: Introducing the concept of a subspace in Sproston (1995, p. 4)
Appendix D

Examples from Coding Analysis
Figure D.1: Coding analysis of document P5
Figure D.2: Coding analysis of document P8
Figure D.3: Coding analysis of document P12
Figure D.4: Code list exported from atlas-ti
Figure D.5: Code list exported from atlas-ti
Appendix E

The Linear Algebra Course Notes (Student Version)
3 Subspaces of $\mathbb{R}^n$

3.1 Reminder: Vectors and linear transformations

A matrix of size $n \times 1$ is called an $n$-component vector.
The set of all $n$-component vectors is denoted by $\mathbb{R}^n$.
Thus, the notation $v \in \mathbb{R}^n$ means

Observation 3.1. The set $\mathbb{R}^n$ of $n$-component vectors has the following properties:

1. Two vectors can be added, the result is a vector.
2. A vector can be multiplied by a number, the result is a vector.
3. The zero vector $0$ is a vector.

Observation 3.2. If $A$ is an $m \times n$ matrix and $v$ is an $n$-component vector, the product $Av$ is defined. It is an $m$-component vector.
The matrix $A$ defines a function $\varphi_A : \mathbb{R}^n \to \mathbb{R}^m$ by $\varphi_A(v) = Av$. The function $\varphi_A$ takes $n$-component vectors as arguments and gives $m$-component vectors as result.
The function $\varphi_A$ defined by a matrix $A$ is called a linear transformation or a linear map.

Observation 3.3. A linear transformation has the following properties:

1. $\varphi_A(v_1 + v_2) = \varphi_A(v_1) + \varphi_A(v_2)$ or $Av_1 + Av_2$
2. $\varphi_A(\lambda v_1) = \lambda \varphi_A(v_1)$ or $A(\lambda v_1) = \lambda Av_1$
3. $\varphi_A(0) = 0$ or $A0 = 0$

for all vectors $v_1, v_2$ and all real numbers $\lambda$.
These properties correspond to the three properties of the set $\mathbb{R}^n$ in Observation 3.1.
When we call $\varphi_A$ a linear transformation, we refer to these properties.
3.2 The null space of a matrix

Example 3.4. Assume that \( A \) is an unknown \( 2 \times 3 \) matrix. We know that

\[
\begin{align*}
    x_1 &= \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} \\
    x_2 &= \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}
\end{align*}
\]

are solutions of the homogeneous linear equation system \( Ax = 0 \). Find more solutions of this equation system.

Solution:

Observation 3.5. For any matrix \( A \), the solution set \( S \) of the homogeneous linear equation system \( Ax = 0 \) has the properties

1. Two vectors in \( S \) can be added, the result is again in \( S \).
2. A vector in \( S \) can be multiplied by a number, the result is in \( S \).
3. The zero vector \( \mathbf{0} \) is in \( S \).

We say: The set \( S \) is closed under addition and under multiplication by numbers.

Compare this to Observation 3.1: The set \( S \) has the same properties as the full set of vectors \( \mathbb{R}^n \)!
3.2. THE NULL SPACE OF A MATRIX

Definition 3.6. A subset $S$ of the set of vectors $\mathbb{R}^n$ is called a subspace if it has the three properties of Observation 3.5.

Remark. Any collection of $n$-component vectors is called a subset of $\mathbb{R}^n$. Not every subset is a subspace. To qualify as a subspace, a subset must have special properties, namely those listed in Observation 3.5. In particular, a subset may be empty. A subspace is never empty. It always contains at least the zero vector.

Definition 3.7. The solution set of the homogeneous linear equation system $Ax = 0$ is called the null space of the matrix $A$.

With these new terms, we can phrase Observation 3.5 as follows:

This observation means to us: If we know a few solutions of a homogeneous linear equation system, we know infinitely many. We can construct further solutions as we did in Example 3.4. Our next question is: Can we obtain all solutions in this way?

Example 3.8. Find the null space of the matrix

$$A = \begin{pmatrix} 1 & 2 & -3 \\ -2 & -4 & 6 \\ 3 & 6 & -9 \end{pmatrix}.$$  

Solution:
3.4 Linear independence

Example 3.16. Find a set of vectors that spans the range of the matrix

\[
A = \begin{pmatrix}
1 & 4 & 5 \\
1 & -2 & -1 \\
2 & 1 & 3
\end{pmatrix}.
\]

Solution:

Example 3.17. Can you find a smaller set of vectors that spans the range of the matrix \( A \) in Example 3.16?

Solution:
Observe that the relation \( v_3 = v_1 + v_2 \) that we used in the example can be rewritten as \( v_1 + v_2 - v_3 = 0 \). This leads to the

**Definition 3.18.** The vectors \( v_1, v_2, \ldots, v_p \) are **linearly dependent** if there are numbers \( \lambda_1, \ldots, \lambda_p \), not all zero, such that

\[
\lambda_1 v_1 + \cdots + \lambda_p v_p = 0.
\] (3.1)

In this case, Eq. (3.1) is called a **linear relation** between the \( v_i \).

The vectors \( v_1, v_2, \ldots, v_p \) are **linearly independent** if they are not linearly dependent.

**Remark.** (a) The homogeneous linear equation system (3.1) always has the solution \( \lambda_1 = \lambda_2 = \cdots = \lambda_p = 0 \). The vectors \( v_1, v_2, \ldots, v_p \) are linearly independent if this is the only solution, i.e. if the equation system (3.1) has a unique solution.

(b) If the vectors \( v_1, v_2, \ldots, v_p \) are linearly dependent, at least one of them can be written as a linear combination of the others.

(c) When we say that the vectors \( v_1, v_2, \ldots, v_p \) are linearly independent, we should more precisely say that the set \( \{ v_1, v_2, \ldots, v_p \} \) is linearly independent. The concept of linear dependence or independence applies to a **collection** of vectors, not to individual vectors within that collection.

**Example 3.19.** (a) Show that the vectors \( v_1, v_2, v_3 \) of the previous example are linearly dependent.

Solution:

(b) Decide if the vectors \( v_1 \) and \( v_2 \) are linearly independent.

Solution:
Definition 3.20. (a) A basis of a subspace is a linear independent set that spans the subspace.

(b) The number of elements in a basis is called the dimension of the subspace.

Definition 3.21. For any matrix $A$, the dimension of its range is called the rank of $A$, denoted by $\text{rank } A$. The dimension of the null space is called the nullity of $A$.

