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Geometrical acoustics approximation for Rayleigh and Lamb waves

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Abstract
The present paper gives a brief review of the geometrical acoustics theory for Rayleigh and Lamb waves in inhomogeneous solids, based mainly on the original results of the present author. Initially, the propagation of Rayleigh waves along arbitrary curved surfaces is considered. The obtained results are applied to the explanation of the so-called smooth topographic waveguides for surface waves. Then, flexural waves in plates of variable thickness are considered. The results of this study are used for the development of the theory of the so-called ‘wedge acoustic waves’, i.e. guided ‘one-dimensional’ waves propagating along sharp edges of elastic wedges. Another important application of geometrical acoustics is the development of the theory of ‘acoustic black holes’ for flexural waves that can absorb almost all of the incident wave energy. Finally, the use of geometrical acoustic to describe waves in non-circular shells of arbitrary shape is considered, with applications to modelling structural vibrations of cars and aircraft.

1. Introduction

Geometrical acoustics approximation (GA) is used widely in underwater acoustics or in solving problems of sound propagation in inhomogeneous atmosphere (1). However, in the acoustics of solids its use is not so frequent, which can be partly explained by the complexity of real inhomogeneous solid structures, such as bodies with curved surfaces having large pits, plates of variable thickness, noncircular shells of arbitrary shape, etc. In the same time, the use of GA to describe wave propagation in such inhomogeneous solids is very efficient, and it can provide useful information about the nature of the defect that can be used for condition monitoring purposes (2). Geometrical acoustics approximation is an asymptotic high-frequency solution (sometimes called ray-tracing solution) to the differential equations and boundary conditions describing wave propagation in any particular medium or structure.

In this paper, a brief review of the developments of the geometrical acoustics theory for waves in inhomogeneous solids over the last three decades is given, based mainly on the original results of the present author. Initially, the propagation of Rayleigh waves along arbitrary curved surfaces is considered. The obtained results are applied to the description of Rayleigh wave propagation in the vicinity of large two-dimensional indentations (pits) and to the explanation of the so-called smooth topographic
waveguides for surface waves. Then, flexural and compression waves in plates of variable thickness are considered. The obtained results are used in particular for the development of the theory of the so-called ‘wedge acoustic waves’, i.e. guided ‘one-dimensional’ waves propagating along sharp edges of elastic wedges of different types. The use of GA in this case permits to consider wedges of arbitrary shape, curved wedges and truncated wedges. Another important application of geometrical acoustics is the development of the theory of ‘acoustic black holes’ for flexural waves. Such objects, that can be easily manufactured, can absorb almost all of the incident wave energy.

Finally, the use of GA to describe waves in non-circular shells of arbitrary shape is considered, with applications to modelling structural vibrations of cars and aircraft.

2. Geometrical acoustics of Rayleigh waves on curved surfaces

It is well known that if a Rayleigh wave propagates along a curved surface its velocity changes due to the effect of curvature \( \left( \frac{\partial^2}{\partial z^2} \right) \). Generally, curved surfaces represent anisotropic and inhomogeneous media for propagating Rayleigh waves. As it obviously follows from geometry, only a spherical surface is both isotropic and homogeneous. A circular cylindrical surface is anisotropic, but homogeneous, and a non-circular cylindrical surface is both anisotropic and inhomogeneous (in the direction perpendicular to the cylindrical axis). Thus, if the surface of a solid body is curved, then there are two main features associated with propagation of high-frequency Rayleigh waves: the anisotropy of the local wave velocity and the wave refraction associated with surface inhomogeneity.