Example 3.22. In the previous example, $\{v_1, v_2\}$ is a basis for the range of $A$. The rank of $A$ is 2.

Note how this basis was found: We started with a spanning set for the subspace. We then left out vectors that could be written as linear combinations of the other vectors until we could go no further.

Remark. (a) The dimension is the minimum number of vectors needed to span a subspace. It measures the complexity of a subspace.

(b) Many different sets can be a basis for a given subspace. All bases must contain the same number of vectors.

Example 3.23. Find a basis for the null space of the matrix

$$
A = \begin{pmatrix}
1 & 2 & -1 & 0 & 2 \\
2 & 4 & -1 & 2 & 1 \\
0 & 0 & 2 & 5 & -4
\end{pmatrix}
$$

of Example 3.12.

Solution:
3.6. HOMOGENEOUS AND INHOMOGENEOUS LINEAR EQUATION SYSTEMS

2. To check if the column vectors are linearly independent, we have to solve the equation
\[ x_1v_1 + x_2v_2 + \cdots + x_nv_n = 0. \] (3.2)

Solving this linear dependence equation is the same as determining the null space of the matrix \( A \). The augmented coefficient matrix of (3.2) is \( (A|0) \). Once we have transformed this matrix to reduced echelon form, we can find a basis for the null space.

3. If the column vectors are not linearly independent, each basis vector in the null space will give us a linear relation between them that allows us to leave out one of the column vectors. The number of vectors left in our basis for range \( A \), will be the total number of column vectors minus the number of vectors in the basis of null \( A \). If we note that the number of vectors in a basis for range \( A \) is the rank of \( A \) and the number of vectors in a basis for the null space is the nullity of \( A \), we can formulate this observation as the important Rank-nullity theorem:

Theorem 3.27. The rank of a matrix plus its nullity equals the number of columns.

For an \( m \times n \) matrix \( A \), \( \text{rank}(A) + \text{nullity}(A) = n \).

If we analyze the calculation more closely, we observe that every vector in our basis for the null space allows us to leave out a column of the matrix \( A \) for which we introduced a free parameter. The remaining column vectors in the basis for the range of \( A \) are those columns (of the original matrix) for which no free parameter was introduced because they contain a pivot in the reduced echelon form.

Theorem 3.28. The rank of a matrix is the number of nonzero rows in its (reduced) echelon form.

Corollary 3.29. The rank of a matrix cannot be larger than the number of its rows.

3.6 Homogeneous and inhomogeneous linear equation systems

Consider an \( m \times n \) matrix \( A \) and an \( m \)-component vector \( b \). Then \( Ax = b \) is a linear equation system for the \( n \)-component vector \( x \). We will compare the solutions of the inhomogeneous system \( Ax = b \) and the homogeneous system \( Ax = 0 \) with the same coefficient matrix.

Example 3.30. Consider
\[
A = \begin{pmatrix}
1 & 2 & -1 & 1 \\
-3 & -6 & 4 & -6 \\
2 & 4 & 0 & -4
\end{pmatrix}
\quad \text{and} \quad
b = \begin{pmatrix}
-1 \\
5 \\
2
\end{pmatrix}.
\]

(a) Find all solutions \( y \) of the homogeneous linear equation system \( Ay = 0 \).
(b) Find all solutions \( \mathbf{x} \) of the inhomogeneous linear equation system \( A\mathbf{x} = \mathbf{b} \).

Solution:

\begin{align*}
\text{Observation 3.31.} & \quad \text{The general solution of the inhomogeneous equation system } A\mathbf{x} = \mathbf{b} \\
& \quad \text{is given by a single ("particular") solution of the inhomogeneous system plus the general solution of the homogeneous equation system } A\mathbf{y} = \mathbf{0}.
\end{align*}

If we know one solution of the inhomogeneous equation system \( A\mathbf{x} = \mathbf{b} \) and all solutions of the homogeneous equation system \( A\mathbf{y} = \mathbf{0} \), we know all solutions of the inhomogeneous system.

\begin{align*}
\text{Observation 3.32.} & \quad \text{If the inhomogeneous equation system } A\mathbf{x} = \mathbf{b} \text{ is consistent, its solution set has the same number of free parameters as the homogeneous equation system } A\mathbf{y} = \mathbf{0} \text{. This number is the nullity of } A.
\end{align*}

In particular:

If the columns of \( A \) are linearly independent, the homogeneous system \( A\mathbf{y} = \mathbf{0} \) has only the solution \( \mathbf{y} = \mathbf{0} \). The solution of the inhomogeneous equation \( A\mathbf{x} = \mathbf{b} \), if it exists, must then be unique.

To understand this observation, we observe further:

\begin{align*}
\text{Observation 3.33. (a)} & \quad \text{Let } \mathbf{x}_1 \text{ and } \mathbf{x}_2 \text{ be two solutions of the inhomogeneous equation system } A\mathbf{x} = \mathbf{b}. \text{ Show that their difference } \mathbf{y} = \mathbf{x}_2 - \mathbf{x}_1 \text{ is a solution to the homogeneous equation system } A\mathbf{y} = \mathbf{0}.
\end{align*}
3.6. HOMOGENEOUS AND INHOMOGENEOUS LINEAR EQUATION SYSTEMS

Solution:

(b) If $x$ is a solution of $Ax = b$ and $y$ is a solution of the homogeneous equation system $Ay = 0$, show that $x' = x + y$ is another solution of $Ax' = b$.

Solution:

Example 3.34. Consider

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -3 \\ 2 & 1 & -1 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 3 \\ -7 \\ -1 \end{pmatrix}.$$  

The vector

$$\mathbf{x} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

satisfies

$$A\mathbf{x} = \begin{pmatrix} 1 \times 1 + 0 \times (-1) + 1 \times 2 \\ 0 \times 1 + 1 \times (-1) - 3 \times 2 \\ 2 \times 1 + 1 \times (-1) - 1 \times 2 \end{pmatrix} = \begin{pmatrix} 3 \\ -7 \\ -1 \end{pmatrix} = \mathbf{b}.$$  

All solutions of the homogeneous system $Ay = 0$ are given by

$$y = \lambda \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}$$

with an arbitrary real number $\lambda$. (Check this!)

Find all solutions of the equation system $Ax = b$.

Solution:

Example 3.35. Show that the solution set

$$S = \left\{ \begin{pmatrix} 1 \\ 0 \\ 2 \\ 0 \\ \lambda \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 0 \\ 3 \\ 1 \\ 0 \end{pmatrix} : \lambda, \mu \in \mathbb{R} \right\}$$

of the inhomogeneous linear equation system $Ax = b$ in Example 3.30 is not a subspace of $\mathbb{R}^4$.

Solution:

Figure E.9: Chapter 3, page 14: Linear Independence
4 Eigenvalues and Eigenvectors

4.1 Eigenvectors

Example 4.1. Consider the matrix
\[
A = \begin{pmatrix}
5 & 5 & -2 \\
-4 & -4 & 2 \\
-3 & -3 & 2 \\
\end{pmatrix}
\]
and the vectors
\[
v_1 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad v_2 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}
\]

(a) Calculate the vectors \( Av_1 \) and \( Av_2 \). Comment on your results.

Solution:

(b) Calculate \( A^{2000}v_2 \).

Solution:

Definition 4.2. For an \( n \times n \) matrix \( A \) a vector \( v \) (not equal to the zero vector) and a number \( \lambda \) that satisfy

\[
Av = \lambda v
\]

are called an eigenvector and the corresponding eigenvalue of \( A \).
4.1. EIGENVECTORS

Remark 4.3. The zero vector satisfies $A0 = \lambda 0$ for any square matrix $A$ and any number $\lambda$. For this reason, the zero vector does not count as an eigenvector. However, the number zero is permitted as an eigenvalue.

Observation 4.4. If $v$ is an eigenvector of the matrix $A$ with eigenvalue $\lambda$, it is also an eigenvalue of $A^n$ for every $n$, and the eigenvalue is $\lambda^n$.

Proof. The argument is the same as in the example: Take the eigenvector-eigenvalue equation $Av = \lambda v$ and multiply if by $A$ from the left, which gives $A^2 v = \lambda^2 v$. Now use the eigenvector equation on the right hand side and get $A^2 v = \lambda^2 v$. Repeating the process $n$ times, we get $A^n v = \lambda^n v$.

Example 4.5. Consider the matrix

$$A = \begin{pmatrix} 0 & -6 & 4 \\ 1 & 5 & 2 \\ 0 & 0 & -2 \end{pmatrix}.$$ 

(a) Show that the vector

$$v = \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}$$

is an eigenvector of $A$. What is the corresponding eigenvalue?

Solution:

(b) Show that $\lambda = 3$ is an eigenvalue of $A$. Find all corresponding eigenvectors.

Solution:
4.1. EIGENVECTORS

(c) Is $-5$ an eigenvalue of $A$? If so, find the corresponding eigenvectors.

Solution:

Observation 4.6. (a) The eigenvectors of an $n \times n$ matrix $A$ corresponding to the eigenvalue $\lambda$ are the non-zero solutions of the homogeneous linear equation system $(A - \lambda I_n)v = 0$.

(b) If the matrix $A - \lambda I_n$ is invertible, the only solution to the eigenvector equation $(A - \lambda I_n)v = 0$ is $v = 0$. In this case, $\lambda$ is not an eigenvalue of $A$. The eigenvalues of $A$ are the values $\lambda$ for which the matrix $A - \lambda I_n$ is not invertible.

(c) The solution set of this equation system is a subspace of $\mathbb{R}^n$, the null space of the matrix $A - \lambda I_n$. It is called the eigenspace of $A$ to the eigenvector $\lambda$. The dimension of the eigenspace is called the geometric multiplicity of the eigenvalue $\lambda$. The geometric multiplicity of an eigenvalue is the number of linearly independent corresponding eigenvectors.

(d) The eigenspace to the eigenvalue $\lambda$ contains all eigenvectors to the eigenvalue $\lambda$, and the zero vector.
4.5 The characteristic polynomial

Example 4.33. Consider the matrix

\[ A = \begin{pmatrix} 4 & 6 \\ -1 & -1 \end{pmatrix}. \]

(a) Show that 2 is an eigenvalue of \( A \). Find all corresponding eigenvectors.

Solution:

(b) Is the matrix \( A - 2I_2 \) invertible?

(Do not do a new calculation. Use what you already know about the matrix \( A \).)

Solution:

Remark. For matrices that are larger than \( 3 \times 3 \), calculating a determinant through row- or column expansions is exceedingly laborious, unless you expand after a row or column that has many zeros. For example, if you expand after rows or columns that do not have any zeros, calculating a \( 5 \times 5 \) determinant gives 5 determinants of size \( 4 \times 4 \), then \( 5 \times 4 \) determinants of size \( 3 \times 3 \), then \( 5 \times 4 \times 3 = 60 \) determinants of size \( 2 \times 2 \). Expanding a \( 6 \times 6 \) determinant gives \( 6 \times 5 \times 4 \times 3 = 360 \) determinants of size \( 2 \times 2 \). These numbers grow rapidly as the determinants get larger. Using Gaussian elimination for these calculations is a lot faster. You can use Gaussian elimination to produce zeros in a determinant and then expand.
4.5. THE CHARACTERISTIC POLYNOMIAL

(c) Use determinants to find all values $\lambda$ for which the matrix $A - \lambda I_n$ is not invertible.
Solution:

(d) Find all eigenvalues and all eigenvectors of the matrix $A$.
Solution:

**Observation 4.34. (a)** If $A$ is an $n \times n$ matrix, the function $P(\lambda) = \det(A - \lambda I_n)$ is a polynomial of degree $n$. It is called the characteristic polynomial of $A$. The eigenvalues of $A$ are the zeros of the characteristic polynomial.

(b) To find all eigenvalues and eigenvectors of a (small) matrix $A$, we can proceed as follows:

1. Calculate the characteristic polynomial $P(\lambda) = \det(A - \lambda I_n)$.
2. Find all zeros of the characteristic polynomial. These are the eigenvalues.
3. For each eigenvalue $\lambda_i$, solve the equation $A\mathbf{v} = \lambda_i \mathbf{v}$ or $(A - \lambda_i I_n)\mathbf{v} = \mathbf{0}$. The solutions are the eigenvectors corresponding to $\lambda_i$.

There will always be infinitely many eigenvectors corresponding to each eigenvalue. The number of free parameters that are needed to describe all eigenvectors is the geometric multiplicity of the eigenvalue. This is also the number of linearly independent eigenvectors associated with the eigenvalue.