The starting point for the geometrical acoustics approximation for Rayleigh waves on curved surfaces of arbitrary form is the high-frequency asymptotic expression for the local Rayleigh wave velocity as a function of two local radii of curvature \( \left( \frac{\partial^2}{\partial z^2} \right) \):

\[
\eta = a_u \frac{1}{k_0 \rho_u} + a_v \frac{1}{k_0 \rho_v},
\]

where \( c_0 \) is the Rayleigh wave velocity on a plane surface, \( k_0 = \omega c_0 \) is the corresponding wave number, \( \rho_u \) and \( \rho_v \) are the radii of the surface curvature in the direction of wave propagation and in the direction perpendicular to it, respectively, \( a_u \) and \( a_v \) are the non-dimensional coefficients that depend on Poisson’s ratio of the medium. Note that expression (1) has been established by several researchers, including the present author (see Ref 2 for details). The values of \( a_u \) and \( a_v \) for all values of Poisson’s ratio can be found in Ref 2. The interesting fact to be mentioned here is that \( a_u \) is always positive, whereas \( a_v \) is always negative. The latter feature is very important, and, as will be discussed in the next section, it is responsible for the existence of the so-called smooth topographic waveguides for Rayleigh waves.

As soon as the two radii of curvature \( \rho_u \) and \( \rho_v \) are known as functions of surface coordinates, the usual formalism of geometrical acoustics (in scalar approximation) can be applied to describe either vertical or horizontal component of a propagating Rayleigh wave in the arbitrary point of the curved surface. In order to do so, one should initially establish the trajectories of all possible rays that can be traced from a chosen point of Rayleigh wave excitation. After the ray trajectories have been established, the solution...
for a wave propagating along any particular trajectory, using a coordinate $s$ measured along the trajectory, can be written in the form \(^{(1, 2)}\)

$$u = A(s) e^{i \int k(s) ds}.$$  

(2)

Here $A(s)$ and $k(s) = \omega c$ are slowly varying functions describing the amplitude and the local wavenumber of the Rayleigh wave.

To calculate ray trajectories of Rayleigh waves propagating over surfaces of variable curvature it is convenient to use the Hamiltonian approach \(^{(2, 3)}\). For example, in the case of a cylindrical indentation (pit) of variable curvature (see Figure 1), as the generalised coordinates $x_i$ used in Hamiltonian formulation, one can take the distance $r$ measured along the surface from the pit’s centre and the polar angle $\theta$.

**Figure 1. Geometry of a cylindrical pit of variable curvature**

Important parameters in this formulation are: $\alpha$, the angle between the position vector $r$ and the Rayleigh wave vector $k$, and $n(r, \alpha) = c_0/c(r, \alpha)$, the index of refraction for a Rayleigh wave on a curved surface.

**Figure 2. Ray trajectories of Rayleigh waves propagating in the vicinity of a cylindrical pit of variable curvature \(^{(3)}\).**

In a similar way, one can write down the ray trajectory equations in the case of one-dimensional inhomogeneous surfaces (a cylinder of varying curvature) \(^{(3)}\). In this case, it is convenient to use surface coordinates $x$ and $z$, along the axis of cylinder and perpendicular to it respectively, so that $n(z, \alpha) = c_0/c(z, \alpha)$. 
The example of calculated trajectories of Rayleigh waves propagating over a smooth pit are shown in Figure 2 (3). One can see that rays are deflected away from the centre of the pit.

In the case of a smooth noncircular cylinder (Figure 3), the calculated ray trajectories of Rayleigh waves propagating from the point of excitation located in the area of maximum curvature are shown in Figure 4 (3). It can be seen that Rayleigh waves in this case become captured within the area of maximum curvature corresponding to the minimum of phase velocity, which constitutes a waveguide effect of such surfaces.

![Figure 3. Geometry of a noncircular cylinder](image)

![Figure 4. Ray trajectories of Rayleigh waves propagating at arbitrary angles in respect of the axis of a noncircular cylinder](image)

3. Topographic waveguides for Rayleigh waves

Solid structures of some types can guide surface acoustic waves. If the waveguide properties are attributed to the influence of the surface geometry, then the associated waveguides are usually called topographic waveguides. The need to take into account guiding properties of surfaces appears in seismology and in different applications of ultrasonic non-destructive testing. For example, in condition monitoring using acoustic emission, it is often necessary to be able to predict the most likely directions of propagation for Rayleigh waves radiated by a developing crack. Since the likely locations of emerging cracks can be anticipated, it is important to be able to predict
possible paths of guided waves propagation, where placement of acoustic emission sensors would be most efficient.

The analysis of topographic waveguides is very difficult, and as a rule it requires numerical approaches. However, there are several important situations that can be considered analytically. Among these situations in particular are smooth topographic waveguides, for which the minimum radius of curvature of the surface is greater than the surface wavelength. Such waveguides can be considered in geometrical acoustics approximation on the basis of the asymptotic expression (1) for the local velocity of Rayleigh waves.

The first geometrical acoustics theory of smooth topographic waveguides and the first physical interpretation of the Rayleigh wave localisation in these and similar smooth topographic structures have been given by the present author more than 20 years ago (4). See also the monograph (2), where a special chapter is devoted to Rayleigh waves on curved surfaces of arbitrary form (Chapter 9), and another separate chapter (Chapter 10) describes waves in topographic waveguides, including smooth topographic waveguides (see Section 10.5). The geometrical acoustics theory of smooth topographic waveguides provides a clear and physically explicit explanation of the reason of the waveguide effect in such structures. Namely, it demonstrates that the existence of localised propagating modes of Rayleigh waves in smooth topographic waveguides can be explained by two factors. The first one is the presence of an internal area with the curvature-dependent local phase velocity of Rayleigh waves being smaller in the direction of guided wave propagation than their velocity for an adjacent flat surface. And the second factor is the possibility of total internal reflection of the curvature-modified Rayleigh waves from the surrounding areas of zero or negative curvature (2, 4).

![Figure 5. Ridge and groove types of smooth topographic waveguides](image)

It is interesting to note that papers still appear in which the authors, who apparently are unfamiliar with geometrical acoustics, try to give physical interpretations of waveguide effect of surface topography. For example, in the recently published paper (5), the authors consider asymptotically and numerically Rayleigh wave propagation in a topographic waveguide formed by a smooth ridge-type elevation over a flat surface. They also attempt to give a physical interpretation to the phenomenon of Rayleigh wave localisation within the elevated area. Although the factual results obtained by the authors seem to be correct, albeit not so novel, their physical interpretation of the localisation phenomenon is not satisfactory and not convincing. Contrary to this, the approach based on geometrical acoustics approximation provides a clear physical
explanation of the guiding properties of surface topography in all structures of this type and gives numerical results for phase velocities of guided modes that are in good agreement with numerical calculation and with experiments.

4. Geometrical acoustics of Lamb modes

Geometrical acoustics approximation can be applied also to Lamb waves propagating in plates of variable thickness. Using this approach, it is possible to consider real life complex structures that are used for condition monitoring. The most important modes of Lamb waves are lowest order antisymmetric and symmetric modes, or flexural and compression waves respectively. In what follows, a particular type of plates of variable thickness will be discussed – plates with linearly variable thickness, i.e. slender elastic wedges. To develop a geometrical acoustics theory of flexural or compression wave propagation in slender wedges one can start from the corresponding reduced equations for flexural and compression waves respectively and seek the solution of these equations in the form similar to equation (2). It can be shown that equation (2) satisfies both these equations asymptotically at relatively high frequencies. In practice though this means that frequencies are well within the range of practically used frequencies.