**Example 4.35.** Find all eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} -4 & -3 & -3 \\ 0 & -1 & 0 \\ 6 & 6 & 5 \end{pmatrix}.$$ 

Is $A$ diagonalizable?
Appendix F

The Linear Algebra Course Notes
(Complete Version)
1.2 Under-determined systems

Example 1.11. Find the solution set $S$ of the linear equation system

$$x_1 - 2x_2 = 1$$

Solution: As there is only one equation, we can solve for only one variable, e.g.

$$x_1 = 1 + 2x_2.$$ 

The second variable remains undetermined, we can give it an arbitrary value, say we set $x_2 = \lambda$. The value of $x_1$ is then determined by

$$x_1 = 1 + 2x_2 = 1 + 2\lambda,$$

and the solution set can be written as

$$S = \{(1 + 2\lambda, \lambda) : \lambda \in \mathbb{R}\}.$$ 

The variable $\lambda$ is called a parameter or a free variable. Every value of the parameter yields a solution of the equation. The name of the parameter is irrelevant. We can just as well write

$$S = \{(1 + 2\mu, \mu) : \mu \in \mathbb{R}\}.$$ 

Alternatively, we can solve for $x_2$ and assign an arbitrary value $\nu$ to the variable $x_1$:

$$x_1 = \nu$$

$$x_2 = -\frac{1}{2} - \frac{1}{2}\nu$$

The solution set then reads

$$S = \{ (\nu, -\frac{1}{2} + \frac{1}{2}\nu) : \nu \in \mathbb{R} \}.$$ 

All these results describe the same solution set $S$.

Observation:
There is always one free variable, no matter how we express the solution set. Because there is one equation, we can solve for one variable. The other variable remains free.

Example 1.12. Find the solution set of the linear equation system

$$x_1 - 2x_2 = 1$$
$$2x_1 - 4x_2 = 2$$

Solution: The two equations are equivalent. Effectively, there is only one equation given. The solution set is the same as for the single equation in the previous example:

$$S = \{(1 + 2\lambda, \lambda) : \lambda \in \mathbb{R}\}.$$ 

Figure F.1: Chapter 1, page 6: Linear Equation Systems (complete course notes)
We say: The equations are not independent because they satisfy

\[(\text{eq. 2}) = 2 \times (\text{eq. 1})\]
or \[2 \times (\text{eq. 1}) - (\text{eq. 2}) = 0.\]

**Observation:**
The number of independent equations can be smaller than the total number of equations. In this example, there is one independent equation.

We can solve for one variable out of two. The other variable remains free.

If we did not notice that the equations are not independent, we would do Gaussian elimination:

\[
\begin{pmatrix}
1 & -2 & 1 \\
2 & -4 & 2
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & -2 & 1 \\
0 & 0 & 0
\end{pmatrix}
\]

The last row corresponds to the equation

\[0 \times x_1 + 0 \times x_2 = 0,
\]

which is true for all values of \(x_1\) and \(x_2\). Gaussian elimination produces a trivial equation, and the “extra” equation drops out.

**Example 1.13.** Find the solution set for the linear equation system

\[
\begin{align*}
x_1 + x_3 &= 1 \\
x_2 + x_3 &= 2 \\
x_1 + x_2 + 2x_3 &= 3
\end{align*}
\]

**Solution:** The equations are not independent: The third is the sum of the first two.

\[(\text{eq. 1}) + (\text{eq. 2}) = (\text{eq. 3})\]
or \[(\text{eq. 1}) + (\text{eq. 2}) - (\text{eq. 3}) = 0.\]

Therefore, any solution of the first two equations must also solve the third.

Use Gaussian elimination for a detailed analysis:

\[
\begin{pmatrix}
1 & 0 & 1 & 1 \\
0 & 1 & 1 & 2 \\
1 & 1 & 2 & 3
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 0 & 1 & 1 \\
0 & 1 & 1 & 2 \\
0 & 1 & 1 & 2
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 0 & 1 & 1 \\
0 & 1 & 1 & 2 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

(reduced EF)
3 Subspaces of $\mathbb{R}^n$

3.1 Reminder: Vectors and linear transformations

A matrix of size $n \times 1$ is called an $n$-component vector. The set of all $n$-component vectors is denoted by $\mathbb{R}^n$.

Thus, the notation $v \in \mathbb{R}^n$ means

$v$ is an $n$-component vector.

Observation 3.1. The set $\mathbb{R}^n$ of $n$-component vectors has the following properties:

1. Two vectors can be added, the result is a vector.
2. A vector can be multiplied by a number, the result is a vector.
3. The zero vector $0$ is a vector.

Observation 3.2. If $A$ is an $m \times n$ matrix and $v$ is an $n$-component vector, the product $Av$ is defined. It is an $m$-component vector.

The matrix $A$ defines a function $\varphi_A : \mathbb{R}^n \to \mathbb{R}^m$ by $\varphi_A(v) = Av$. The function $\varphi_A$ takes $n$-component vectors as arguments and gives $m$-component vectors as result.

The function $\varphi_A$ defined by a matrix $A$ is called a linear transformation or a linear map.

Observation 3.3. A linear transformation has the following properties:

1. $\varphi_A(v_1 + v_2) = \varphi_A(v_1) + \varphi_A(v_2)$ or $A(v_1 + v_2) = Av_1 + Av_2$
2. $\varphi_A(\lambda v_1) = \lambda \varphi_A(v_1)$ or $A(\lambda v_1) = \lambda Av_1$
3. $\varphi_A(0) = 0$ or $A0 = 0$

for all vectors $v_1, v_2$ and all real numbers $\lambda$.

These properties correspond to the three properties of the set $\mathbb{R}^n$ in Observation 3.1. When we call $\varphi_A$ a linear transformation, we refer to these properties.
3.2 The null space of a matrix

Example 3.4. Assume that \( A \) is an unknown \( 2 \times 3 \) matrix. We know that
\[
\begin{bmatrix}
1 \\
3
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
-2 \\
1
\end{bmatrix}
\]
are solutions of the homogeneous linear equation system \( Ax = 0 \). Find more solutions of this equation system.