Figure 6. Geometry of a slender solid wedge

5. Geometrical acoustics theory of wedge acoustic waves

One of the important applications of geometrical acoustic to Lamb modes is the development of the analytical theory of localised elastic waves propagating along the tip of an elastic wedge, also known as wedge acoustic waves. The approximate theory of localised elastic waves in slender solid wedges has been developed by the present author (6, 7) (see also the monograph 2), and it is based on the geometrical acoustics approach considering a slender wedge as a plate with a local variable thickness \( d = x\Theta \), where \( \Theta \) is the wedge apex angle and \( x \) is the distance from the wedge tip measured in the middle plane (see Figure 6). The velocities \( c \) of the localised wedge modes propagating in \( y \) direction are determined in the geometrical acoustics approximation as solutions of the Bohr - Sommerfeld type equation

\[
\int_0^x \left[ k^2(x) - \beta^2 \right]^{1/2} dx = n, \tag{3}
\]
where $\beta = \omega/c$ is yet unknown wavenumber of a wedge mode, $k(x)$ is a local wavenumber of a flexural wave in a plate of variable thickness, $n = 1, 2, 3, \ldots$ is the mode number, and $x_t$ is the so called ray turning point being determined from the equation $k^2(x) - \beta^2 = 0$.

For example, in the case of a wedge in vacuum $k(x) = 12^{1/4} k_p^{1/2} (\Theta x)^{-1/2}$, $x_t = 2 \sqrt{3} k_p / \Theta \beta^2$, and $k_p = \omega/c_p$, where $\omega$ is circular frequency, $c_p = 2 c_t (1 - c_t^2 / c_l^2)^{1/2}$ is the so called plate wave velocity, $c_l$ and $c_t$ are propagation velocities of longitudinal and shear acoustic waves in plate material. Then, taking analytically the integral in (2) and solving the resulting algebraic equation yields the extremely simple analytical expression for wedge wave velocities (6, 7):

$$c = \frac{c_p n \Theta}{\sqrt{3}}. \quad (4)$$

The expression (4) agrees well with the other (numerically based) theories and with the available experimental results. The analytical expressions for amplitude distributions of wedge modes are rather cumbersome (6) and are not displayed here. Figure 7 illustrates the first three modes calculated in geometrical acoustics approximation (6).

![Figure 7. First three modes of wedge acoustic waves calculated using geometrical acoustics approximation (6)](image)

The structure of the wedge modes shown in Figure 7 agrees well with the results of numerical calculations, with the exception of the clearly seen singularities in the areas marked by dashed vertical lines. Beyond these lines, which show the locations of caustics (ray congestions), the modes do not penetrate into the depth of a wedge. The above singularities manifest the well-known limitation of all geometrical acoustics (optics) theories that become invalid near caustics (1, 2). The above-mentioned geometrical acoustics theory of wedge acoustic waves can be generalised to consider localised modes of quadratically-shaped elastic wedges (8), wedges immersed in liquids (9), cylindrical and conical wedge-like structures (curved wedges) (10), wedges of general power-law shape (11), and wedges made of anisotropic materials (12).
6. Acoustic black holes for flexural waves

Another important application of geometrical acoustics to waves in plates of variable thickness is the theory of the so-called ‘acoustic black holes’ for flexural waves that can absorb almost 100% of the incident wave energy. This effect can be used for damping resonant vibrations \(^{(13)}\). To understand the phenomenon of acoustic black holes, or the acoustic black hole effect, one should start with the simplest one-dimensional case of plane flexural wave propagation in the normal direction towards the edge of a free wedge with the local thickness \(h(x)\) described by a power-law relationship \(h(x) = \varepsilon x^m\), where \(m\) is a positive rational number and \(\varepsilon\) is a constant (see Figure 8).

\[ k(x) = \left(\frac{\varepsilon_{ps}}{2}\right)^{1/2} \left(\frac{1}{\varepsilon^m}\right)^{1/2} \]

where \(\varepsilon_{ps} = \omega/c_p\) is the wavenumber of a symmetrical plate wave, \(c_p = 2c_l\sqrt{1-c_l^2/c_t^2}\) is its phase velocity, and \(c_l\) and \(c_t\) are longitudinal and shear wave velocities in a wedge material, and \(\omega = 2\pi f\) is circular frequency. One can see that the integral in (5) diverges for \(m \geq 2\). This indicates that the total phase \(\Phi\) becomes infinite under these circumstances, which means that the wave never reaches the edge. Therefore, it never reflects back either, i.e. the wave becomes trapped at the edge, thus indicating that the above mentioned ideal wedges of power-law profile for \(m \geq 2\) materialise acoustic ‘black holes’ for incident flexural waves \(^{(14,15)}\) (see also Ref 16 that gives a slightly different interpretation of this phenomenon).

Real fabricated wedges, however, are not ideally sharp, and they always have truncated edges. And this adversely affects their performance as ‘black holes’. If for ideal wedges of power-law shape (with \(m \geq 2\)) it follows from (5) that even an infinitely small material attenuation, described by the imaginary part of \(k(x)\), would be sufficient

![Figure 8. Wedge of power-law profile](image)
for the whole wave energy to be absorbed, this is not so for truncated wedges. Indeed, for truncated wedges the lower integration limit in (5) must be changed from $\theta$ to a certain value $x_0$ describing the length of truncation, thus resulting in the amplitude of the total reflection coefficient $R_0$ being expressed in the form (6)

$$R_0 = \exp(-2\int_{x_0}^{x} \text{Im}k(x)dx).$$

According to (6), for low values of wave attenuation in such typical wedge materials as steel, even very small truncations $x_0$ result in the reflection coefficients $R_0$ becoming as large as 50-70 %, which makes it impossible to use such wedges in practice.

In order to improve the situation for real wedges (with truncations), it has been proposed by the present author to cover wedge surfaces near the edges by thin absorbing layers (films) of thickness $\delta$, e.g. by polymeric films. To analyse the effect of thin absorbing films on flexural wave propagation in a wedge using geometrical acoustics approximation one should consider first the effect of such films on flexural wave propagation in plates of constant thickness. The simplest way of approaching this problem is to use the already known solutions for plates covered by absorbing layers of arbitrary thickness obtained by different authors with regard to the description of damped vibrations in sandwich plates. Using this approach, one can derive the analytical expressions for the reflection coefficients of flexural waves from the edges of truncated wedges covered by thin absorbing layers.

Typical calculated values of the reflection coefficient $R_0$ for flexural waves in real quadratic wedges are as follows: in the presence of the damping film $R_0 = 2 – 4 \%$, whereas in the absence of the damping film $R_0 = 50 – 70 \%$. Thus, in the presence of the damping film the value of the reflection coefficient is much smaller than for a wedge with the same value of truncation, but without a film. Note that almost all absorption of the incident wave energy takes place in the vicinity of the sharp edge of a wedge. The above predictions have been confirmed experimentally and numerically.

The next important step is to consider the case of two-dimensional acoustic black holes, such as nearly or slightly protruding cylindrically symmetrical indentations (pits) of power-law profile drilled in a regular thin plate of constant thickness. To consider bending wave propagation over two-dimensional pits of power-law profile it is convenient to use a geometrical acoustics approach in Hamiltonian formulation, similar to that described in Section 2 for Rayleigh wave propagation across smooth large-scale surface irregularities. The analysis shows that the presence of a pit influences the rays of flexural waves that propagate through it. In particular, in the case of symmetrical pits of power-law profile with $m \geq 2$, all rays incident at angles $\alpha_0$ less than a certain critical value $\alpha_{cr}$ (that depends on the distance $r_0$ from the source to the centre of the pit), including a direct ray for which $\alpha_0 = 0$, will deflect towards the centre of the pit, approaching its centre almost in the normal direction. If a piece of polymeric film is attached to the central area of a pit, then all such rays will be efficiently absorbed by the pit, which thus materialises a two-dimensional acoustic black hole.