Solution: We can use the linearity properties of Observation 3.3.
\[
x_1 + x_2 = \begin{bmatrix}
-1 \\
4
\end{bmatrix}
is a solution because \( A(x_1 + x_2) = Ax_1 + Ax_2 = 0 + 0 = 0 \).
\[
2x_1 = \begin{bmatrix}
2 \\
6
\end{bmatrix}
is a solution because \( A(2x_1) = 2Ax_1 = 2 \times 0 = 0 \).
\[
-3x_2 = \begin{bmatrix}
6 \\
-9
\end{bmatrix}
is a solution for the same reason.
More generally: every multiple of \( x_1 \) and every multiple of \( x_2 \) is a solution.
\[
2x_1 - 3x_2 = \begin{bmatrix}
8 \\
3
\end{bmatrix}
is a solution because \( 2x_1 \) and \( -3x_2 \) are solutions.
More generally: Every vector of the form \( \lambda_1 x_1 + \lambda_2 x_2 \) is a solution.
In particular: The zero vector \( \mathbf{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \) is a solution.

Observation 3.5. For any matrix \( A \), the solution set \( S \) of the homogeneous linear equation system \( Ax = 0 \) has the properties

1. Two vectors in \( S \) can be added, the result is again in \( S \).
2. A vector in \( S \) can be multiplied by a number, the result is in \( S \).
3. The zero vector \( \mathbf{0} \) is in \( S \).

We say: The set \( S \) is closed under addition and under multiplication by numbers.
Compare this to Observation 3.1: The set \( S \) has the same properties as the full set of vectors \( \mathbb{R}^n \).
3.2. THE NULL SPACE OF A MATRIX

**Definition 3.6.** A subset $S$ of the set of vectors $\mathbb{R}^n$ is called a subspace if it has the three properties of Observation 3.5.

**Remark.** Any collection of $n$-component vectors is called a subset of $\mathbb{R}^n$. Not every subset is a subspace. To qualify as a subspace, a subset must have special properties, namely those listed in Observation 3.5.

In particular, a subset may be empty. A subspace is never empty. It always contains at least the zero vector.

**Definition 3.7.** The solution set of the homogeneous linear equation system $Ax = 0$ is called the null space of the matrix $A$.

With these new terms, we can phrase Observation 3.5 as follows:

The null space of a matrix is a subspace.

This observation means to us: If we know a few solutions of a homogeneous linear equation system, we know infinitely many. We can construct further solutions as we did in Example 3.4. Our next question is: Can we obtain all solutions in this way?

**Example 3.8.** Find the null space of the matrix 

$$A = \begin{pmatrix} 1 & 2 & -3 \\ -2 & -4 & 6 \\ 3 & 6 & -9 \end{pmatrix}.$$ 

**Solution:** Use Gaussian elimination to solve the equation system $Ax = 0$:

$$\begin{pmatrix} 1 & 2 & -3 & 0 \\ -2 & -4 & 6 & 0 \\ 3 & 6 & -9 & 0 \end{pmatrix} \xrightarrow{r_2 + 2r_1, r_3 - 3r_1} \begin{pmatrix} 1 & 2 & -3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Introduce parameters: $x_2 = \lambda$, $x_3 = \mu$, then $x_1 = -2\lambda + 3\mu$. The general solution is

$$x = \begin{pmatrix} -2\lambda + 3\mu \\ \lambda \\ \mu \end{pmatrix} = \lambda \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}.$$ 

The null space is

$$\text{null } A = \left\{ \begin{pmatrix} -2\lambda + 3\mu \\ \lambda \\ \mu \end{pmatrix} : \lambda, \mu \in \mathbb{R} \right\} = \left\{ \lambda \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} : \lambda, \mu \in \mathbb{R} \right\}.$$

Thus, all solutions can be written as $x = \lambda v_1 + \mu v_2$ with the two vectors

$$v_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad v_2 = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}.$$ 

We say: The vectors $v_1$ and $v_2$ span the null space of $A$. 

---

**Figure F.5:** Chapter 3, page 3: Subspace
3.4 Linear independence

Example 3.16. Find a set of vectors that spans the range of the matrix

\[ A = \begin{pmatrix} 1 & 4 & 5 \\ 1 & -2 & -1 \\ 2 & 1 & 3 \end{pmatrix}. \]

Solution: A vector \( \mathbf{b} \) is in the range of \( A \) if there is a vector \( \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \) such that

\[ \mathbf{b} = A \mathbf{x} = \begin{pmatrix} x_1 + 4x_2 + 5x_3 \\ x_1 - 2x_2 - x_3 \\ 2x_1 + x_2 + 3x_3 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + x_2 \begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix} + x_3 \begin{pmatrix} 5 \\ -1 \\ 3 \end{pmatrix}. \]

Thus, a vector \( \mathbf{b} \) is in the range of \( A \) if and only if it can be written as a linear combination of the column vectors of \( A \)

\[ \mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix}, \quad \text{and} \quad \mathbf{v}_3 = \begin{pmatrix} 5 \\ -1 \\ 3 \end{pmatrix}. \]

The column vectors of \( A \) span the range.

Example 3.17. Can you find a smaller set of vectors that spans the range of the matrix \( A \) in Example 3.16?

Solution: The column vectors satisfy \( \mathbf{v}_3 = \mathbf{v}_1 + \mathbf{v}_2 \). We can therefore rewrite

\[ x_1 \mathbf{v}_1 + x_2 \mathbf{v}_2 + x_3 \mathbf{v}_3 = x_1 \mathbf{v}_1 + x_2 \mathbf{v}_2 + x_3 (\mathbf{v}_1 + \mathbf{v}_2) \]

\[ = (x_1 + x_3) \mathbf{v}_1 + (x_2 + x_3) \mathbf{v}_2 \]

\[ = y_1 \mathbf{v}_1 + y_2 \mathbf{v}_2 \]

with \( y_1 = x_1 + x_3 \) and \( y_2 = x_2 + x_3 \). Thus, every linear combination of \( \mathbf{v}_1 \), \( \mathbf{v}_2 \) and \( \mathbf{v}_3 \) can be written as a linear combination of \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \) alone. The vectors \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \) span the range of \( A \).
Observe that the relation \( v_3 = v_1 + v_2 \) that we used in the example can be rewritten as \( v_1 + v_2 - v_3 = 0 \). This leads to the

**Definition 3.18.** The vectors \( v_1, v_2, \ldots, v_p \) are **linearly dependent** if there are numbers \( \lambda_1, \ldots, \lambda_p \), not all zero, such that

\[
\lambda_1 v_1 + \cdots + \lambda_p v_p = 0.
\]  

(3.1)

In this case, Eq. (3.1) is called a **linear relation** between the \( v_i \).