For practical purposes, the above-mentioned two-dimensional acoustic black hole (a power-law pit) can be placed at any point of a plate- or shell-like structure. The effect of such a black hole will be in eliminating the rays intersecting with it from contributing to the overall frequency response function of a structure, which will result in substantial damping of resonant vibrations. To amplify the effect of two-dimensional acoustic
black holes one can use combinations or arrays of several black holes distributed over the structure, if this does not compromise the main functions of the latter (19).

One of the most important advantages of the acoustic black holes as dampers of structural vibrations is that they are efficient even for relatively thin and narrow strips of attached absorbing layers (14, 15, 17, 18). This is in contrast with the traditional techniques requiring covering the entire surfaces of structures by relatively thick layers of absorbing materials (13). This important feature of acoustic black holes can be very attractive for many practical applications, e.g. in transport engineering, where the reduction of mass of a vehicle is highly desirable for fuel economy and environmental reasons.

7. Waves in noncircular cylindrical shells.

Non-circular cylindrical shells are relatively simple structures that are used widely for their own right and as elements of more complex structures, especially in aeronautical, automotive and oceanic engineering. They are also used frequently for modelling structural vibrations and structure-borne vehicle interior noise in all means of transportation. Note that obtaining analytical solutions for non-circular cylindrical shells is extremely difficult since it requires solving a system of governing differential equations with variable coefficients that describe the effects of variable curvature.

![Figure 9. Waveguide propagation in a non-circular cylindrical shell](image)

The geometrical acoustics approach to the description of waveguide properties of shells and calculation of their resonant vibrations simplifies the problem substantially. As in the case of Rayleigh waves and flexural waves in elastic wedges, the physical reason for waveguide propagation along quasi-flat areas of non-circular cylindrical shells is the difference between flexural wave velocities in their quasi-flat and curved parts. In particular, for waveguide propagation being possible it is necessary that the velocity of flexural waves in the adjacent curved areas is higher than in the quasi-flat area. This is always the case for flexural wave propagation in near-axial directions of curved shells.

It is convenient to consider waveguide propagation of flexural waves in a simple non-circular shell comprising an infinitely long flat plate (strip) of thickness \( h \) and width \( a \) bounded by fragments of two cylindrical shells having equal radii \( R \) (Figure 9). Like in similar cases for Rayleigh waves, such a structure can be considered as a three-layered anisotropic medium for flexural waves. The middle layer (a flat isotropic
strip) is characterised by a lower phase velocity of flexural waves in comparison with the velocities in the near-axial direction of the adjacent cylindrical shells. Waveguide propagation in such a three-layered medium can be described in the same way as guided Rayleigh waves or guided flexural waves in elastic wedges have been described in the previous sections. For example, using Bohr-Sommerfeld type equation similar to (3).

It is well known (21) that flexural waves in shells, being in fact curvature-modified Lamb modes, are governed by bending and membrane effects, which makes the expressions for their flexural type wavenumbers (that are anisotropic) rather complex. Their functional appearance depends on the characteristic parameters of the shell, in particular on its ring frequency $\omega_r$ and on its mean radius of curvature $R$.

Starting from the simplified dispersion equation for flexural type waves in a circular cylindrical shell (21), one can derive analytical expressions for the wavenumbers of guided flexural type waves in cylindrical shells. These expressions can be used for developing the geometrical acoustics theory of topographic waveguides formed by noncircular shells. The results for velocities of guided waves and for the values of resonant frequencies of vibration of finite shells obtained in this way are in good agreement with the numerical calculations and with the experimental measurements (20).

8. Conclusions

It has been demonstrated that geometrical acoustics approximation for Rayleigh waves and for Lamb waves of the lowest order, especially flexural waves, is a powerful tool that provides a researcher with the possibility to develop theoretical descriptions of wave propagation in a variety of important complex solid structures.

In many cases geometrical acoustics also gives a clear and physically explicit explanation of the physical mechanisms behind the wave phenomena in question.

References