**Remark.** (a) The homogeneous linear equation system (3.1) always has the solution \( \lambda_1 = \lambda_2 = \cdots = \lambda_p = 0 \). The vectors \( v_1, v_2, \ldots, v_p \) are **linearly independent** if this is the **only** solution, i.e. if the equation system (3.1) has a **unique** solution.

(b) If the vectors \( v_1, v_2, \ldots, v_p \) are linearly dependent, at least one of them can be written as a linear combination of the others.

(c) When we say that the vectors \( v_1, v_2, \ldots, v_p \) are linearly independent, we should more precisely say that the set \( \{v_1, v_2, \ldots, v_p\} \) is linearly independent. The concept of linear dependence or independence applies to a **collection** of vectors, not to individual vectors within that collection.

**Example 3.19.** (a) Show that the vectors \( v_1, v_2, v_3 \) of the previous example are linearly dependent.

**Solution:** We have to solve the linear equation system

\[
\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 = 0,
\]

which reads in full

\[
\lambda_1 + 4\lambda_2 + 5\lambda_3 = 0 \quad \lambda_1 - 2\lambda_2 - \lambda_3 = 0 \quad 2\lambda_1 + \lambda_2 + 3\lambda_3 = 0.
\]

The array corresponding to this system of equations is

\[
\begin{pmatrix}
  1 & 4 & 5 & 0 \\
  1 & -2 & -1 & 0 \\
  2 & 1 & 3 & 0
\end{pmatrix}
\]

Gaussian elimination

\[
\begin{pmatrix}
  1 & 0 & 1 & 0 \\
  0 & 1 & 1 & 0 \\
  0 & 0 & 0 & 0
\end{pmatrix}
\]

The solution set is \( \{(\mu, -\mu, \mu) : \mu \in \mathbb{R}\} \). Hence the given vectors are linearly dependent. For \( \mu = 1 \) we obtain the linear relation

\[
-v_1 - v_2 + v_3 = 0,
\]

i.e.

\[
v_3 = v_1 + v_2,
\]

which was already used in the previous example.

(b) Decide if the vectors \( v_1 \) and \( v_2 \) are linearly independent.

**Solution:** The condition \( \lambda_1 v_1 + \lambda_2 v_2 = 0 \) gives rise to the equations

\[
\lambda_1 + 4\lambda_2 = 0 \quad \lambda_1 - 2\lambda_2 = 0 \quad 2\lambda_1 + \lambda_2 = 0
\]
The array corresponding to this system of equations is
\[
\begin{pmatrix}
1 & 4 & 0 \\
1 & -2 & 0 \\
2 & 1 & 0
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

The only solution is thus \( \lambda_1 = \lambda_2 = 0 \). Hence \( v_1 \) and \( v_2 \) are linearly independent.

**Definition 3.20.** (a) A basis of a subspace is a linear independent set that spans the subspace.

(b) The number of elements in a basis is called the dimension of the subspace.

**Definition 3.21.** For any matrix \( A \), the dimension of its range is called the rank of \( A \), denoted by rank \( A \). The dimension of the null space is called the nullity of \( A \).

**Example 3.22.** In the previous example, \( \{v_1, v_2\} \) is a basis for the range of \( A \). The rank of \( A \) is 2.

Note how this basis was found: We started with a spanning set for the subspace. We then left out vectors that could be written as linear combinations of the other vectors until we could go no further.

**Remark.** (a) The dimension is the minimum number of vectors needed to span a subspace. It measures the complexity of a subspace.

(b) Many different sets can be a basis for a given subspace. All bases must contain the same number of vectors.

**Example 3.23.** Find a basis for the null space of the matrix
\[
A = \begin{pmatrix}
1 & 2 & -1 & 0 & 2 \\
2 & 4 & -1 & 2 & 1 \\
0 & 2 & 5 & -4
\end{pmatrix}
\]
of Example 3.12.

**Solution:** We have seen in Example 3.12 that the vectors
\[
v_1 = \begin{pmatrix}
-2 \\
1 \\
0 \\
0 \\
0
\end{pmatrix}, \quad v_2 = \begin{pmatrix}
-2 \\
0 \\
-2 \\
1 \\
0
\end{pmatrix}, \quad v_3 = \begin{pmatrix}
1 \\
0 \\
3 \\
0 \\
1
\end{pmatrix}
\]
span the null space of \( A \). To see if these vectors are linearly independent, we solve the linear equation system \( \lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 = 0 \), or explicitly

\[
\begin{align*}
-2\lambda_1 - 2\lambda_2 + \lambda_3 &= 0 \\
\lambda_1 &= 0 \\
-2\lambda_2 + 3\lambda_3 &= 0 \\
\lambda_2 &= 0 \\
\lambda_3 &= 0
\end{align*}
\]

This has only the solution \( \lambda_1 = \lambda_2 = \lambda_3 = 0 \). Therefore, the vectors \( v_1, v_2, v_3 \) are linearly independent. The set \( \{v_1, v_2, v_3\} \) is a basis.
3.6. HOMOGENEOUS AND INHOMOGENEOUS LINEAR EQUATION SYSTEMS

2. To check if the column vectors are linearly independent, we have to solve the equation

\[ x_1 v_1 + x_2 v_2 + \cdots + x_n v_n = 0. \]  \hfill (3.2)

Solving this linear dependence equation is the same as determining the null space of the matrix \( A \). The augmented coefficient matrix of (3.2) is \( (A|0) \). Once we have transformed this matrix to reduced echelon form, we can find a basis for the null space.

3. If the column vectors are not linearly independent, each basis vector in the null space will give us a linear relation between them that allows us to leave out one of the column vectors. The number of vectors left in our basis for range \( A \), will be the total number of column vectors minus the number of vectors in the basis of null \( A \). If we note that the number of vectors in a basis for range \( A \) is the rank of \( A \) and the number of vectors in a basis for the null space is the nullity of \( A \), we can formulate this observation as the important Rank-nullity theorem.

**Theorem 3.27.** The rank of a matrix plus its nullity equals the number of columns. For an \( m \times n \) matrix \( A \), \( \text{rank} A + \text{nullity} A = n \).

If we analyze the calculation more closely, we observe that every vector in our basis for the null space allows us to leave out a column of the matrix \( A \) for which we introduced a free parameter. The remaining column vectors in the basis for the range of \( A \) are those columns (of the original matrix) for which no free parameter was introduced because they contain a pivot in the reduced echelon form.

**Theorem 3.28.** The rank of a matrix is the number of nonzero rows in its (reduced) echelon form.

**Corollary 3.29.** The rank of a matrix cannot be larger than the number of its rows.

3.6 Homogeneous and inhomogeneous linear equation systems

Consider an \( m \times n \) matrix \( A \) and an \( m \)-component vector \( b \). Then \( Ax = b \) is a linear equation system for the \( n \)-component vector \( x \). We will compare the solutions of the inhomogeneous system \( Ax = b \) and the homogeneous system \( Ax = 0 \) with the same coefficient matrix.

**Example 3.30.** Consider

\[
A = \begin{pmatrix}
1 & 2 & -1 & 1 \\
-3 & -6 & 4 & -6 \\
2 & 4 & 0 & -4
\end{pmatrix}
\quad \text{and} \quad
b = \begin{pmatrix}
-1 \\
5 \\
2
\end{pmatrix}.
\]

(a) Find all solutions \( y \) of the homogeneous linear equation system \( Ay = 0 \).
3.6. HOMOGENEOUS AND INHOMOGENEOUS LINEAR EQUATION SYSTEMS

Solution: Use Gaussian elimination:
\[
\begin{pmatrix}
1 & 2 & -1 & 1 & 0 \\
-3 & -6 & 4 & -6 & 0 \\
2 & 4 & 0 & -4 & 0
\end{pmatrix}
\overset{\text{Gaussian elimination}}{\rightarrow}
\begin{pmatrix}
1 & 2 & 0 & -2 & 0 \\
0 & 1 & -3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

The general solution is therefore
\[
y = \begin{pmatrix}
-2\lambda + 2\mu \\
\lambda \\
3\mu \\
\mu
\end{pmatrix} = \lambda \begin{pmatrix}
-2 \\
1 \\
0 \\
0
\end{pmatrix} + \mu \begin{pmatrix}
2 \\
0 \\
3 \\
1
\end{pmatrix}.
\]

(b) Find all solutions \(x\) of the inhomogeneous linear equation system \(Ax = b\).

Solution: Use Gaussian elimination:
\[
\begin{pmatrix}
1 & 2 & -1 & 1 & -1 \\
-3 & -6 & 4 & -6 & 5 \\
2 & 4 & 0 & -4 & 2
\end{pmatrix}
\overset{\text{Gaussian elimination}}{\rightarrow}
\begin{pmatrix}
1 & 2 & 0 & -2 & 1 \\
0 & 1 & -3 & 2 & 1 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

The general solution is therefore
\[
x = \begin{pmatrix}
1 - 2\lambda + 2\mu \\
\lambda \\
2 + 3\mu \\
\mu
\end{pmatrix} = \begin{pmatrix}
1 \\
0 \\
2 \\
0
\end{pmatrix} + \lambda \begin{pmatrix}
-2 \\
1 \\
0 \\
0
\end{pmatrix} + \mu \begin{pmatrix}
2 \\
0 \\
3 \\
1
\end{pmatrix}.
\]

Observation 3.31. The general solution of the inhomogeneous equation system \(Ax = b\) is given by a single ("particular") solution of the inhomogeneous system plus the general solution of the homogeneous equation system \(Ay = 0\).

If we know one solution of the inhomogeneous equation system \(Ax = b\) and all solutions of the homogeneous equation system \(Ay = 0\), we know all solutions of the inhomogeneous system.

Observation 3.32. If the inhomogeneous equation system \(Ax = b\) is consistent, its solution set has the same number of free parameters as the homogeneous equation system \(Ay = 0\). This number is the nullity of \(A\).

In particular:
If the columns of \(A\) are linearly independent, the homogeneous system \(Ay = 0\) has only the solution \(y = 0\). The solution of the inhomogeneous equation \(Ax = b\), if it exists, must then be unique.

To understand this observation, we observe further:

Observation 3.33. (a) Let \(x_1\) and \(x_2\) be two solutions of the inhomogeneous equation system \(Ax = b\). Show that their difference \(y = x_2 - x_1\) is a solution to the homogeneous equation system \(Ax = 0\).
3.6. HOMOGENEOUS AND INHOMOGENEOUS LINEAR EQUATION SYSTEMS

Solution:
\[ Ay = A(x_2 - x_1) = Ax_2 - Ax_1 = b - b = 0. \]

(b) If \( x \) is a solution of \( Ax = b \) and \( y \) is a solution of the homogeneous equation system \( Ay = 0 \), show that \( x' = x + y \) is another solution of \( Ax' = b \).

Solution:
\[ Ax' = A(x + y) = Ax + Ay = b + 0 = b. \]

Example 3.34. Consider
\[ A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -3 \\ 2 & 1 & -1 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 3 \\ -7 \\ -1 \end{pmatrix}. \]

The vector \( x = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \) satisfies
\[ Ax = \begin{pmatrix} 1 \times 1 + 0 \times (-1) + 1 \times 2 \\ 0 \times 1 + 1 \times (-1) - 3 \times 2 \\ 2 \times 1 + 1 \times (-1) - 1 \times 2 \end{pmatrix} = \begin{pmatrix} 3 \\ -7 \\ -1 \end{pmatrix} = b. \]

All solutions of the homogeneous system \( Ay = 0 \) are given by
\[ y = \lambda \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix} \]
with an arbitrary real number \( \lambda \). (Check this!)

Find all solutions of the equation system \( Ax = b \).

Solution: The vector
\[ x' = x + y = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix} \]
satisfies \( Ax' = b \) for every value of the parameter \( \lambda \), and these are all solutions.

Example 3.35. Show that the solution set
\[ S = \left\{ \begin{pmatrix} 1 \\ 0 \\ 2 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 0 \\ 0 \\ 3 \\ 1 \end{pmatrix} : \lambda, \mu \in \mathbb{R} \right\} \]
of the inhomogeneous linear equation system \( Ax = b \) in Example 3.30 is not a subspace of \( \mathbb{R}^4 \).

Solution: We have to show that the set \( S \) does not satisfy the three properties of Observation 3.5 that define a subspace. In fact, it satisfies none of them.
4 Eigenvalues and Eigenvectors

4.1 Eigenvectors

Example 4.1. Consider the matrix

\[ A = \begin{pmatrix} 5 & 5 & -2 \\ -4 & -4 & 2 \\ -3 & -3 & 2 \end{pmatrix} \]

and the vectors

\[ v_1 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad v_2 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \]

(a) Calculate the vectors \( Av_1 \) and \( Av_2 \). Comment on your results.

Solution:

\[ Av_1 = \begin{pmatrix} 8 \\ -6 \\ -4 \end{pmatrix}, \quad Av_2 = \begin{pmatrix} -2 \\ 2 \\ 2 \end{pmatrix} = 2v_2. \]

(b) Calculate \( A^{2000}v_2 \).

Solution:

\[ Av_2 = 2v_2 \]
\[ A^2v_2 = A(2v_2) = 2Av_2 = 4v_2 = 2^2v_2 \]
\[ A^3v_2 = A(2^2v_2) = 2^3Av_2 = 2^3v_2 \]
\[ \cdots \]
\[ A^{2000}v_2 = 2^{2000}v_2 = \begin{pmatrix} 2^{2000} \\ 2^{2000} \\ 2^{2000} \end{pmatrix} \]

We could not easily calculate \( A^{2000}v_1 \).

Definition 4.2. For an \( n \times n \) matrix \( A \) a vector \( v \) (not equal to the zero vector) and a number \( \lambda \) that satisfy

\[ Av = \lambda v \]

are called an eigenvector and the corresponding eigenvalue of \( A \).

---

Figure F.12: Chapter 4, page 1: Eigenvectors (complete course notes)
Remark 4.3. The zero vector satisfies $A\mathbf{0} = \lambda\mathbf{0}$ for any square matrix $A$ and any number $\lambda$. For this reason, the zero vector does not count as an eigenvector. However, the number zero is permitted as an eigenvalue.

Observation 4.4. If $\mathbf{v}$ is an eigenvector of the matrix $A$ with eigenvalue $\lambda$, it is also an eigenvalue of $A^n$ for every $n$, and the eigenvalue is $\lambda^n$.

Proof. The argument is the same as in the example: Take the eigenvector-eigenvalue equation $A\mathbf{v} = \lambda\mathbf{v}$ and multiply if by $A$ from the left, which gives $A^2\mathbf{v} = \lambda A\mathbf{v}$. Now use the eigenvector equation on the right hand side and get $A^2\mathbf{v} = \lambda^2\mathbf{v}$. Repeating the process $n$ times, we get $A^n\mathbf{v} = \lambda^n\mathbf{v}$.

Example 4.5. Consider the matrix

$$A = \begin{pmatrix} 0 & -6 & 4 \\ 1 & 5 & 2 \\ 0 & 0 & -2 \end{pmatrix}.$$

(a) Show that the vector

$$\mathbf{v} = \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}$$

is an eigenvector of $A$. What is the corresponding eigenvalue?

Solution: Calculate

$$A\mathbf{v} = \begin{pmatrix} 0 \times 3 - 6 \times (-1) + 4 \times 0 \\ 1 \times 3 + 5 \times (-1) + 2 \times 0 \\ 0 \times 3 + 0 \times (-1) - 2 \times 0 \end{pmatrix} = \begin{pmatrix} 6 \\ -2 \\ 0 \end{pmatrix} = 2 \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix} = 2\mathbf{v}.$$

Thus, $\mathbf{v}$ is an eigenvector, and the corresponding eigenvalue is 2.

(b) Show that $\lambda = 3$ is an eigenvalue of $A$. Find all corresponding eigenvectors.

Solution: We need to find a non-zero vector $\mathbf{w}$ that satisfies $A\mathbf{w} = 3\mathbf{w}$. If we write

$$\mathbf{w} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

in components, this equation reads

$$-6y + 4z = 3x$$
$$x + 5y + 2z = 3y$$
$$-2z = 3z$$
4.1. EIGENVECTORS

This is a linear equation system! We rewrite it as

\[-3x - 6y + 4z = 0\]
\[x + 2y + 2z = 0\]
\[5z = 0\]

and solve it:

\[
\begin{pmatrix}
-3 & -6 & 4 & 0 \\
1 & 2 & 2 & 0 \\
0 & 0 & -5 & 0
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 2 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

The general solution is

\[w = \alpha \begin{pmatrix}
-2 \\
1 \\
0
\end{pmatrix}\]

with a free parameter \(\alpha\). All vectors of this form, with \(\alpha \neq 0\), are therefore eigenvectors.

Observation: The linear equation system \(Aw = 3w\) can be rewritten as \((A - 3I_3)w = 0\).

(c) Is \(-5\) an eigenvalue of \(A\)? If so, find the corresponding eigenvectors.

Solution: We have to decide if there is a non-zero vector \(v\) such that \(Av = -5v\) or \((A + 5I_3)v = 0\). We solve this equation system by Gaussian elimination:

\[
\begin{pmatrix}
5 & -6 & 4 & 0 \\
1 & 10 & 2 & 0 \\
0 & 0 & 3 & 0
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{pmatrix}
\]

The only solution is \(v = 0\). There is no non-zero solution. \(-5\) is not an eigenvalue.

Observation 4.6. (a) The eigenvectors of an \(n \times n\) matrix \(A\) corresponding to the eigenvalue \(\lambda\) are the non-zero solutions of the homogeneous linear equation system \((A - \lambda I_n)v = 0\).

(b) If the matrix \(A - \lambda I_n\) is invertible, the only solution to the eigenvector equation \((A - \lambda I_n)v = 0\) is \(v = 0\). In this case, \(\lambda\) is not an eigenvalue of \(A\). The eigenvalues of \(A\) are the values \(\lambda\) for which the matrix \(A - \lambda I_n\) is not invertible.

(c) The solution set of this equation system is a subspace of \(\mathbb{R}^n\), the null space of the matrix \(A - \lambda I_n\). It is called the eigenspace of \(A\) to the eigenvector \(\lambda\). The dimension of the eigenspace is called the geometric multiplicity of the eigenvalue \(\lambda\). The geometric multiplicity of an eigenvalue is the number of linearly independent corresponding eigenvectors.

(d) The eigenspace to the eigenvalue \(\lambda\) contains all eigenvectors to the eigenvalue \(\lambda\) and the zero vector.
Bibliography


